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ORIGINAL PAPER

Error bounds and gap functions for various variational type problems

Aviv Gibali[1] · **Salahuddin**[2]

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Abstract
In this work we study various gap functions for the generalized multivalued mixed variational-hemivariational inequality problems by using the $(\tau_{\mathscr{M}}, \sigma_{\mathscr{M}})$-relaxed cocoercive mapping and Hausdorff Lipschitz continuity. Moreover, we establish global error bounds for such inequalities using the characteristic of the Clarke generalized gradient method. As application, we present a stationary nonsmooth semipermeability problem.

Keywords Generalized multivalued mixed variational-hemivariational inequality problems · Gap function · Regularized gap function · Global error bounds · Semipermeability problem

Mathematics Subject Classification 47J20 · 49J40 · 49J45 · 74M10 · 74M15

1 Introduction

Fichera [1,2] and Stampacchia [3] initiated the theory of variational inequality problem for mathematical modelling problems arising from mechanics to investigate the regularity problem for partial differential equations. The variational inequalities and quasi variational inequalities have many application in different areas such as economics, management and engineering sciences. The theory of variational inequality can also be utilized as a core problem in optimization and nonlinear analysis to analyse the tremendous problems of complementarity and equilibrium in operational science, we often meet the variational inequality problem for finding $x \in \mathcal{C}$ such that

$$\langle \mathscr{B}(x), y - x \rangle_{\mathbb{X}} \geq 0, \quad \forall y \in \mathcal{C}, \tag{1}$$

✉ Aviv Gibali
avivg@braude.ac.il

Salahuddin
smohammad@jazanu.edu.sa

1 Department of Mathematics, ORT Braude College, P.O. Box 78, Karmiel 2161002, Israel

2 Department of Mathematics, Jazan University, Jazan 45142, Kingdom of Saudi Arabia

where $\mathcal{C}$ is a nonempty closed convex subset of a normed space $\mathbb{X}$ representing constraints, $\mathscr{B} : \mathbb{X} \longrightarrow \mathbb{X}^*$ is a given operator, and $\langle \cdot, \cdot \rangle_{\mathbb{X}}$ denotes the duality pairing between $\mathbb{X}$ and its dual $\mathbb{X}^*$.

It is well known that the variational inequality (1) can be solved by transforming it into an equivalent optimization problem for the so-called merit function $\mu(\cdot; \alpha) : \mathbb{X} \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\mu(x; \alpha) = \sup\{\langle \mathscr{B}(x), x - z \rangle_{\mathbb{X}} - \alpha \|x - z\|_{\mathbb{X}}^2 \mid z \in \mathcal{C}\} \quad \text{for} \quad x \in \mathcal{C}, \tag{2}$$

where α is a nonnegative parameter. Here, we note that

(i) If $\alpha = 0$ and $\mathbb{X}$ is finite dimensional, then (2) was first studied by Auslender in [4].
(ii) If $\alpha > 0$ and $\mathbb{X}$ is finite dimensional, then (2) was studied by Fukushima in [5].

The function $\mu(\cdot, 0)$ is usually known as the gap function, and the function $\mu(\cdot, \alpha)$ for $\alpha > 0$ is a regularized gap function.

Also, we notes that for all $\alpha > 0$, the function $\mu(\cdot, \alpha)$ is nonnegative on $\mathcal{C}$, and $\mu(x^*; \alpha) = 0$ whenever x^* satisfies the variational inequality (1), *see* [6].

The theory of gap function was first introduced for the study of optimization problem and subsequently applied to variational inequalities, quasi variational inequalities and quasi vector variational inequalities. The concept of gap function plays an numerous role in the development of iterative algorithms, an evaluation of their convergence properties and useful stopping methods for iterative algorithms. Error bounds are very important and used because they provide a measure of the distance between a solution set and a feasible arbitrary point. A comprehensive survey of theory and rich applications about error bounds can be found in [7]. Solodov [8] developed some merit function associated with a generalized mixed variational inequality, and used those functions to achieve mixed variational error limits. Aussel et al. [9] introduced a new inverse quasi variational inequality, obtained local (global) error bounds for inverse quasi variational inequality in terms of certain gap functions to demonstrate the applicability of inverse quasi variational inequality. Focused on the Fukushima [10] concept, the regularized function of the Moreau–Yosida type has been introduced by Yamashita and Fukushima in [11]. They also suggested the so-called error bounds for variational inequalities *via* the regularized gap functions. Recently, there have been many studies on gap functions for different models on different topics such as iterative algorithms [12], the Painlevé–Kuratowski convergence [13] and error bounds [14–16]. In 2020, Chang et al. [17] introduce the mixed set-valued vector inverse quasi-variational inequality problems and to obtain error bounds for this kind of mixed set-valued vector inverse quasi-variational inequality problems in terms of the residual gap function, the regularized gap function, and the D-gap function. These bounds provide effective estimated distances between an arbitrary feasible point and the solution set of mixed set-valued vector inverse quasi-variational inequality problem. Recently, Chang et al. [18] studied the three types of gap functions, i.e., the residual gap function, the regularized gap function and the global gap function by using the relaxed monotonicity and Hausdorff Lipschitz continuity and obtained the error bounds for generalized vector inverse-variational inequality problems.

Hemivariational variational inequalities, which were first introduced by Panagiotopoulos [19,20], deal with certain mechanical problems involving nonconvex and nonsmooth energy functions. The theory of variational-hemivariational inequalities is known as a generalization of variational inequalities and hemivariational inequalities to the case involving both the convex and the nonconvex potentials, and based on the notion of the Clarke generalized gradient for locally Lipschitz functions, *see,* [21–25].

Inspired by the recent works [26–33], in this paper, we suggest the gap functions and regularized gap functions for a class of generalized multivalued mixed variational-hemivariational inequality problems. We also discuss the gap functions for the Minty version of these inequalities by utilizing the Lipschitz continuity, $(\tau_{\mathscr{M}}, \sigma_{\mathscr{M}})$-relaxed cocoercive mapping and Hausdorff Lipschitz continuous mapping and also provides two new global error bounds for the generalized multivalued mixed variational-hemivariational inequality problems. Finally, as application, we present a semipermeability problem for stationary heat problem is given to illustrate our main results.

2 Preliminaries

Throughout this paper, let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a real Banach space with the dual $\mathbb{X}^*$, and $\langle\cdot,\cdot\rangle_{\mathbb{X}}$ be the duality pairing between $\mathbb{X}^*$ and $\mathbb{X}$. Let $CB(\mathbb{X})$ be the family of all nonempty closed and bounded sets in $\mathbb{X}$.

Definition 1 A function $\mathscr{A} : \mathbb{X} \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be

(a) proper, if $\mathscr{A} \neq +\infty$.
(b) convex, if $\mathscr{A}(tx + (1-t)y) \leq t\mathscr{A}(x) + (1-t)\mathscr{A}(y), \forall x, y \in \mathbb{X},\ t \in [0, 1]$.
(c) lower semicontinuous (l.s.c.) at $x \in \mathbb{X}$, if for any sequence $\{x_n\} \subset \mathbb{X}$ such that

$$x_n \longrightarrow x,$$

it holds

$$\mathscr{A}(x) \leq \liminf \mathscr{A}(x_n).$$

(d) upper semicontinuous (u.s.c.) at $x \in \mathbb{X}$, if for any sequence $\{x_n\} \subset \mathbb{X}$ such that

$$x_n \longrightarrow x,$$

it holds

$$\limsup \mathscr{A}(x_n) \leq \mathscr{A}(x).$$

(e) l.s.c (resp. u.s.c.) on $\mathbb{X}$, if $\mathscr{A}$ is l.s.c (resp. u.s.c.) at every $x \in \mathbb{X}$.

Definition 2 Let $g : \mathbb{X} \to \mathbb{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous function. The convex subdifferential $\partial_c g : \mathbb{X} \to \mathbb{X}^*$ of g is defined by

$$\partial_c g(x) = \left\{x^* \in \mathbb{X}^* \middle| \langle x^*, y - x\rangle_{\mathbb{X}} \leq g(y) - g(x), \quad \forall\, y \in \mathbb{X}\right\} \quad \forall\, x \in \mathbb{X}.$$

An element $x^* \in \partial_c g(x)$ is called a subgradient of g at $x \in \mathbb{X}$.

Definition 3 A function $\mathscr{A} : \mathbb{X} \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in \mathbb{X}$, there exist a neighbourhood U of x and a constant $\varepsilon_x > 0$ such that

$$|\mathscr{A}(z_1) - \mathscr{A}(z_2)| \leq \varepsilon_x \|z_1 - z_2\|_{\mathbb{X}}, \quad \forall\, z_1, z_2 \in \mathrm{U}.$$

Let $\mathscr{A} : \mathbb{X} \to \mathbb{R}$ be a locally Lipschitz function, the Clarke generalized directional derivative of $\mathscr{A}$ at the point $x \in \mathbb{X}$ in the direction $y \in \mathbb{X}$ defined by

$$\mathscr{A}^\circ(x; y) = \limsup_{z \longrightarrow x, t \longrightarrow 0^+} \frac{\mathscr{A}(z + ty) - \mathscr{A}(z)}{t}.$$

The generalized gradient of $\mathscr{A}$ at $x \in \mathbb{X}$ is a subset of $\mathbb{X}$ defined by

$$\partial\mathscr{A}(x) = \left\{x^* \in X^* | \mathscr{A}^\circ(x; y) \geq \langle x^*, y\rangle_{\mathbb{X}} \quad \forall\, y \in \mathbb{X}\right\}.$$

Lemma 1 *Let $\mathbb{X}$ be a real Banach space and $\mathscr{A} : \mathbb{X} \to \mathbb{R}$ be a locally Lipschitz function, then the following assumptions are satisfied:*

(a) *For each $x \in \mathbb{X}$, the function $\mathbb{X} \ni y \rightarrowtail \mathscr{A}^\circ(x; y) \in \mathbb{R}$ is finite, positively homogeneous and subadditive, and*

$$\left|\mathscr{A}^\circ(x; y)\right| \leq \varepsilon_x \|y\|_{\mathbb{X}}\, \forall\, y \in \mathbb{X},$$

where $\varepsilon_x > 0$ is a Lipschitz constant of $\mathscr{A}$ near x.

(b) *The function $\mathbb{X} \times \mathbb{X} \ni (x, y) \rightarrowtail \mathscr{A}^\circ(x; y) \in \mathbb{R}$ is upper semicontinuous.*

(c) *For every $x, y \in \mathbb{X}$, it holds*

$$\mathscr{A}^\circ(x; y) = \max\{\langle\varsigma, y\rangle_{\mathbb{X}} \mid \varsigma \in \partial\mathscr{A}(x)\}.$$

Definition 4 An operator $\mathscr{P} : \mathbb{X} \to CB(\mathbb{X}^*)$ is said to be pseudomonotone, if $\mathscr{P}$ is a bounded operator and for every sequence $\{p_n\} \subseteq \mathbb{X}$ converging weakly to $p \in \mathbb{X}$ such that

$$\limsup_{n\longrightarrow\infty}\langle p_n, x_n - x\rangle \leq 0,\ \forall p_n \in \mathscr{P}(x_n)$$

we have

$$\langle p, x - y\rangle \leq \liminf_{n\longrightarrow\infty}\langle p_n, x_n - y\rangle, \quad \forall\, y \in \mathbb{X}, \quad p \in \mathscr{P}(x).$$

Definition 5 An operator $\mathscr{M} : \mathbb{X}\times\mathbb{X} \to \mathbb{X}^*$ is said to be pseudomonotone with both variable, if $\mathscr{M}$ is a bounded operator and for every sequence $\{x_n\} \subseteq \mathbb{X}$ converging weakly to $x \in \mathbb{X}$ such that

$$\limsup_{n\longrightarrow\infty}\langle\mathscr{M}(x_n, x_n), x_n - x\rangle \leq 0,$$

we have

$$\langle\mathscr{M}(x, x), x - y\rangle \leq \liminf_{n\longrightarrow\infty}\langle\mathscr{M}(x_n, x_n), x_n - y\rangle,\ \forall\, y \in \mathbb{X}.$$

Let $\mathbb{X}$ be a reflexive Banach space and $\mathcal{C}$ be a nonempty subset of $\mathbb{X}$. Let $\mathscr{P}, \mathscr{Q} : \mathcal{C} \longrightarrow CB(\mathbb{X}^*)$ be the multivalued mappings, $\mathscr{M} : CB(\mathbb{X}^*)\times CB(\mathbb{X}^*) \longrightarrow CB(\mathbb{X}^*)$ be an operator and $\phi : \mathcal{C} \times \mathcal{C} \longrightarrow \mathbb{R}$ and $J : \mathbb{X} \longrightarrow \mathbb{R}$ be the functionals, and $f \in \mathbb{X}$. Our purpose of this paper is to study following constrained generalized multivalued mixed variational-hemivatiational inequality problem for finding $x \in \mathcal{C}$, $p \in \mathscr{P}(x)$ and $q \in \mathscr{Q}(x)$ such that

$$\begin{aligned}&\langle\mathscr{M}(p, q) - f, y - x\rangle_{\mathbb{X}} + \phi(x, y) - \phi(x, x) + J^\circ(x; y - x)\\ &\geq 0,\ \forall\, y \in \mathcal{C},\ p \in \mathscr{P}(x), q \in \mathscr{Q}(x),\end{aligned} \tag{3}$$

with the following assertions:

(1) the operator $\mathscr{M} : CB(\mathbb{X}^*) \times CB(\mathbb{X}^*) \longrightarrow CB(\mathbb{X}^*)$ is satisfying

(1(a)) $\mathscr{M}$ is continuous mapping with both variables.
(1(b)) $\mathscr{M}$ is pseudomonotone.
(1(c)) $\mathscr{M}$ is Lipschitz continuous with respect to first variable with constant $\alpha_{\mathscr{M}} > 0$ and second variable with constant $\beta_{\mathscr{M}} > 0$ such that

$$\begin{aligned}&\|\mathscr{M}(x_1, y_1) - \mathscr{M}(x_2, y_2)\|_{\mathbb{X}^*}\\ &\leq \alpha_{\mathscr{M}}\|x_1 - x_2\|_{\mathbb{X}} + \beta_{\mathscr{M}}\|y_1 - y_2\|_{\mathbb{X}}, \quad \forall\, x_1, x_2, y_1, y_2 \in \mathbb{X}.\end{aligned} \tag{4}$$

(1(d)) $\mathscr{M}$ is $(\tau_{\mathscr{M}}, \sigma_{\mathscr{M}})$-relaxed cocoercive if there exist two constants $\tau_{\mathscr{M}}, \sigma_{\mathscr{M}} > 0$ such that

$$\langle \mathscr{M}(x_1, y_1) - \mathscr{M}(x_2, y_2), y_1 - y_2 \rangle_{\mathbb{X}} \geq -\tau_{\mathscr{M}} \|\mathscr{M}(x_1, y_1) - \mathscr{M}(x_2, y_2)\|^2_{\mathbb{X}^*} + \sigma_{\mathscr{M}} \|y_1 - y_2\|^2_{\mathbb{X}}, \quad \text{for all } x_1, x_2, y_1, y_2 \in \mathbb{X}. \tag{5}$$

(2) the operator $\mathscr{P}, \mathscr{Q} : \mathbb{X} \longrightarrow CB(\mathbb{X}^*)$ is satisfying

(2(a)) both $\mathscr{P}, \mathscr{Q}$ are pseudomonotone.

(2(b)) $\mathscr{P}$ is Hausdorff Lipschitz continuous then there exists $\alpha_{\mathscr{P}} > 0$ such that

$$\|p_1 - p_2\|_{\mathbb{X}^*} \leq \mathcal{H}(\mathscr{P}(x_1), \mathscr{P}(x_2))_{\mathbb{X}^*} \leq \alpha_{\mathscr{P}} \|x_1 - x_2\|_{\mathbb{X}}, \quad \forall\, x_1, x_2 \in \mathbb{X}, \tag{6}$$

where $\mathscr{H}(\cdot, \cdot)$ is the Hausdorff metric on $CB(\mathbb{X})$.

(2(c)) $\mathscr{Q}$ is Hausdorff Lipschitz continuous then there exists $\alpha_{\mathscr{Q}} > 0$ such that

$$\|q_1 - q_2\|_{\mathbb{X}^*} \leq \mathcal{H}(\mathscr{Q}(x_1), \mathscr{Q}(x_2))_{\mathbb{X}^*} \leq \alpha_{\mathscr{Q}} \|x_1 - x_2\|_{\mathbb{X}}, \quad \forall\, x_1, x_2 \in \mathbb{X}. \tag{7}$$

(3) $\phi : \mathcal{C} \times \mathcal{C} \longrightarrow \mathbb{R}$ is such that

(3(a)) for each $x \in \mathcal{C}$, $\phi(x, \cdot) : \Omega \longrightarrow \mathbb{R}$ is convex and lower semicontinuous.

(3(b)) there exists $\alpha_{\phi} > 0$ such that

$$\phi(x_1, y_2) - \phi(x_1, y_1) + \phi(x_2, y_1) - \phi(x_2, y_2) \leq \alpha_{\phi} \|x_1 - x_2\|_{\mathbb{X}} \|y_1 - y_2\|_{\mathbb{X}}, \quad \forall\, x_1, x_2, y_1, y_2 \in \mathcal{C}. \tag{8}$$

(4) $J : \mathbb{X} \longrightarrow \mathbb{R}$ is a locally Lipschitz function such that

(4(a)) $\|\partial J(y)\|_{\mathbb{X}^*} \leq \omega_0 + \omega_1 \|y\|_{\mathbb{X}}, \ \forall\, y \in \mathbb{X}$ with some $\omega_0, \omega_1 \geq 0$.

(4(b)) there exists $\alpha_J \geq 0$ such that

$$J^{\circ}(y_1; y_2 - y_1) + J^{\circ}(y_2; y_1 - y_2) \leq \alpha_J \|y_1 - y_2\|^2_{\mathbb{X}}, \quad \forall\, y_1, y_2 \in \mathcal{C}. \tag{9}$$

(5) $\mathcal{C}$ is a nonempty, closed and convex subset of $\mathbb{X}$ and

$$f \in \mathbb{X}. \tag{10}$$

For (3), we have the following existence and uniqueness result.

Theorem 1 *Assume that (1)–(5) hold. If, in addition, the following condition is satisfied*

$$\tau_{\mathscr{M}} (\alpha_{\mathscr{M}} \alpha_{\mathscr{P}} + \beta_{\mathscr{M}} \alpha_{\mathscr{Q}})^2 + \alpha_j + \alpha_{\phi} < \sigma_{\mathscr{M}}, \tag{11}$$

then (3) has a unique solution. Moreover, x solves (3) if and only if it solves the following generalized multivalued Minty mixed variational-hemivariational inequality problems for finding $x \in \mathcal{C}$, $r \in \mathscr{P}(y)$ and $s \in \mathscr{Q}(y)$ such that

$$\langle \mathscr{M}(r, s) - f, y - x \rangle_{\mathbb{X}} + \phi(x, y) - \phi(x, x) + J^{\circ}(y; y - x) \geq 0 \quad \forall\, y \in \mathcal{C}. \tag{12}$$

Proof Let $x \in \mathcal{C}$ be the unique solution of (3). First, we note that the assumption (4(b)) is equivalent to the following relaxed monotonicity condition of the generalized gradient

$$\langle \partial J(y) - \partial J(x), y - x \rangle_{\mathbb{X}} \geq -\alpha_j \|y - x\|^2_{\mathbb{X}} \quad \forall\, y, x \in \mathbb{X}. \tag{13}$$

Next from the condition (11) together with (13), and the Lipscitz continuity of $\mathscr{M}$, $(\tau_{\mathscr{M}}, \sigma_{\mathscr{M}})$-relaxed cocoercive of $\mathscr{M}$, and Hausdorff Lipschitz continuity of $\mathscr{P}$ and $\mathscr{Q}$, we have

$$
\begin{aligned}
&\langle \mathscr{M}(r,s) - \mathscr{M}(p,q), y-x\rangle_{\mathbb{X}} + \langle \varsigma_y - \varsigma_x, y-x\rangle_{\mathbb{X}} \\
&\quad \geq -\tau_{\mathscr{M}} \|\mathscr{M}(r,s) - \mathscr{M}(p,q)\|^2_{\mathbb{X}^*} + \sigma_{\mathscr{M}} \|y-x\|^2_{\mathbb{X}} - \alpha_j \|y-x\|^2_{\mathbb{X}}, \\
&\quad \geq -\tau_{\mathscr{M}} (\alpha_{\mathscr{M}} \|r-p\|_{\mathbb{X}^*} + \beta_{\mathscr{M}} \|s-q\|_{\mathbb{X}^*})^2 + (\sigma_{\mathscr{M}} - \alpha_j)\|y-x\|^2_{\mathbb{X}}, \\
&\quad \geq -\tau_{\mathscr{M}} (\alpha_{\mathscr{M}} \mathcal{H}(\mathscr{P}(y), \mathscr{P}(x))_{\mathbb{X}^*} + \beta_{\mathscr{M}} \mathcal{H}(\mathscr{Q}(y), \mathscr{Q}(x))_{\mathbb{X}^*})^2 + (\sigma_{\mathscr{M}} - \alpha_j)\|y-x\|^2_{\mathbb{X}}, \\
&\quad \geq -\tau_{\mathscr{M}} (\alpha_{\mathscr{M}} \alpha_{\mathscr{P}} \|y-x\|_{\mathbb{X}} + \beta_{\mathscr{M}} \alpha_{\mathscr{Q}} \|y-x\|_{\mathbb{X}})^2 + (\sigma_{\mathscr{M}} - \alpha_j)\|y-x\|^2_{\mathbb{X}}, \\
&\quad \geq -\tau_{\mathscr{M}} (\alpha_{\mathscr{M}} \alpha_{\mathscr{P}} + \beta_{\mathscr{M}} \alpha_{\mathscr{Q}})^2 \|y-x\|^2_{\mathbb{X}} + (\sigma_{\mathscr{M}} - \alpha_j)\|y-x\|^2_{\mathbb{X}}, \\
&\quad \geq (-\tau_{\mathscr{M}} (\alpha_{\mathscr{M}} \alpha_{\mathscr{P}} + \beta_{\mathscr{M}} \alpha_{\mathscr{Q}})^2 + \sigma_{\mathscr{M}} - \alpha_j)\|y-x\|^2_{\mathbb{X}}, \\
&\forall \varsigma_y \in \partial J(y), \varsigma_x \in \partial J(x), \ \forall x, y \in \mathcal{C}, p \in \mathscr{P}(x), q \in \mathscr{Q}(x), r \in \mathscr{P}(y), s \in \mathscr{Q}(y). \qquad (14)
\end{aligned}
$$

Let $y \in \mathcal{C}$ be arbitrary. From (14), Lemma 1(c) and the definition of generalized gradient, we have

$$
\begin{aligned}
&\langle \mathscr{M}(r,s) - f, y-x\rangle_{\mathbb{X}} + \phi(x,y) - \phi(x,x) + J^\circ(y; y-x) \\
&\quad \geq \langle \mathscr{M}(r,s) - f + \varsigma_y, y-x\rangle_{\mathbb{X}} + \phi(x,y) - \phi(x,x) \\
&\quad \geq \langle \mathscr{M}(p,q) - f + \varsigma_x, y-x\rangle_{\mathbb{X}} + \phi(x,y) - \phi(x,x) + (-\tau_{\mathscr{M}} (\alpha_{\mathscr{M}} \alpha_{\mathscr{P}} \\
&\qquad + \beta_{\mathscr{M}} \alpha_{\mathscr{Q}})^2 + \sigma_{\mathscr{M}} - \alpha_j)\|y-x\|^2_{\mathbb{X}} \\
&\quad \geq \langle \mathscr{M}(p,q) - f + \varsigma_x, y-x\rangle_{\mathbb{X}} + \phi(x,y) - \phi(x,x) \\
&\quad = \langle \mathscr{M}(p,q) - f, y-x\rangle_{\mathbb{X}} + \phi(x,y) - \phi(x,x) + J^\circ(x; y-x) \geq 0, \forall \varsigma_y \in \partial J(y),
\end{aligned}
$$

where $\varsigma_x \in \partial J(x)$ is such that

$$J^\circ(x; y-x) = \langle \varsigma_x, y-x\rangle_{\mathbb{X}}.$$

Since $y \in \mathcal{C}$ is arbitrary, hence $x \in \mathcal{C}$ is a solution of (12).

Conversely, let $x \in \mathcal{C}$ be a solution to the problem (12). For any $y \in \mathcal{C}$ and $t \in (0,1)$, we denote $y_t = ty + (1-t)x \in \mathcal{C}$. Inserting y_t into (12), we have

$$
\begin{aligned}
0 &\leq t\langle \mathscr{M}(r_t, s_t) - f, y-x\rangle_{\mathbb{X}} + \phi(x, y_t) - \phi(x,x) + J^\circ(y_t; y_t - x) \\
&\leq t\langle \mathscr{M}(r_t, s_t) - f, y-x\rangle_{\mathbb{X}} + t\phi(x,y) - t\phi(x,x) \\
&\quad + tJ^\circ(y_t; y-x), \ \forall r_t \in \mathscr{P}(y_t), s_t \in \mathscr{Q}(y_t),
\end{aligned}
$$

here we utilized the convexity of

$$y \rightarrowtail \phi(x,y)$$

and the positive homogeneity of

$$y \rightarrowtail J^\circ(x; y).$$

Hence,

$$
\begin{aligned}
&\langle \mathscr{M}(r_t, s_t) - f, y-x\rangle_{\mathbb{X}} + \phi(x,y) - \phi(x,x) \\
&+ J^\circ(y_t; y-x) \geq 0, \ \ \forall r_t \in \mathscr{P}(y_t), s_t \in \mathscr{Q}(y_t).
\end{aligned} \qquad (15)
$$

Since $\mathscr{M}$ is pseudomonotone, therefore it is demicontinuous, *see* [25]. Passing to the upper limit as $t \longrightarrow 0^+$ in (15), it gives

$$\begin{aligned}
&\langle \mathscr{M}(p,q) - f, y - x\rangle_{\mathbb{X}} + \phi(x,y) - \phi(x,x) + J^{\circ}(x; y-x) \\
&\quad \geq \limsup_{t\longrightarrow 0^+} \langle \mathscr{M}(r_t,s_t) - f, y - x\rangle_{\mathbb{X}} + \phi(x,y) - \phi(x,x) + \limsup_{t\longrightarrow 0^+} J^{\circ}(y_t; y-x) \\
&\quad \geq \limsup_{t\longrightarrow 0^+} \left\{\langle \mathscr{M}(r_t,s_t) - f, y - x\rangle_{\mathbb{X}} + \phi(x,y) - \phi(x,x) + J^{\circ}(y_t; y-x)\right\} \\
&\quad \geq 0, \ \ \forall r_t \in \mathscr{P}(y_t), s_t \in \mathscr{Q}(y_t).
\end{aligned}$$

Here we utilized Lemma 1(b). Since $y \in \mathcal{C}$ is an arbitrary, hence, we conclude that $x \in \mathcal{C}$ is a solution of (3) and proof is completed. □

3 Main results

In this section, we discuss the gap function, regularized gap function and the Moreau-Yosida type regularized gap function utilizing Lipschitz continuity, $(\tau_{\mathscr{M}}, \sigma_{\mathscr{M}})$-relaxed cocoercivity and Hausdorff Lipschitz continuity associates with (3).

Definition 6 A real-valued function $\mu : \mathcal{C} \longrightarrow \mathbb{R}$ is said to be a gap function for (3), if it satisfies the following assertions:

(a) $\mu(x) \geq 0, \ \forall x \in \mathcal{C}$.
(b) $x^* \in \mathcal{C}$ is such that

$$\mu(x^*) = 0$$

if and only if x^* is a solution of (3).

Consider the functions $\Phi^f, \ \Phi^f_* : \mathcal{C} \longrightarrow \mathbb{R}$ defined by

$$\Phi^f(x) = \sup_{y\in\mathcal{C}} \left\{\langle \mathscr{M}(p,q) - f, x - y\rangle_{\mathbb{X}} + \phi(x,x) - \phi(x,y) - J^{\circ}(x; y-x)\right\}, \tag{16}$$

for all $x \in \mathcal{C}$, $p \in \mathscr{P}(x), q \in \mathscr{Q}(x)$;

$$\Phi^f_*(x) = \sup_{y\in\mathcal{C}} \left\{\langle \mathscr{M}(r,s) - f, x - y\rangle_{\mathbb{X}} + \phi(x,x) - \phi(x,y) - J^{\circ}(y; y-x)\right\}, \tag{17}$$

for all $x \in \mathcal{C}, r \in \mathscr{P}(y), s \in \mathscr{Q}(y)$.

The following lemma shows that functions Φ^f and Φ^f_* are gap functions for (3).

Lemma 2 *Assume that the assumptions of Theorem 1 hold. Then, the functions Φ^f and Φ^f_* defined by (16) and (17) are two gap functions for (3).*

Proof First of all, we prove that Φ^f is a gap function for (3). It is not difficult to demonstrate in an analogous way that the function Φ^f_* is also a gap function for (3). We will review two conditions of Definition 6.

(a) In fact, it is obvious that

$$\Phi^f(x) \geq 0, \quad \forall\, x \in \mathcal{C}.$$

Since then this property has been retained by for all $x \in \mathcal{C}$,

$$\Phi^f(x) \geq \langle \mathscr{M}(p,q) - f, x - x\rangle_{\mathbb{X}} + \phi(x,x) - \phi(x,x) - J^{\circ}(x; x-x)$$

$$= -j^{\circ}(x; 0)$$
$$= 0, \forall p \in \mathscr{P}(x), q \in \mathscr{Q}(x). \tag{18}$$

(b) Suppose that $x^* \in \mathcal{C}$ is such that

$$\Phi^f(x^*) = 0,$$

that is

$$\sup_{y \in \mathcal{C}} \left\{ \langle \mathscr{M}(p^*, q^*) - f, x^* - y \rangle_{\mathbb{X}} + \phi(x^*, x^*) - \phi(x^*, y) - J^{\circ}(x^*; y - x^*) \right\} = 0,$$
$$\forall p^* \in \mathscr{P}(x^*), q^* \in \mathscr{Q}(x^*). \tag{19}$$

This together with the fact

$$\langle \mathscr{M}(p^*, q^*) - f, x^* - x^* \rangle_{\mathbb{X}} + \phi(x^*, x^*) - \phi(x^*, x^*)$$
$$-J^{\circ}(x^*; x^* - x^*) = 0, \forall p^* \in \mathscr{P}(x^*), q^* \in \mathscr{Q}(x^*)$$

implies that (19) is equivalent to

$$\langle \mathscr{M}(p^*, q^*) - f, y - x^* \rangle_{\mathbb{X}} + \phi(x^*, y) - \phi(x^*, x^*)$$
$$-J^{\circ}(x^*; y - x^*) \geq 0, \ \forall\, y \in \mathcal{C}, \ p^* \in \mathscr{P}(x^*), q^* \in \mathscr{Q}(x^*).$$

Therefore, we infer that x^* is a solution of (3) if and only if

$$\Phi^f(x^*) = 0.$$

□

Let $\curlywedge > 0$ be a fixed parameter. We consider the following functions $\Phi^{f,\curlywedge}, \Phi_*^{f,\curlywedge} : \mathcal{C} \longrightarrow \mathbb{R}$ defined by

$$\Phi^{f,\curlywedge}(x) = \sup_{y \in \mathcal{C}} \{ \langle \mathscr{M}(p, q) - f, x - y \rangle_{\mathbb{X}} + \phi(x, x) - \phi(x, y)$$
$$-J^{\circ}(x; y - x) - \frac{1}{2\curlywedge} \|x - y\|_{\mathbb{X}}^2 \Big\},$$
$$\text{for all } \ x \in \mathcal{C}, p \in \mathscr{P}(x), q \in \mathscr{Q}(x), \tag{20}$$
$$\Phi_*^{f,\curlywedge}(x) = \sup_{y \in \mathcal{C}} \{ \langle \mathscr{M}(r, s) - f, x - y \rangle_{\mathbb{X}} + \phi(x, x) - \phi(x, y)$$
$$-J^{\circ}(y; y - x) - \frac{1}{2\curlywedge} \|x - y\|_{\mathbb{X}}^2 \Big\},$$
$$\text{for all } x \in \mathcal{C}, r \in \mathscr{P}(y), s \in \mathscr{Q}(y). \tag{21}$$

In what follows, the functions $\Phi^{f,\curlywedge}$ and $\Phi_*^{f,\curlywedge}$ are called the regularized gap functions for (3).

Theorem 2 *Suppose the assertions of Theorem 1 hold. Then, for any* $\curlywedge > 0$*, the functions* $\Phi^{f,\curlywedge}$ *and* $\Phi_*^{f,\curlywedge}$ *are two gap functions for* (3).

Proof Now, we prove that $\Phi^{f,\curlywedge}$ is a gap function for (3). Applying the analogous techniques, it is not difficult to show that $\Phi_*^{f,\curlywedge}$ is also a gap function for (3). We will verify two assumptions of Definition 6.

(a) For each $\curlywedge > 0$ fixed, it is trivial that for each $x \in \mathcal{C}$ it holds

$$\Phi^{f,\curlywedge}(x) \geq 0.$$

Therefore for $x \in \mathcal{C}$,

$$\begin{aligned}\Phi^{f,\curlywedge}(x) &= \langle \mathscr{M}(p,q) - f, x - x\rangle_{\mathbb{X}} + \phi(x,x) - \phi(x,x) - J^\circ(x; x - x)\\ &\quad - \frac{1}{2\curlywedge}\|x - x\|^2_{\mathbb{X}} = -J^\circ(x; 0)\\ &= 0, \forall p \in \mathscr{P}(x), q \in \mathscr{Q}(x).\end{aligned}$$

(b) Assume that $x^* \in \mathcal{C}$ is such that

$$\Phi^{f,\curlywedge}(x^*) = 0,$$

and for all $p^* \in \mathscr{P}(x^*), q^* \in \mathscr{Q}(x^*)$,

$$\begin{aligned}&\sup_{y\in\mathcal{C}}\Big\{\langle \mathscr{M}(p^*,q^*) - f, x^* - y\rangle_{\mathbb{X}} + \phi(x^*,x^*)\\ &\quad -\phi(x^*,y) - J^\circ(x^*; y - x^*) - \frac{1}{2\curlywedge}\|x^* - y\|^2_{\mathbb{X}}\Big\} = 0.\end{aligned}$$

This imply that

$$\begin{aligned}&\langle \mathscr{M}(p^*,q^*) - f, y - x^*\rangle_{\mathbb{X}} - \phi(x^*,x^*) + \phi(x^*,y) + J^\circ(x^*; y - x^*)\\ &\quad \geq -\frac{1}{2\curlywedge}\|x^* - y\|^2_{\mathbb{X}}, \quad \forall y \in \mathcal{C}, p^* \in \mathscr{P}(x^*), q^* \in \mathscr{Q}(x^*).\end{aligned} \tag{22}$$

For any $z \in \mathcal{C}$ and $t \in (0, 1)$, we put $y = y_t = (1-t)x^* + tz \in \mathcal{C}$ in (22) to obtain

$$\begin{aligned}&t\langle \mathscr{M}(p^*,q^*) - f, z - x^*\rangle_{\mathbb{X}} - t\phi(x^*,x^*) + t\phi(x^*,z) + tJ^\circ(x^*; z - x^*)\\ &\quad \geq \langle \mathscr{M}(p^*,q^*) - f, y_t - x^*\rangle_{\mathbb{X}} - \phi(x^*,x^*) + \phi(x^*,y_t) + J^\circ(x^*; y_t - x^*)\\ &\quad \geq -\frac{1}{2\curlywedge}\|x^* - y_t\|^2_{\mathbb{X}}\\ &\quad = -\frac{t^2}{2\curlywedge}\|x^* - z\|^2_{\mathbb{X}}, \forall p^* \in \mathscr{P}(x^*), q^* \in \mathscr{Q}(x^*),\end{aligned}$$

here we utilized the convexity of

$$y \rightarrowtail \phi(x,y)$$

and positive homogeneity of

$$y \rightarrowtail J^\circ(x; y).$$

Hence, we have

$$\begin{aligned}&\langle \mathscr{M}(p^*,q^*) - f, z - x^*\rangle_{\mathbb{X}} - \phi(x^*,x^*) + \phi(x^*,z) - J^\circ(x^*; z - x^*)\\ &\quad \geq -\frac{t}{2\curlywedge}\|x^* - z\|^2_{\mathbb{X}}, \ \forall z \in \mathcal{C}, p^* \in \mathscr{P}(x^*), q^* \in \mathscr{Q}(x^*).\end{aligned}$$

Letting $t \longrightarrow 0^+$ for the above inequality, we get

$$\begin{aligned}&\langle \mathscr{M}(p^*,q^*) - f, z - x^*\rangle_{\mathbb{X}} - \phi(x^*,x^*) + \phi(x^*,z)\\ &\quad + J^\circ(x^*; z - x^*) \geq 0, \ \forall z \in \mathcal{C}, p^* \in \mathscr{P}(x^*), q^* \in \mathscr{Q}(x^*).\end{aligned}$$

Hence, x^* is also a solution of (3).

Conversely, suppose that $x^* \in \mathcal{C}$ is a solution of (3), that is,

$$\langle \mathscr{M}(p^*, q^*) - f, y - x^* \rangle_{\mathbb{X}} - \phi(x^*, x^*) + \phi(x^*, y) \\ + J^\circ(x^*; y - x^*) \geq 0, \ \forall y \in \mathcal{C}, \ p^* \in \mathscr{P}(x^*), q^* \in \mathscr{Q}(x^*).$$

This ensures that

$$\sup_{y \in \mathcal{C}} \Big\{ \langle \mathscr{M}(p^*, q^*) - f, x^* - y \rangle_{\mathbb{X}} + \phi(x^*, x^*) - \phi(x^*, y) \\ - J^\circ(x^*; y - x^*) - \frac{1}{2\curlywedge} \|x^* - y\|^2_{\mathbb{X}} \Big\} \leq 0,$$

for all $p^* \in \mathscr{P}(x^*)$, $q^* \in \mathscr{Q}(x^*)$. The latter combined with the fact

$$\Phi^{f,\curlywedge}(x) \geq 0, \quad \forall x \in \mathcal{C}$$

and imply that

$$\Phi^{f,\curlywedge}(x^*) = 0.$$

The proof is completes. □

Latter on we will prove that the regularized gap functions $\Phi^{f,\curlywedge}$ and $\Phi_*^{f,\curlywedge}$ are lower semi-continuous.

Lemma 3 *Assume that the assumptions of Theorem 1 are satisfied. If, in addition,* $\phi : \mathcal{C} \times \mathcal{C} \longrightarrow \mathbb{R}$ *is continuous, then, for each* $\curlywedge > 0$*, the functions* $\Phi^{f,\curlywedge}$ *and* $\Phi_*^{f,\curlywedge}$ *are both lower semicontinuous.*

Proof We can prove that $\Phi^{f,\curlywedge}$ is a lower semicontinuous for each $\curlywedge > 0$. It is not difficult to use a similar argument to verify that $\Phi_*^{f,\curlywedge}$ has the same property.

Consider the function $\hat{\Phi}^{f,\curlywedge} : \mathcal{C} \times \mathcal{C} \longrightarrow \mathbb{R}$ defined by

$$\hat{\Phi}^{f,\curlywedge}(x, y) = \langle \mathscr{M}(p, q) - f, x - y \rangle_{\mathbb{X}} + \phi(x, x) - \phi(x, y) - J^\circ(x; y - x) \\ - \frac{1}{2\curlywedge} \|x - y\|^2_{\mathbb{X}}, \forall p \in \mathscr{P}(x), q \in \mathscr{Q}(x).$$

Since the operator $\mathscr{M} : CB(\mathbb{X}^*) \times CB(\mathbb{X}^*) \longrightarrow CB(\mathbb{X}^*)$ and $\mathscr{P}, \mathscr{Q} : \mathbb{X} \longrightarrow CB(\mathbb{X}^*)$ are demicontinuous being pseudomonotone. This means that the function

$$x \rightarrowtail \langle \mathscr{M}(p, q), x \rangle_{\mathbb{X}}$$

is continuous. The latter together with the lower semicontinuity of

$$(x, y) \rightarrowtail -J^\circ(x; y),$$

and the continuity of

$$(x, y) \rightarrowtail \phi(x, y)$$

and

$$x \rightarrowtail \|x\|_{\mathbb{X}}$$

guarantees that

$$x \rightarrowtail \hat{\Phi}^{f,\curlywedge}(x, y)$$

is lower semicontinuous for all $y \in \mathcal{C}$. Next, we see that

$$\Phi^{f,\curlywedge}(x) = \sup_{y\in\mathcal{C}} \hat{\Phi}^{f,\curlywedge}(x,y), \ \forall\, x \in \mathcal{C}.$$

Let $\{x_n\} \subset \mathcal{C}$ be such that

$$x_n \longrightarrow x \text{ as } \ n \longrightarrow \infty.$$

Now

$$\|p_n - p\| \leq \mathcal{H}(\mathscr{P}(x_n), \mathscr{P}(x)) \leq \alpha_{\mathscr{P}} \|x_n - x\| \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

implies that

$$\|p_n - p\| \longrightarrow 0, \ \text{ as } n \longrightarrow \infty.$$

Hence

$$p_n \longrightarrow p \in \mathbb{X}.$$

Similarly

$$\|q_n - q\| \leq \mathcal{H}(\mathscr{Q}(x_n), \mathscr{Q}(x)) \leq \alpha_{\mathscr{Q}} \|x_n - x\| \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

implies that

$$\|q_n - q\| \longrightarrow 0, \ \text{ as } n \longrightarrow \infty.$$

Hence

$$q_n \longrightarrow q \in \mathbb{X}.$$

Then, we have

$$\begin{aligned}\liminf_{n\longrightarrow\infty} \Phi^{f,\curlywedge}(x_n) &= \liminf_{n\longrightarrow\infty} \sup_{y\in\mathcal{C}} \hat{\Phi}^{f,\curlywedge}(x_n, y)\\ &\geq \liminf_{n\longrightarrow\infty} \hat{\Phi}^{f,\curlywedge}(x_n, z)\\ &\geq \hat{\Phi}^{f,\curlywedge}(x, z), \forall z \in \mathcal{C}.\end{aligned}$$

Passing to supremum with $z \in \mathcal{C}$ for the above inequality, it gives

$$\begin{aligned}\liminf_{n\longrightarrow\infty} \Phi^{f,\curlywedge}(x_n) &\geq \sup_{z\in\mathcal{C}} \hat{\Phi}^{f,\curlywedge}(x, z)\\ &= \Phi^{f,\curlywedge}(x).\end{aligned}$$

Therefore, the function $\Phi^{f,\curlywedge}$ is lower semicontinuous and proof is completed. □

Let $\curlywedge, \Upsilon > 0$ be two parameters. Moreover, let us consider the following functions

$$\Pi_{\Phi^{f,\curlywedge,\Upsilon}}, \ \Pi_{\Phi_*^{f,\curlywedge,\Upsilon}} : \mathcal{C} \longrightarrow \mathbb{R}$$

defined by

$$\Pi_{\Phi^{f,\curlywedge,\Upsilon}}(x) = \inf_{z\in\mathcal{C}} \left\{\Phi^{f,\curlywedge}(z) + \Upsilon\|x - z\|^2_{\mathbb{X}}\right\}, \forall x \in \mathcal{C}, \tag{23}$$

$$\Pi_{\Phi_*^{f,\curlywedge,\Upsilon}}(x) = \inf_{z\in\mathcal{C}} \left\{\Phi_*^{f,\curlywedge}(z) + \Upsilon\|x - z\|^2_{\mathbb{X}}\right\}, \forall x \in \mathcal{C}. \tag{24}$$

In the sequel, we invoke the functions $\varPi_{\varPhi^{f,\curlywedge,\curlyvee}}$ and $\varPi_{\varPhi_*^{f,\curlywedge,\curlyvee}}$ to be the Moreau-Yosida regularized gap functions for (3). Subsequently, we will verify that these functions are two gap functions for (3).

Theorem 3 *Assume that the assumptions of Lemma 3 are satisfied. Then, for all* $\curlywedge, \curlyvee > 0$, *the functions* $\varPi_{\varPhi^{f,\curlywedge,\curlyvee}}$ *and* $\varPi_{\varPhi_*^{f,\curlywedge,\curlyvee}}$ *are two gap functions for* (3).

Proof We can prove that $\varPi_{\varPhi^{f,\curlywedge,\curlyvee}}$ is a gap function for (3). It is possible to prove, in an analogous way, that $\varPi_{\varPhi_*^{f,\curlywedge,\curlyvee}}$ is also a gap function for (3).

(a) For any $\curlywedge, \curlyvee > 0$ fixed, recall that $\varPhi^{f,\curlywedge,\curlyvee}$ is a gap function for (3), hence

$$\varPhi^{f,\curlywedge,\curlyvee}(x) \geq 0, \ \forall\, x \in \mathcal{C}.$$

In consequence,

$$\varPi_{\varPhi^{f,\curlywedge,\curlyvee}}(x) \geq 0, \ \forall\, x \in \mathcal{C}.$$

(b) Suppose that $x \in \mathcal{C}$ is a solution of (3). Theorem 2 show that

$$\varPhi^{f,\curlywedge,\curlyvee}(x^*) = 0.$$

Moreover, the inequality

$$\begin{aligned}\varPi_{\varPhi^{f,\curlywedge,\curlyvee}}(x^*) &= \inf_{z\in\mathcal{C}} \left\{ \varPhi^{f,\curlywedge}(z) + \curlyvee \|x^* - z\|_{\mathbb{X}}^2 \right\} \\ &\leq \varPi^{f,\curlywedge}(x^*) + \curlyvee \|x^* - x^*\|_{\mathbb{X}}^2 \\ &= 0,\end{aligned}$$

and the fact

$$\varPi_{\varPhi^{f,\curlywedge,\curlyvee}}(x^*) \geq 0$$

imply that

$$\varPi_{\varPhi^{f,\curlywedge,\curlyvee}}(x^*) = 0.$$

Conversely, let $x^* \in \mathcal{C}$ be such that

$$\varPi_{\varPhi^{f,\curlywedge,\curlyvee}}(x^*) = 0,$$

and

$$\inf_{z\in\mathcal{C}} \left\{ \varPhi^{f,\curlywedge}(z) + \curlyvee \|x^* - z\|_{\mathbb{X}}^2 \right\} = 0.$$

Therefore, there exists a minimizing sequence $\{z_n\}$ in $\mathcal{C}$ such that

$$\begin{aligned}0 &\leq \varPhi^{f,\curlywedge}(z_n) + \curlyvee \|x^* - z_n\|_{\mathbb{X}}^2 \\ &< \frac{1}{n}.\end{aligned} \tag{25}$$

It is obvious that

$$\varPhi^{f,\curlywedge}(z_n) \longrightarrow 0$$

and

$$\|x^* - z_n\|_{\mathbb{X}} \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

implies

$$z_n \longrightarrow x^*, \text{ as } n \longrightarrow +\infty.$$

From Lemma 3 and nonnegativity of $\Phi^{f,人}$, we have

$$\begin{aligned} 0 &\le \Phi^{f,人}(x^*) \\ &\le \liminf_{n\longrightarrow+\infty} \Phi^{f,人}(z_n) \\ &= 0. \end{aligned}$$

Thus

$$\Phi^{f,人}(x^*) = 0.$$

Because $\Phi^{f,人}$ is a gap function. Therefore, x^* is a solution of (3), and proof is completed. □

Here, we conclude with two global error bounds for (3) associated with the regularized gap function $\Phi^{f,人,\curlyvee}$ and the Moreau–Yosida regularized gap function $\Pi_{\Phi^{f,人,\curlyvee}}$, *respectively*. These global error estimates measure the distance between any admissible point and the unique solution of (3).

Theorem 4 *Let $x^* \in \mathcal{C}$ be the unique solution of (3) and $人 > 0$ be such that*

$$\sigma_{\mathscr{M}} - \tau_{\mathscr{M}}(\alpha_{\mathscr{M}}\alpha_{\mathscr{P}} + \beta_{\mathscr{M}}\alpha_{\mathscr{Q}})^2 - \alpha_\phi - \alpha_J > \frac{1}{2人}. \tag{26}$$

Assume that the assertions of Theorem 1 hold. Then, for each $x \in \mathcal{C}$, we have

$$\|x - x^*\|_{\mathbb{X}} \le \sqrt{\frac{\Phi^{f,人}(x)}{\sigma_{\mathscr{M}} - \tau_{\mathscr{M}}(\alpha_{\mathscr{M}}\alpha_{\mathscr{P}} + \beta_{\mathscr{M}}\alpha_{\mathscr{Q}})^2 - \alpha_\phi - \alpha_J - \frac{1}{2人}}}. \tag{27}$$

Proof Let $x^* \in \mathcal{C}$ be the unique solution of (3), that is

$$\begin{aligned} &\langle \mathscr{M}(p^*, q^*) - f, y - x^*\rangle_{\mathbb{X}} + \phi(x^*, y) - \phi(x^*, x^*) \\ &+ J^\circ(x^*; y - x^*) \ge 0, \ \forall y \in \mathcal{C}, \ p^* \in \mathscr{P}(x^*), q^* \in \mathscr{Q}(x^*). \end{aligned} \tag{28}$$

For any $x \in \mathcal{C}$ fixed, we put $y = x$ in (28), we obtain

$$\begin{aligned} &\langle \mathscr{M}(p^*, q^*) - f, x - x^*\rangle_{\mathbb{X}} + \phi(x^*, x) - \phi(x^*, x^*) \\ &+ J^\circ(x^*; x - x^*) \ge 0, \forall p^* \in \mathscr{P}(x^*), q^* \in \mathscr{Q}(x^*). \end{aligned} \tag{29}$$

By virtue of the definition of $\Phi^{f,人}$, one has

$$\begin{aligned} \Phi^{f,人}(x) \ge &\langle \mathscr{M}(p, q) - f, x - x^*\rangle_{\mathbb{X}} + \phi(x, x) - \phi(x, x^*) \\ &- J^\circ(x; x^* - x) - \frac{1}{2人}\|x - x^*\|^2_{\mathbb{X}}. \end{aligned} \tag{30}$$

It follows from the Lipschitz continuity of $\mathscr{M}$ with respect to first variable with constant $\alpha_{\mathscr{M}} > 0$ and second variable with constant $\beta_{\mathscr{M}} > 0$, $(\tau_{\mathscr{M}}, \sigma_{\mathscr{M}})$-relaxed cocoercivity of $\mathscr{M}$ with respect to the constants $\tau_{\mathscr{M}}, \sigma_{\mathscr{M}} > 0$, Hausdorff Lipschitz continuity of $\mathscr{P}, \mathscr{Q}$ with respect to constants $\alpha_{\mathscr{P}} > 0, \alpha_{\mathscr{Q}} > 0$, *respectively* and assumptions (3(b)) and (4(b)), we have

$$\langle \mathscr{M}(p, q) - f, x - x^*\rangle_{\mathbb{X}} + \phi(x, x) - \phi(x, x^*) - J^\circ(x; x^* - x) - \frac{1}{2人}\|x - x^*\|^2_{\mathbb{X}}$$

$$\geq \langle \mathscr{A}(p^*, q^*) - f, x - x^* \rangle_{\mathbb{X}} + \phi(x^*, x) - \phi(x^*, x^*) + J^{\circ}(x^*; x - x^*)$$
$$+ (\sigma_{\mathscr{M}} - \tau_{\mathscr{M}}(\alpha_{\mathscr{M}}\alpha_{\mathscr{P}} + \beta_{\mathscr{M}}\alpha_{\mathscr{Q}})^2 - \alpha_{\phi} - \alpha_J - \frac{1}{2\curlywedge})\|x - x^*\|_{\mathbb{X}}^2$$
$$\geq \left(\sigma_{\mathscr{M}} - \tau_{\mathscr{M}}(\alpha_{\mathscr{M}}\alpha_{\mathscr{P}} + \beta_{\mathscr{M}}\alpha_{\mathscr{Q}})^2 - \alpha_{\phi} - \alpha_J - \frac{1}{2\curlywedge}\right)\|x - x^*\|_{\mathbb{X}}^2$$
$$\forall p \in \mathscr{P}(x), p^* \in \mathscr{P}(x^*), q \in \mathscr{Q}(x), q^* \in \mathscr{Q}(x^*), \tag{31}$$

where the last inequality is obtained by using (29). Combining (30) and (31), we have

$$\Phi^{f,\curlywedge}(x) \geq \left(\sigma_{\mathscr{M}} - \tau_{\mathscr{M}}(\alpha_{\mathscr{M}}\alpha_{\mathscr{P}} + \beta_{\mathscr{M}}\alpha_{\mathscr{Q}})^2 - \alpha_{\phi} - \alpha_J - \frac{1}{2\curlywedge}\right)\|x - x^*\|_{\mathbb{X}}^2. \tag{32}$$

Hence, the desired inequality (27) is valid. □

Theorem 5 *Let $x^* \in \mathcal{C}$ be the unique solution of (3) and $\curlywedge > 0$ be such that*

$$\sigma_{\mathscr{M}} - \tau_{\mathscr{M}}(\alpha_{\mathscr{M}}\alpha_{\mathscr{P}} + \beta_{\mathscr{M}}\alpha_{\mathscr{Q}})^2 - \alpha_{\phi} - \alpha_J \geq \frac{1}{2\curlywedge}. \tag{33}$$

Assume that the assumptions of Theorem 1 hold. Then, for each $x \in \mathcal{C}$ and all $\curlyvee > 0$, we have

$$\|x - x^*\|_{\mathbb{X}} \leq \sqrt{\frac{2\Pi_{\Phi^{f,\curlywedge,\curlyvee}}(x)}{\min\left\{\sigma_{\mathscr{M}} - \tau_{\mathscr{M}}(\alpha_{\mathscr{M}}\alpha_{\mathscr{P}} + \beta_{\mathscr{M}}\alpha_{\mathscr{Q}})^2 - \alpha_{\phi} - \alpha_J - \frac{1}{2\curlywedge}, \curlyvee\right\}}}. \tag{34}$$

Proof Let $x^* \in \mathcal{C}$ be the unique solution of (3). By the definition of the function

$$\Pi_{\Phi^{f,\curlywedge,\curlyvee}}(x) = \inf_{z\in\mathcal{C}}\left\{\Phi^{f,\curlywedge}(z) + \curlyvee\|x - z\|_{\mathbb{X}}^2\right\}$$
$$\geq \inf_{z\in\mathcal{C}}\left\{\left(\sigma_{\mathscr{M}} - \tau_{\mathscr{M}}(\alpha_{\mathscr{M}}\alpha_{\mathscr{P}} + \beta_{\mathscr{M}}\alpha_{\mathscr{Q}})^2 - \alpha_{\phi} - \alpha_J - \frac{1}{2\curlywedge}\right)\right.$$
$$\left.\|x^* - z\|_{\mathbb{X}}^2 + \curlyvee\|x - z\|_{\mathbb{X}}^2\right\}$$
$$\geq \min\left\{\sigma_{\mathscr{M}} - \tau_{\mathscr{M}}(\alpha_{\mathscr{M}}\alpha_{\mathscr{P}} + \beta_{\mathscr{M}}\alpha_{\mathscr{Q}})^2 - \alpha_{\phi} - \alpha_J - \frac{1}{2\curlywedge}, \curlyvee\right\}$$
$$\inf_{z\in\mathcal{C}}\left\{\|x^* - z\|_{\mathbb{X}}^2 + \|x - z\|_{\mathbb{X}}^2\right\}$$
$$\geq \frac{1}{2}\min\left\{\sigma_{\mathscr{M}} - \tau_{\mathscr{M}}(\alpha_{\mathscr{M}}\alpha_{\mathscr{P}} + \beta_{\mathscr{M}}\alpha_{\mathscr{Q}})^2 - \alpha_{\phi} - \alpha_J - \frac{1}{2\curlywedge}, \curlyvee\right\}$$
$$\|x - x^*\|_{\mathbb{X}}^2, \ \forall x \in \mathcal{C}.$$

Hence

$$\|x - x^*\|_{\mathbb{X}} \leq \sqrt{\frac{2\Pi_{\Phi^{f,\curlywedge,\curlyvee}}(x)}{\min\left\{\sigma_{\mathscr{M}} - \tau_{\mathscr{M}}(\alpha_{\mathscr{M}}\alpha_{\mathscr{P}} + \beta_{\mathscr{M}}\alpha_{\mathscr{Q}})^2 - \alpha_{\phi} - \alpha_J - \frac{1}{2\curlywedge}, \curlyvee\right\}}}, \quad \forall x \in \mathcal{C},$$

which completes the proof of the theorem. □

4 Application

The goal of this section is to investigate a boundary value problem with the generalized gradient and an obstacle effect which illustrates the applicability of the abstract results.

Let $\mho$ be a bounded domain in $\mathbb{R}^\ell$ ($\ell = 2, 3$) with Lipschitz continuous boundary λ. The boundary is divided into two mutually disjoint measurable parts λ_1 and λ_2 such that $\text{meas}(\lambda_1) > 0$. Consider the following nonlinear mixed boundary value problem with constraints.

For finding a function $x : \mho \longrightarrow \mathbb{R}$ such that

$$- div\, a(\mathbf{u}, \nabla x) + \partial g(\mathbf{u}, x) \ni f(\mathbf{u}), \text{ in } \mho, \tag{35}$$

where ∂g and $\partial_c \mathscr{A}$ denote the generalized gradient and the convex subdifferential of the functions $g : \mho \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathscr{A} : \lambda_2 \times \mathbb{R} \longrightarrow \mathbb{R}$, *respectively* with respect to their second variables, while the conormal derivative

$$\frac{\partial x}{\partial \nu_a} = (a(\mathbf{u}, \nabla x), \boldsymbol{\nu})_{\mathbb{R}^\ell}$$

represents the heat flux through the part λ_2, and $\boldsymbol{\nu}$ stands for the outward unit normal on λ. The function x represents the electric potential, the function $a = a(\mathbf{u}, \nabla x)$ is the dielectric coefficient and $f = f(\mathbf{u})$ is a given source term. The material which occupies $\mho$ is non-isotropic and heterogeneous, and thus a effectively depends on $\mathbf{u}$.

$$x(\mathbf{u}) \le \psi(\mathbf{u}), \quad \text{in } \mho, \tag{36}$$

represents an additional unilateral constraint for the solution,

$$x = 0, \quad \text{on} \quad \lambda_1 \tag{37}$$

$$-\frac{\partial x}{\partial \nu_a} \in \kappa(x)\partial_c \mathscr{A}(\mathbf{u}, x), \quad \text{on} \quad \lambda_2. \tag{38}$$

We remark that in general there is no function $\mathscr{A}$ such that

$$\partial \tilde{\mathscr{A}} = \kappa \partial_c \mathscr{A}.$$

This means that if $g \equiv 0$, then the weak form of (35), stated in (36) below, reduces to quasi-variational inequality.

We need the following standard functional space. Let $\mathbb{X}$ be defined by

$$\mathbb{X} = \left\{ y \in H^1(\mho) \middle|\ y = 0 \quad \text{on} \ \lambda_1 \right\}.$$

Since $\text{meas}(\lambda_1) > 0$, the space $\mathbb{X}$ is endowed with the inner product and corresponding norm given by

$$\langle x, y \rangle_{\mathbb{X}} = \int_{\mho} (\nabla x(\mathbf{u}), \nabla y(\mathbf{u}))_{\mathbb{R}^\ell}\, d\mathbf{u}$$

and

$$\|y\|_{\mathbb{X}} = \left(\int_{\mho} \|\nabla y(\mathbf{u})\|^2_{\mathbb{R}^\ell} d\mathbf{u} \right)^{\frac{1}{2}}, \quad \forall x, y \in \mathbb{X}.$$

Let $\lambda_0 : \mathbb{X} \longrightarrow L^2(\lambda)$ be the trace operator and $\mathcal{C}$ be the admissible set defined by

$$\mathcal{C} = \left\{ y \in \mathbb{X} \middle|\ y(\mathbf{u}) \le \psi(\mathbf{u}) \quad \text{for a.e. } \mathbf{u} \in \mho \right\}.$$

The unique solvability of (35), we suggest the following hypotheses:

(A) $a : \mho \times \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell$ is such that

(A(a)) $a(\cdot, \mathbf{w})$ is measurable on $\mho$ for all $\mathbf{w} \in \mathbb{R}^\ell$ with

$$a(\mathbf{u}, \mathbf{0}) = \mathbf{0} \text{ for a.e.} \mathbf{u} \in \mho. \tag{39}$$

(A(b)) $a(\mathbf{u}, \cdot)$ is continuous on $\mathbb{R}^\ell$ for a.e. $\mathbf{u} \in \mho$.

(A(c)) for all $\mathbf{w} \in \mathbb{R}^\ell$, a.e. $\mathbf{u} \in \mho$ with $\alpha_a > 0$, we have

$$\|a(\mathbf{u}, \mathbf{w})\|_{\mathbb{R}^\ell} \leq \alpha_a(1 + \|\mathbf{w}\|_{\mathbf{R}^\ell}). \tag{40}$$

(A(d)) for all $\mathbf{w_1}, \mathbf{w_2} \in \mathbb{R}^\ell$ and a.e. $\mathbf{u} \in \mho$ with $\tau_a > 0$ and $\sigma_a > 0$, we have

$$(a(\mathbf{u}, \mathbf{w}_1) - a(\mathbf{u}, \mathbf{w}_2)) \cdot (\mathbf{w}_1 - \mathbf{w}_2) \geq (-\tau_a + \sigma_a)\|\mathbf{w}_1 - \mathbf{w}_2\|^2_{\mathbb{R}^\ell}. \tag{41}$$

(B) $g : \mho \times \mathbb{R} \longrightarrow \mathbb{R}$ is such that

(B(a)) $g(\cdot, \imath)$ is measurable on $\mho$ for all $\imath \in \mathbb{R}$ and there exists $\tilde{e} \in L^2(\mho)$ such that

$$g(\cdot, \tilde{e}(\cdot)) \in L^1(\mho). \tag{42}$$

(B(b)) $g(\mathbf{u}, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $\mathbf{u} \in \mho$.

(B(c)) there exist $\bar{\omega}_0, \bar{\omega}_1 \geq 0$ such that

$$\left|\partial g(\mathbf{u}, \imath)\right| \leq \bar{\omega}_0 + \bar{\omega}_1|\imath|, \ \forall \imath \in \mathbb{R} \text{ and a.e. } \mathbf{u} \in \mho. \tag{43}$$

(B(d)) there exists $\alpha_g \geq 0$ such that

$$g^\circ(\mathbf{u}, \imath_1; \imath_2 - \imath_1) + g^\circ(\mathbf{u}, \imath_2; \imath_1 - \imath_2) \leq \alpha_g|\imath_1 - \imath_2|^2, \ \forall \imath_1, \imath_2 \in \mathbb{R} \text{ and a.e. } \mathbf{u} \in \mho. \tag{44}$$

(C) $\mathscr{A} : \lambda_2 \times \mathbb{R} \longrightarrow \mathbb{R}$ is such that

(C(a)) $\mathscr{A}(\cdot, \imath)$ is measurable on λ_2 for all $\imath \in \mathbb{R}$.

(C(ii)) $\mathscr{A}(\mathbf{u}, \cdot)$ is convex on $\mathbb{R}$ for a.e. $\mathbf{u} \in \mho$.

(C(iii)) there exists $\varepsilon_{\mathscr{A}} > 0$ such that

$$\left|\mathscr{A}(\mathbf{u}, \imath_1) - \mathscr{A}(\mathbf{u}, \imath_2)\right| \leq \varepsilon_{\mathscr{A}}|\imath_1 - \imath_2|, \ \ \forall \imath_1, \imath_2 \in \mathbb{R} \text{ and a.e. } \mathbf{u} \in \lambda_2. \tag{45}$$

(D) $\kappa : \lambda_2 \times \mathbb{R} \longrightarrow \mathbb{R}$ is such that

(D(a)) $\kappa(\cdot, \imath)$ is measurable on λ_2 for all $\imath \in \mathbb{R}$.

(D(b)) there exists $\varepsilon_\kappa > 0$ such that

$$\left|\kappa(\mathbf{u}, \imath_1) - \kappa(\mathbf{u}, \imath_2)\right| \leq \varepsilon_\kappa|\imath_1 - \imath_2|, \ \forall \imath_1, \imath_2 \in \mathbb{R} \text{ and a.e. } \mathbf{u} \in \lambda_2. \tag{46}$$

(D(c)) $\kappa(\mathbf{u}, 0) = 0$ for a.e. $\mathbf{u} \in \mho$.

(E) $\psi \in \mathbb{X}$ and

$$f \in L^2(\mho). \tag{47}$$

Now, using the standard method based on the Green Theorem, *see* [25,27], we have the following variational formulation of (35) for finding $x \in \mathcal{C}$ such that

$$\int_\mho (a(\mathbf{u}, \nabla x), \nabla(y - x))_{\mathbb{R}^\ell} \, d\mathbf{u} + \int_{\lambda_2} (\kappa(x)\mathscr{A}(\mathbf{u}, y) - \kappa(x)\mathscr{A}(\mathbf{u}, x)) \, d\lambda$$
$$+ \int_\mho g^\circ(\mathbf{u}, x; y - x) d\mathbf{u} \geq \int_\mho f(y - x) d\mathbf{u}, \ \forall y \in \mathcal{C}. \tag{48}$$

Theorem 6 *Assume that the assumptions* (A)–(E) *are satisfied. If, in addition, the inequality holds*

$$\sigma_a - \tau_a - \alpha_g - \varepsilon_{\mathscr{A}}\varepsilon_\kappa \| \curlywedge_0 \|^2 \geq 0, \tag{49}$$

then (48) *has an unique solution* $x^* \in \mathcal{C}$.

Proof Consider the operator $\mathscr{P}, \mathscr{Q} : \mathcal{C} \longrightarrow CB(\mathbb{X}^*)$ are multivalued functions, $\mathscr{M} : CB(\mathbb{X}^*) \times CB(\mathbb{X}^*) \rightarrow CB(\mathbb{X}^*)$ is bi-function and the functions $\phi : \mathcal{C} \times \mathcal{C} \longrightarrow \mathbb{R}$ and $J : \mathbb{X} \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} \langle \mathscr{M}(p,q), y\rangle_{\mathbb{X}} &= \int_{\mho} (a(\mathbf{u}, \nabla x), \nabla y)_{\mathbb{R}^\ell} d\mathbf{u}, \forall p \in \mathscr{P}(x), q \in \mathscr{Q}(x) \\ \phi(x,y) &= \int_{\lambda_2} \kappa(x)\mathscr{A}(y) d\lambda, \\ J(y) &= \int_{\mho} g(\mathbf{u}, y) d\mathbf{u}, \ \forall x, y \in \mathbb{X}. \end{aligned}$$

It is easy to show that all conditions of Theorem 1 are satisfied with

$$\tau_{\mathscr{M}}(\alpha_{\mathscr{M}}\alpha_{\mathscr{P}} + \beta_{\mathscr{M}}\alpha_{\mathscr{Q}})^2 = \tau_a, \ \ \sigma_{\mathscr{M}} = \sigma_a, \ \alpha_J = \alpha_g, \omega_0 = \bar{\omega}_0, \omega_1 = \bar{\omega}_1 \text{ and }$$
$$\alpha_\phi = \varepsilon_{\mathscr{A}}\varepsilon_\kappa \| \curlywedge_0 \|^2.$$

Using Theorem 1 and the fact

$$J^\circ(x; y) \leq \int_{\mho} g^\circ(x; y) d\mathbf{u}, \ \forall x, y \in \mathbb{X}.$$

Therefore, we can conclude that (48) admits a solution. Moreover, the condition (49) guarantees that (48) is uniquely solvable.
Next, for any parameter $\curlywedge > 0$, we introduce the function $\tilde{\Phi}^{f,\curlywedge} : \mathcal{C} \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} \tilde{\Phi}^{f,\curlywedge}(x) = \sup_{y \in \mathcal{C}} \Big\{ &\int_{\mho} a(\mathbf{u}, \nabla x) \cdot \nabla(x-y) d\mathbf{u} + \int_{\lambda_2} (\kappa(x)\mathscr{A}(\mathbf{u}, x) - \kappa(x)\mathscr{A}(\mathbf{u}, y))\, d\lambda \\ &- \int_{\mho} f(x-y) d\mathbf{u} - \int_{\mho} g^\circ(\mathbf{u}, x; y-x) d\mathbf{u} - \frac{1}{2\curlywedge}\|x-y\|^2_{\mathbb{X}} \Big\}. \end{aligned} \tag{50}$$

□

From Theorems 2–3, 4–5 and 6, we directly obtain the following error estimates.

Theorem 7 *Let* $x^* \in \mathcal{C}$ *be the unique solution of* (35)–(38). *Under the hypotheses of Theorem 6, we have*

(i) *for each* $\curlywedge > 0$ *and* $f \in L^2(\mho)$, $\tilde{\Phi}^{f,\curlywedge} : \mathcal{C} \longrightarrow \mathbb{R}$ *is a regularized gap function for* (48).
(ii) *If* $\curlywedge > 0$ *is such that*

$$\sigma_a - \tau_a - \alpha_g - \varepsilon_{\mathscr{A}}\varepsilon_\kappa \| \curlywedge_0 \|^2 > \frac{1}{2\curlywedge}. \tag{51}$$

Then for each $x \in \mathcal{C}$, *it holds*

$$\|x - x^*\|_{\mathbb{X}} \leq \sqrt{\frac{\tilde{\Phi}^{f,\curlywedge}(x)}{\sigma_a - \tau_a - \alpha_g - \varepsilon_{\mathscr{A}}\varepsilon_\kappa \| \curlywedge_0 \|^2 - \frac{1}{2\curlywedge}}}. \tag{52}$$

Theorem 8 *Let $x^* \in \mathcal{C}$ be the unique solution of* (48). *Under the hypotheses of Theorem* 6, *we have*

(i) *for any* $\curlywedge, \curlyvee > 0$, *the function* $\tilde{\Pi}_{\tilde{\Phi}^{f,\curlywedge,\curlyvee}} : \mathcal{C} \longrightarrow \mathbb{R}$ *defined by*

$$\tilde{\Pi}_{\tilde{\Phi}^{f,\curlywedge,\curlyvee}}(x) = \inf_{z \in \mathcal{C}} \left\{ \tilde{\Phi}^{f,\curlywedge}(z) + \curlyvee \|x - z\|^2_{\mathbb{X}} \right\} \tag{53}$$

is the Moreau–Yosida regularized gap function for (48).

(ii) *for any* $\curlyvee > 0$, *if* $\curlywedge > 0$ *is such that*

$$\sigma_a - \tau_a - \alpha_g - \varepsilon_{\mathscr{A}} \varepsilon_\kappa \| \curlywedge_0 \|^2 > \frac{1}{2\curlywedge}. \tag{54}$$

Then for each $x \in \mathcal{C}$ *the following bounds holds*

$$\|x - x^*\|_{\mathbb{X}} \leq \sqrt{\frac{2\tilde{\Pi}_{\tilde{\Phi}^{f,\curlywedge,\curlyvee}(x)}}{\min \left\{ \sigma_a - \tau_a - \alpha_g - \varepsilon_{\mathscr{A}} \varepsilon_\kappa \| \curlywedge_0 \|^2 - \frac{1}{2\curlywedge}, \curlyvee \right\}}}. \tag{55}$$

5 Conclusions

In this work, motivated by old and new results such as [26–29] and more, we study the gap functions and regularized gap functions for a class of generalized multivalued mixed variational-hemivariational inequality problems. Extensions to Minty version of these inequalities is also discussed as well as two new global error bounds for these problems. Application for this new results is presented and it is clear that this work extend and generalize related results in the literature.

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