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ORIGINAL PAPER

Two new extragradient methods for solving equilibrium problems

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Abstract
In this paper, we are concern with the classical equilibrium problem in real Hilbert spaces and introduce two new extragradient variants for it. By taking into account several fixed point theory techniques, we obtain simple structure methods that converge strongly and hence demonstrate the theoretical advantage of our methods. Moreover, our convergence assumptions are weaker than those assumed in related works in the literature. Primary numerical examples with comparisons illustrate the behaviour of our proposed scheme and show its advantages.

Keywords Strong convergence · Lipschitz-type constants · Equilibrium problem · Variational inequalities · Fixed point problems

Mathematics Subject Classification 65Y05 · 65K15 · 68W10 · 47H05 · 47H10

1 Introduction

In this work we study the classical *equilibrium problem* originally introduced by Muu and Oettli [27] and has been elaborated further by Blum and Oettli [4] (see also [11]). Given a non-empty, close and convex subset $\mathbb{K}$ of a real Hilbert space $\mathbb{E}$ and let $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ be a bifunction such that $f(y, y) = 0$, for all $y \in \mathbb{K}$. With this data, the equilibrium problem is formulated as follows.

$$\text{Find} \quad \wp^* \in \mathbb{K} \quad \text{such that } f(\wp^*, y) \geq 0, \quad \forall\, y \in \mathbb{K}. \tag{EP}$$

We denote the solution set of (EP) by Ω. Equilibrium problems attract much interest due to their generality in unifying various mathematical problems such as fixed point problems, vector and scalar minimization problems, variational inequalities, complementarity problems, and many more, see e.g., [3,4,7,12,13,27]. An important historical remark is that (EP) is also acknowledged as the well-known Ky Fan inequality due to his contribution [11].

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The study of equilibrium problems is divided roughly into two parts, the first is theoretical research of the existence of solutions to (EP) and the second is concern with the development of iterative methods for finding such solutions. Regarding the second direction the interested reader is refereed to the many existing results, see e.g., [8–10,15–17,26,28,30,33,38]. Moreover, techniques for non-monotone problems can be found in [19,20,32,34].

For our purposes we recall Tran et al. [37] iterative scheme that is formulated as follows.

$$\begin{cases} x_n \in \mathbb{K}, \\ y_n = \arg\min_{y\in\mathbb{K}}\{\zeta f(x_n, y) + \frac{1}{2}\|x_n - y\|^2\}, \\ x_{n+1} = \arg\min_{y\in\mathbb{K}}\{\zeta f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\}, \end{cases} \tag{1}$$

where ζ is some constant depending on the Lipschitz constant of the involved bifunction. This method is known as the two-step extragradient scheme taking its name from the work of Korpelevich [21] focus on saddle points. This method holds two major drawback, the first is the constant step size that require the knowledge or approximation of the Lipschitz constant of the involved bifunction and it only converges weakly in Hilbert spaces. In most cases, the Lipschitz constants are unknown or difficult to compute because it is difficult to check for every three elements in the underlying abstract space [1,29]. From the computational point of view it might be difficult to estimate the Lipschitz constant a-priori, and hence the convergence rate and applicability of the method could be effected.

Hence, a natural question arises:

Is it possible to introduce a strong convergent extragradient algorithm with adaptive stepsize rule for solving pseudomonotone equilibrium problems (EP)?

Motivated by the above, as well the works in [6,24,37], we answer the above question by introducing two strong convergence extragradient-type methods for solving pseudomonotone equilibrium problems in real Hilbert spaces. Moreover, we avoid the need to know the Lipschitz constant of the involved bifunction by using an adaptive stepsize rule.

The outline of our work is as follows. In Sect. 2 we recall some basic results and definitions. Then in Sect. 3 we introduce and analyse our new methods and afterwards in Sect. 4 we present some mathematical applications of our main results and finally in Sect. 5 we illustrate and compare the behaviour of our algorithms.

2 Preliminaries

Let $\mathbb{K}$ be a non-empty, close and convex subset of a real Hilbert space $\mathbb{E}$. The *metric projection* $P_{\mathbb{K}}(x)$ of $x \in \mathbb{E}$ onto a closed and convex subset $\mathbb{K}$ of $\mathbb{E}$ is defined by

$$P_{\mathbb{K}}(x) = \arg\min_{y\in\mathbb{K}}\|y - x\|. \tag{2}$$

Some useful properties of the metric projection are given next.

Lemma 2.1 e.g., [22] *The metric projection* $P_{\mathbb{K}} : \mathbb{E} \to \mathbb{K}$ *satisfy the following.*

(i) $\|y_1 - P_{\mathbb{K}}(y_2)\|^2 + \|P_{\mathbb{K}}(y_2) - y_2\|^2 \le \|y_1 - y_2\|^2, \ \ y_1 \in \mathbb{K}, y_2 \in \mathbb{E}$.

(ii) $y_3 = P_{\mathbb{K}}(y_1)$ *if and only if* $\langle y_1 - y_3, y_2 - y_3\rangle \le 0, \ \forall\, y_2 \in \mathbb{K}$.

(iii) $\|y_1 - P_{\mathbb{K}}(y_1)\| \le \|y_1 - y_2\|, \;\; y_2 \in \mathbb{K}, y_1 \in \mathbb{E}.$

Definition 2.2 Let $\mathbb{K}$ be a subset of a real Hilbert space $\mathbb{E}$ and $\chi : \mathbb{K} \to \mathbb{R}$ a given convex function.

(1) The *subdifferential of set* χ at $x \in \mathbb{K}$ is defined by

$$\partial\chi(x) = \{z \in \mathbb{E} : \chi(y) - \chi(x) \ge \langle z, y - x\rangle, \; \forall\, y \in \mathbb{K}\}. \tag{3}$$

(2) The *normal cone* at $x \in \mathbb{K}$ is defined by

$$N_{\mathbb{K}}(x) = \{z \in \mathbb{E} : \langle z, y - x\rangle \le 0, \; \forall\, y \in \mathbb{K}\}. \tag{4}$$

Lemma 2.3 [36] *Let $\chi : \mathbb{K} \to \mathbb{R}$ be a sub-differentiable, lower semi-continuous and function on $\mathbb{K}$. An element $x \in \mathbb{K}$ is a minimizer of a function χ iff $0 \in \partial\chi(x) + N_{\mathbb{K}}(x)$, where $\partial\chi(x)$ stands for the sub-differential of χ at $x \in \mathbb{K}$ and $N_{\mathbb{K}}(x)$ the normal cone of $\mathbb{K}$ at x.*

Lemma 2.4 [40] *Assume that $\{\gamma_n\} \subset (0, +\infty)$ is a sequence satisfying $\gamma_{n+1} \le (1 - \tau_n)\gamma_n + \tau_n\delta_n$, for all $n \in \mathbb{N}$. Moreover, $\{\tau_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ are sequences such that $\lim_{n\to\infty} \tau_n = 0$, $\sum_{n=1}^{\infty} \tau_n = \infty$ and $\limsup_{n\to\infty} \delta_n \le 0$. Therefore, $\lim_{n\to\infty} \gamma_n = 0$.*

Lemma 2.5 [23] *Assume that $\{\gamma_n\} \subset \mathbb{R}$ be a sequence and there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\gamma_{n_i} < \gamma_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then, there is a non decreasing sequence $m_k \subset \mathbb{N}$ such as $m_k \to \infty$ as $k \to \infty$, and the subsequent conditions are fulfilled by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$\gamma_{m_k} \le \gamma_{m_{k+1}} \text{ and } \gamma_k \le \gamma_{m_{k+1}}.$$

Indeed, $m_k = \max\{j \le k : \gamma_j \le \gamma_{j+1}\}$.

Lemma 2.6 [2] *For all $y_1, y_2 \in \mathbb{E}$ and $\delta \in \mathbb{R}$, then subsequent relationship hold.*

(i) $\|\delta y_1 + (1-\delta)y_2\|^2 = \delta\|y_1\|^2 + (1-\delta)\|y_2\|^2 - \delta(1-\delta)\|y_1 - y_2\|^2.$

(ii) $\|y_1 + y_2\|^2 \le \|y_1\|^2 + 2\langle y_2, y_1 + y_2\rangle.$

3 Main results

In this section, we introduce and analysed our two extragradient-type methods for solving pseudomonotone equilibrium problems in real Hilbert spaces. For the convergence theorems of the methods we assume the following conditions.

(f1) The bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ is *pseudomonotone* on $\mathbb{K}$, that is

$$f(y_1, y_2) \ge 0 \Longrightarrow f(y_2, y_1) \le 0, \quad \forall\, y_1, y_2 \in \mathbb{K}. \tag{5}$$

(f2) The bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ is Lipschitz-type continuity [25] on $\mathbb{K}$, that is, if there exist constants $c_1, c_2 > 0$ such that

$$f(y_1, y_3) \le f(y_1, y_2) + f(y_2, y_3) + c_1\|y_1 - y_2\|^2 + c_2\|y_2 - y_3\|^2, \quad \forall\, y_1, y_2, y_3 \in \mathbb{K} \tag{6}$$

(f3) $\limsup_{n\to\infty} f(y_n, y) \le f(y^*, y)$ for all $y \in \mathbb{K}$ and $\{y_n\} \subset \mathbb{K}$ satisfy $y_n \rightharpoonup y^*$.

(f4) $f(x, \cdot)$ is convex and sub-differentiable on $\mathbb{E}$ for each fixed $x \in \mathbb{E}$.

Algorithm 1 (The first extragradient-type method for solving (EP))

Step 0: Choose $x_0 \in \mathbb{K}$, $0 < \zeta < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$, $\{\phi_n\} \subset (a, b) \subset (0, 1 - \varphi_n)$ and $\{\varphi_n\} \subset (0, 1)$ satisfies the condition such that

$$\lim_{n\to\infty} \varphi_n = 0 \text{ and } \sum_{n=1}^{\infty} \varphi_n = +\infty.$$

Step 1: Compute

$$y_n = \arg\min\{\zeta f(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in \mathbb{K}\}.$$

If $x_n = y_n$, then STOP. Otherwise, go to Step 2.
Step 2: Compute

$$p_n = \arg\min\{\zeta f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in \mathbb{E}_n\},$$

where $\omega_n \in \partial_2 f(x_n, y_n)$ satisfying $x_n - \zeta\omega_n - y_n \in N_{\mathbb{K}}(y_n)$ and construct a half-space

$$\mathbb{E}_n = \{z \in \mathbb{E} : \langle x_n - \zeta\omega_n - y_n, z - y_n\rangle \le 0\}.$$

Step 3: Compute

$$x_{n+1} = (1 - \phi_n - \varphi_n)x_n + \phi_n p_n.$$

Set $n := n + 1$ and go back to Step 1.

Theorem 3.1 *Suppose that condition* (f1)–(f4) *are satisfied and solution set* Ω *is non-empty. Then, any sequence* $\{x_n\}$ *generated by Algorithm 1 converges strongly to* $\wp^* = P_\Omega(0)$.

Proof We start by proving the boundedness of the sequence $\{x_n\}$. By using Lemma 2.3, we get

$$0 \in \partial_2\Big(\zeta f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\Big)(p_n) + N_{\mathbb{E}_n}(p_n).$$

Thus, there is $\omega \in \partial f(y_n, p_n)$ and $\overline{\omega} \in N_{\mathbb{E}_n}(p_n)$ such that $\zeta\omega + p_n - x_n + \overline{\omega} = 0$. Thus, we have

$$\langle x_n - p_n, y - p_n\rangle = \zeta\langle\omega, y - p_n\rangle + \langle\overline{\omega}, y - p_n\rangle, \ \forall y \in \mathbb{E}_n.$$

Given that $\overline{\omega} \in N_{\mathbb{E}_n}(p_n)$ and $\langle\overline{\omega}, y - p_n\rangle \le 0$, for all $y \in \mathbb{E}_n$. Therefore, we have

$$\zeta\langle\omega, y - p_n\rangle \ge \langle x_n - p_n, y - p_n\rangle, \quad \forall y \in \mathbb{E}_n. \tag{7}$$

Since $\omega \in \partial f(y_n, p_n)$, then

$$f(y_n, y) - f(y_n, p_n) \ge \langle\omega, y - p_n\rangle, \quad \forall y \in \mathbb{E}. \tag{8}$$

Combining expressions (7) and (8), we obtain

$$\zeta f(y_n, y) - \zeta f(y_n, p_n) \ge \langle x_n - p_n, y - p_n\rangle, \quad \forall y \in \mathbb{E}_n. \tag{9}$$

Substituting $y = \wp^*$ in (9), we obtain

$$\zeta f(y_n, \wp^*) - \zeta f(y_n, p_n) \ge \langle x_n - p_n, \wp^* - p_n\rangle. \tag{10}$$

Since $\wp^* \in \Omega$, then $f(\wp^*, y_n) \geq 0$ implies that $f(y_n, \wp^*) \leq 0$ and together with Assumption (f1), we obtain

$$\langle x_n - p_n, p_n - \wp^* \rangle \geq \zeta f(y_n, p_n). \tag{11}$$

Following Assumption (f2), we have

$$f(x_n, p_n) \leq f(x_n, y_n) + f(y_n, p_n) + c_1 \|x_n - y_n\|^2 + c_2 \|y_n - p_n\|^2. \tag{12}$$

Combining (11) and (12), we get that

$$\langle x_n - p_n, p_n - \wp^* \rangle \geq \zeta \big\{ f(x_n, p_n) - f(x_n, y_n) \big\} - c_1 \zeta \|x_n - y_n\|^2 - c_2 \zeta \|y_n - p_n\|^2. \tag{13}$$

By the definition of $\mathbb{E}_n$ and the fact that $p_n \in \mathbb{E}_n$, we get $\langle x_n - \zeta \omega_n - y_n, p_n - y_n \rangle \leq 0$, which implies that

$$\langle x_n - y_n, p_n - y_n \rangle \leq \zeta \langle \omega_n, p_n - y_n \rangle. \tag{14}$$

Since $\omega_n \in \partial_2 f(x_n, y_n)$, we obtain

$$f(x_n, y) - f(x_n, y_n) \geq \langle \omega_n, y - y_n \rangle, \quad \forall\, y \in \mathbb{E}.$$

By replacing $y = p_n$, we obtain

$$f(x_n, p_n) - f(x_n, y_n) \geq \langle \omega_n, p_n - y_n \rangle. \tag{15}$$

It follows from inequalities (14) and (15) that

$$\zeta \big\{ f(x_n, p_n) - f(x_n, y_n) \big\} \geq \langle x_n - y_n, p_n - y_n \rangle. \tag{16}$$

From (13) and (16), we have

$$\langle x_n - p_n, p_n - \wp^* \rangle \geq \langle x_n - y_n, p_n - y_n \rangle - c_1 \zeta \|x_n - y_n\|^2 - c_2 \zeta \|y_n - p_n\|^2. \tag{17}$$

Now we obtain the following equalities.

$$2\langle x_n - p_n, p_n - \wp^* \rangle = \|x_n - \wp^*\|^2 - \|p_n - x_n\|^2 - \|p_n - \wp^*\|^2$$

and

$$2\langle y_n - x_n, y_n - p_n \rangle = \|x_n - y_n\|^2 + \|p_n - y_n\|^2 - \|x_n - p_n\|^2.$$

The above together with (17), imply that

$$\|p_n - \wp^*\|^2 \leq \|x_n - \wp^*\|^2 - (1 - 2c_1\zeta)\|x_n - y_n\|^2 - (1 - 2c_2\zeta)\|p_n - y_n\|^2. \tag{18}$$

Since $0 < \zeta < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$ with expression (18) implies that

$$\|p_n - \wp^*\|^2 \leq \|x_n - \wp^*\|^2. \tag{19}$$

Since $\wp^* \in \Omega$, we get

$$\begin{aligned} \big\|x_{n+1} - \wp^*\big\| &= \big\|(1 - \phi_n - \varphi_n)x_n + \phi_n p_n - \wp^*\big\| \\ &= \big\|(1 - \phi_n - \varphi_n)(x_n - \wp^*) + \phi_n(p_n - \wp^*) - \varphi_n \wp^*\big\| \\ &\leq \big\|(1 - \phi_n - \varphi_n)(x_n - \wp^*) + \phi_n(p_n - \wp^*)\big\| + \varphi_n \big\|\wp^*\big\|. \end{aligned} \tag{20}$$

Next, we compute the following:

$$\begin{aligned}
&\left\|(1-\phi_n-\varphi_n)(x_n-\wp^*)+\phi_n(p_n-\wp^*)\right\|^2\\
&\quad=(1-\phi_n-\varphi_n)^2\left\|x_n-\wp^*\right\|^2+\phi_n^2\left\|p_n-\wp^*\right\|^2+2\left\langle(1-\phi_n-\varphi_n)(x_n-\wp^*),\phi_n(p_n-\wp^*)\right\rangle\\
&\quad\le(1-\phi_n-\varphi_n)^2\left\|x_n-\wp^*\right\|^2+\phi_n^2\left\|p_n-\wp^*\right\|^2+2\phi_n(1-\phi_n-\varphi_n)\left\|x_n-\wp^*\right\|\left\|p_n-\wp^*\right\| \qquad (21)\\
&\quad\le(1-\phi_n-\varphi_n)^2\left\|x_n-\wp^*\right\|^2+\phi_n^2\left\|p_n-\wp^*\right\|^2\\
&\qquad+\phi_n(1-\phi_n-\varphi_n)\left\|x_n-\wp^*\right\|^2+\phi_n(1-\phi_n-\varphi_n)\left\|p_n-\wp^*\right\|^2\\
&\quad\le(1-\phi_n-\varphi_n)(1-\varphi_n)\left\|x_n-\wp^*\right\|^2+\phi_n(1-\varphi_n)\left\|p_n-\wp^*\right\|^2 \qquad (22)\\
&\quad\le(1-\phi_n-\varphi_n)(1-\varphi_n)\left\|x_n-\wp^*\right\|^2+\phi_n(1-\varphi_n)\left\|x_n-\wp^*\right\|^2\\
&\quad=(1-\varphi_n)^2\left\|x_n-\wp^*\right\|^2. \qquad (23)
\end{aligned}$$

Thus, we have

$$\left\|(1-\phi_n-\varphi_n)(x_n-\wp^*)+\phi_n(p_n-\wp^*)\right\|\le(1-\varphi_n)\left\|x_n-\wp^*\right\|. \qquad (24)$$

Combining (20) and (24), we get

$$\begin{aligned}
\left\|x_{n+1}-\wp^*\right\|&\le(1-\varphi_n)\left\|x_n-\wp^*\right\|+\varphi_n\left\|\wp^*\right\|\\
&\le\max\left\{\left\|x_n-\wp^*\right\|,\left\|\wp^*\right\|\right\}\\
&\le\max\left\{\left\|x_0-\wp^*\right\|,\left\|\wp^*\right\|\right\} \qquad (25)
\end{aligned}$$

and the boundedness of $\{x_n\}$ is obtained. Now we continue with the strong convergence of the sequence $\{x_n\}$. Indeed, by using definition of $\{x_{n+1}\}$, we have

$$\begin{aligned}
\left\|x_{n+1}-\wp^*\right\|^2&=\left\|(1-\phi_n-\varphi_n)x_n+\phi_n p_n-\wp^*\right\|^2\\
&=\left\|(1-\phi_n-\varphi_n)(x_n-\wp^*)+\phi_n(p_n-\wp^*)-\varphi_n\wp^*\right\|^2\\
&=\left\|(1-\phi_n-\varphi_n)(x_n-\wp^*)+\phi_n(p_n-\wp^*)\right\|^2+\varphi_n^2\left\|\wp^*\right\|^2\\
&\quad-2\left\langle(1-\phi_n-\varphi_n)(x_n-\wp^*)+\phi_n(p_n-\wp^*),\ \varphi_n\wp^*\right\rangle. \qquad (26)
\end{aligned}$$

From (22), we have

$$\begin{aligned}
&\left\|(1-\phi_n-\varphi_n)(x_n-\wp^*)+\phi_n(p_n-\wp^*)\right\|^2\\
&\quad\le(1-\phi_n-\varphi_n)(1-\varphi_n)\left\|x_n-\wp^*\right\|^2+\phi_n(1-\varphi_n)\left\|p_n-\wp^*\right\|^2. \qquad (27)
\end{aligned}$$

Combining (26) and (27) (for some $K_2 > 0$), we obtain

$$\begin{aligned}
&\|x_{n+1} - \wp^*\|^2 \\
&\le (1 - \phi_n - \varphi_n)(1 - \varphi_n)\|x_n - \wp^*\|^2 + \phi_n(1 - \varphi_n)\|p_n - \wp^*\|^2 + \varphi_n K_2 \\
&\le (1 - \phi_n - \varphi_n)(1 - \varphi_n)\|x_n - \wp^*\|^2 + \varphi_n K_2 \\
&\quad + \phi_n(1 - \varphi_n)\big[\|x_n - \wp^*\|^2 - (1 - 2c_1\zeta)\|x_n - y_n\|^2 - (1 - 2c_2\zeta)\|p_n - y_n\|^2\big] \\
&= (1 - \varphi_n)^2\|x_n - \wp^*\|^2 + \varphi_n K_2 \\
&\quad - \phi_n(1 - \varphi_n)\big[(1 - 2c_1\zeta)\|x_n - y_n\|^2 + (1 - 2c_2\zeta)\|p_n - y_n\|^2\big] \\
&\le \|x_n - \wp^*\|^2 + \varphi_n K_2 \\
&\quad - \phi_n(1 - \varphi_n)\Big[(1 - 2c_1\zeta)\|x_n - y_n\|^2 + (1 - 2c_2\zeta)\|p_n - y_n\|^2\Big].
\end{aligned} \tag{28}$$

By following the conditions (f1) and (f2), the solution set Ω is a closed and convex set, see for example, [14,37]). Given that $\wp^* = P_\Omega(0)$, and by Lemma 2.1 (ii), we have

$$\langle 0 - \wp^*, y - \wp^*\rangle \le 0, \ \ \forall\, y \in \Omega. \tag{29}$$

Now we divide the rest of the proof into the following two parts:
Case 1: Suppose that there is a fixed number $N_1 \in \mathbb{N}$ such that

$$\|x_{n+1} - \wp^*\| \le \|x_n - \wp^*\|, \ \ \forall\, n \ge N_1. \tag{30}$$

Then $\lim_{n\to\infty} \|x_n - \wp^*\|$ exists. From (28), we have

$$\begin{aligned}
&\phi_n(1 - \varphi_n)\Big[(1 - 2c_1\zeta)\|x_n - y_n\|^2 + (1 - 2c_2\zeta)\|p_n - y_n\|^2\Big] \\
&\quad \le \|x_n - \wp^*\|^2 + \varphi_n K_2 - \|x_{n+1} - \wp^*\|^2.
\end{aligned} \tag{31}$$

The existence of $\lim_{n\to\infty} \|x_n - \wp^*\|$ and $\varphi_n \to 0$, we infer that

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|p_n - y_n\| = 0. \tag{32}$$

It follows that

$$\lim_{n\to\infty} \|x_n - p_n\| \le \lim_{n\to\infty} \|x_n - y_n\| + \lim_{n\to\infty} \|y_n - p_n\| = 0. \tag{33}$$

It follows from (33) and $\varphi_n \to 0$, that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1 - \phi_n - \varphi_n)x_n + \phi_n p_n - x_n\| \\
&= \|x_n - \varphi_n x_n + \phi_n p_n - \phi_n x_n - x_n\| \\
&\le \phi_n\|p_n - x_n\| + \varphi_n\|x_n\|,
\end{aligned} \tag{34}$$

which gives that

$$\|x_{n+1} - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{35}$$

We deduce that $\{y_n\}$ and $\{p_n\}$ are bounded. The reflexivity of $\mathbb{E}$ and the boundedness of $\{x_n\}$ guarantee that there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\} \rightharpoonup \hat{x} \in \mathbb{E}$ as $k \to \infty$. Next, we need to show that $\hat{x} \in \Omega$. Due to (9), the Lipschitz-type continuous of f and (16), we get

$$\begin{aligned}\zeta f(y_{n_k}, y) &\geq \zeta f(y_{n_k}, p_{n_k}) + \langle x_{n_k} - p_{n_k}, y - p_{n_k}\rangle \\ &\geq \zeta f(x_{n_k}, p_{n_k}) - \zeta f(x_{n_k}, y_{n_k}) - c_1\zeta\|x_{n_k} - y_{n_k}\|^2 \\ &\quad - c_2\zeta\|y_{n_k} - p_{n_k}\|^2 + \langle x_{n_k} - p_{n_k}, y - p_{n_k}\rangle \\ &\geq \langle x_{n_k} - y_{n_k}, p_{n_k} - y_{n_k}\rangle - c_1\zeta\|x_{n_k} - y_{n_k}\|^2 \\ &\quad - c_2\zeta\|y_{n_k} - p_{n_k}\|^2 + \langle x_{n_k} - p_{n_k}, y - p_{n_k}\rangle, \end{aligned} \tag{36}$$

where y is an arbitrary point in $\mathbb{E}_n$. The boundedness of $\{x_n\}$ and from (32), (33) right-hand side converge to zero. Since $\zeta > 0$, Assumption (f3) and $y_{n_k} \rightharpoonup z$, we have

$$0 \leq \limsup_{k\to\infty} f(y_{n_k}, y) \leq f(\hat{x}, y), \ \forall y \in \mathbb{E}_n, \tag{37}$$

which implies that $f(\hat{x}, y) \geq 0$, $\forall y \in \mathbb{K}$, and hence $\hat{x} \in \Omega$. Next, we consider

$$\begin{aligned}&\limsup_{n\to\infty}\langle \wp^*, \wp^* - x_n\rangle \\ &\quad = \limsup_{k\to\infty}\langle \wp^*, \wp^* - x_{n_k}\rangle = \langle \wp^*, \wp^* - \hat{x}\rangle \leq 0.\end{aligned} \tag{38}$$

We have $\lim_{n\to\infty}\left\|x_{n+1} - x_n\right\| = 0$. We can infer that

$$\begin{aligned}&\limsup_{n\to\infty}\langle \wp^*, \wp^* - x_{n+1}\rangle \\ &\quad \leq \limsup_{n\to\infty}\langle \wp^*, \wp^* - x_n\rangle + \limsup_{n\to\infty}\langle \wp^*, x_n - x_{n+1}\rangle \leq 0.\end{aligned} \tag{39}$$

Next, assume that $t_n = (1 - \phi_n)x_n + \phi_n p_n$. Thus, we get

$$x_{n+1} = t_n - \varphi_n x_n = (1 - \varphi_n)t_n - \varphi_n(x_n - t_n) = (1 - \varphi_n)t_n - \varphi_n\phi_n(x_n - p_n). \tag{40}$$

where $x_n - t_n = x_n - (1 - \phi_n)x_n - \phi_n p_n = \phi_n(x_n - p_n)$. Therefore, we have

$$\begin{aligned}&\left\|x_{n+1} - \wp^*\right\|^2 \\ &\quad = \left\|(1 - \varphi_n)t_n + \phi_n\varphi_n(p_n - x_n) - \wp^*\right\|^2 \\ &\quad = \left\|(1 - \varphi_n)(t_n - \wp^*) + \left[\phi_n\varphi_n(p_n - x_n) - \varphi_n\wp^*\right]\right\|^2 \\ &\quad \leq (1 - \varphi_n)^2\left\|t_n - \wp^*\right\|^2 + 2\big\langle\phi_n\varphi_n(p_n - x_n) - \varphi_n\wp^*, \\ &\qquad (1 - \varphi_n)(t_n - \wp^*) + \phi_n\varphi_n(p_n - x_n) - \varphi_n\wp^*\big\rangle \\ &\quad = (1 - \varphi_n)^2\left\|t_n - \wp^*\right\|^2 + 2\big\langle\phi_n\varphi_n(p_n - x_n) - \varphi_n\wp^*, \ t_n - \varphi_n t_n - \varphi_n(x_n - t_n) - \wp^*\big\rangle \\ &\quad = (1 - \varphi_n)\left\|t_n - \wp^*\right\|^2 + 2\phi_n\varphi_n\big\langle p_n - x_n, \ x_{n+1} - \wp^*\big\rangle + 2\varphi_n\big\langle\wp^*, \ \wp^* - x_{n+1}\big\rangle \\ &\quad \leq (1 - \varphi_n)\left\|t_n - \wp^*\right\|^2 + 2\phi_n\varphi_n\left\|p_n - x_n\right\|\left\|x_{n+1} - \wp^*\right\| + 2\varphi_n\big\langle\wp^*, \ \wp^* - x_{n+1}\big\rangle.\end{aligned} \tag{41}$$

We next evaluate

$$\begin{aligned}
&\left\|t_n-\wp^*\right\|^2\\
&=\left\|(1-\phi_n)x_n+\phi_n p_n-\wp^*\right\|^2\\
&=\left\|(1-\phi_n)(x_n-\wp^*)+\phi_n(p_n-\wp^*)\right\|^2\\
&=(1-\phi_n)^2\left\|x_n-\wp^*\right\|^2+\phi_n^2\left\|p_n-\wp^*\right\|^2+2\left\langle(1-\phi_n)(x_n-\wp^*),\phi_n(p_n-\wp^*)\right\rangle\\
&\le(1-\phi_n)^2\left\|x_n-\wp^*\right\|^2+\phi_n^2\left\|p_n-\wp^*\right\|^2+2\phi_n(1-\phi_n)\left\|x_n-\wp^*\right\|\left\|p_n-\wp^*\right\|\\
&\le(1-\phi_n)^2\left\|x_n-\wp^*\right\|^2+\phi_n^2\left\|p_n-\wp^*\right\|^2+\phi_n(1-\phi_n)\left\|x_n-\wp^*\right\|^2\\
&\quad+\phi_n(1-\phi_n)\left\|p_n-\wp^*\right\|^2\\
&=(1-\phi_n)\left\|x_n-\wp^*\right\|^2+\phi_n\left\|p_n-\wp^*\right\|^2\\
&\le(1-\phi_n)\left\|x_n-\wp^*\right\|^2+\phi_n\left\|x_n-\wp^*\right\|^2\\
&=\left\|x_n-\wp^*\right\|^2.
\end{aligned}\tag{42}$$

Combining (41) and (42) yields

$$\begin{aligned}
&\left\|x_{n+1}-\wp^*\right\|^2\\
&\quad\le(1-\varphi_n)\left\|x_n-\wp^*\right\|^2+\varphi_n\Big[2\phi_n\left\|p_n-x_n\right\|\left\|x_{n+1}-\wp^*\right\|+2\varphi_n\left\langle\wp^*,\ \wp^*-x_{n+1}\right\rangle\Big].
\end{aligned}\tag{43}$$

Due to (39), (43) and Lemma 2.4, we deduce that $\left\|x_n-\wp^*\right\|\to 0$ as $n\to\infty$.
Case 2: Assume that there is a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i}-\wp^*\|\le\|x_{n_{i+1}}-\wp^*\|,\ \ \forall i\ \in\mathbb{N}.$$

By Lemma 2.5, there is a sequence $\{m_k\}\subset\mathbb{N}$ ($\{m_k\}\to\infty$), such that

$$\|x_{m_k}-\wp^*\|\le\|x_{m_{k+1}}-\wp^*\|\quad\text{and}\quad\|x_k-\wp^*\|\le\|x_{m_{k+1}}-\wp^*\|,\ \ \forall k\in\mathbb{N}.\tag{44}$$

From (31), we have

$$\begin{aligned}
&\phi_{m_k}(1-\varphi_{m_k})\Big[(1-2c_1\zeta)\|x_{m_k}-y_{m_k}\|^2+(1-2c_2\zeta)\|p_{m_k}-y_{m_k}\|^2\Big]\\
&\quad\le\|x_{m_k}-\wp^*\|^2+\varphi_{m_k}K_2-\|x_{m_k+1}-\wp^*\|^2.
\end{aligned}\tag{45}$$

Due to $\varphi_{m_k}\to 0$, we deduce the following:

$$\lim_{n\to\infty}\|x_{m_k}-y_{m_k}\|=\lim_{n\to\infty}\|p_{m_k}-y_{m_k}\|=0.\tag{46}$$

It follows that

$$\begin{aligned}
\left\|x_{m_k+1}-x_{m_k}\right\|&=\left\|(1-\phi_{m_k}-\varphi_{m_k})x_{m_k}+\phi_{m_k}p_{m_k}-x_{m_k}\right\|\\
&=\left\|x_{m_k}-\varphi_{m_k}x_{m_k}+\phi_{m_k}p_{m_k}-\phi_{m_k}x_{m_k}-x_{m_k}\right\|\\
&\le\phi_{m_k}\left\|p_{m_k}-x_{m_k}\right\|+\varphi_{m_k}\left\|x_{m_k}\right\|\longrightarrow 0.
\end{aligned}\tag{47}$$

Using similar argument as in Case 1, we get

$$\limsup_{k\to\infty}\langle\wp^*,x_{m_k+1}-\wp^*\rangle\le 0.\tag{48}$$

Now, using (43) and (44), we have

$$\begin{aligned}\|x_{m_k+1} - \wp^*\|^2 &\le (1-\varphi_{m_k})\|x_{m_k} - \wp^*\|^2 \\ &\quad + \varphi_{m_k}\Big[2\phi_{m_k}\|p_{m_k} - x_{m_k}\|\|x_{m_k+1} - \wp^*\| + 2\varphi_{m_k}\langle \wp^*, \wp^* - x_{m_k+1}\rangle\Big] \\ &\le (1-\varphi_{m_k})\|x_{m_{k+1}} - \wp^*\|^2 \\ &\quad + \varphi_{m_k}\Big[2\phi_{m_k}\|p_{m_k} - x_{m_k}\|\|x_{m_k+1} - \wp^*\| + 2\varphi_{m_k}\langle \wp^*, \wp^* - x_{m_k+1}\rangle\Big].\end{aligned} \tag{49}$$

It follows that

$$\|x_{m_k+1} - \wp^*\|^2 \le 2\phi_{m_k}\|p_{m_k} - x_{m_k}\|\|x_{m_k+1} - \wp^*\| + 2\varphi_{m_k}\langle \wp^*, \wp^* - x_{m_k+1}\rangle. \tag{50}$$

Since $\varphi_{m_k} \to 0$ and $\|x_{m_k} - \wp^*\|$ is bounded, (48) and (50), yield

$$\|x_{m_k+1} - \wp^*\|^2 \to 0, \text{ as } k \to \infty. \tag{51}$$

The above implies that

$$\lim_{n\to\infty} \|x_k - \wp^*\|^2 \le \lim_{n\to\infty} \|x_{m_k+1} - \wp^*\|^2 \le 0. \tag{52}$$

Consequently, $x_n \to \wp^*$ and the desired result is obtained. □

Next, we introduce a variant of Algorithm 1 in which the constant step size ζ is chosen adaptively and thus yield a sequence ζ_n that does not require the knowledge of the Lipschitz-like parameters of the bifunction f.

We start by a simple result concerning the sequence $\{\zeta_n\}$.

Lemma 3.2 *The sequence $\{\zeta_n\}$ generated according to Algorithm 2 is monotonically decreasing with the lower bound* $\min\left\{\frac{\eta}{2\max\{c_1,c_2\}}, \zeta_0\right\}$.

Proof Assuming that $f(x_n, p_n) - f(x_n, y_n) - f(y_n, p_n) > 0$, such that

$$\begin{aligned}\frac{\eta(\|x_n - y_n\|^2 + \|p_n - y_n\|^2)}{2[f(x_n, p_n) - f(x_n, y_n) - f(y_n, p_n)]} &\ge \frac{\eta(\|x_n - y_n\|^2 + \|p_n - y_n\|^2)}{2[c_1\|x_n - y_n\|^2 + c_2\|p_n - y_n\|^2]} \\ &\ge \frac{\eta}{2\max\{c_1, c_2\}}.\end{aligned} \tag{53}$$

□

Lemma 3.3 [31] *Assume that the bifunction $f : \mathbb{E}\times\mathbb{E} \to \mathbb{R}$ satisfies the conditions* (f1)–(f4)*; then for every $\wp^* \in \Omega \neq \emptyset$, we have*

$$\|p_n - \wp^*\|^2 \le \|x_n - \wp^*\|^2 - \Big(1 - \frac{\eta\zeta_n}{\zeta_{n+1}}\Big)\|x_n - y_n\|^2 - \Big(1 - \frac{\eta\zeta_n}{\zeta_{n+1}}\Big)\|p_n - y_n\|^2.$$

Theorem 3.4 *Suppose that condition* (f1)–(f4) *are hold and solution set Ω is non-empty. Then, any sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $\wp^* = P_\Omega(0)$.*

Proof By the definition of ζ_n, there is a number $N_0 \in \mathbb{N}$ such that

$$\|p_n - \wp^*\|^2 \le \|x_n - \wp^*\|^2, \quad \forall n \ge N_0. \tag{54}$$

The rest of the proof follows from similar arguments in the proof of Theorem 3.1 and hence omitted. □

Algorithm 2 (The second extragradient-type method for solving (EP))

Step 0: Choose $x_0 \in \mathbb{K}$, $\zeta_0 > 0$, $\eta \in (0,1)$, $\{\phi_n\} \subset (a,b) \subset (0, 1-\varphi_n)$ and $\{\varphi_n\} \subset (0,1)$ such that following conditions hold:

$$\lim_{n\to\infty} \varphi_n = 0 \text{ and } \sum_{n=1}^{\infty} \varphi_n = +\infty.$$

Step 1: Compute

$$y_n = \arg\min\{\zeta_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in \mathbb{K}\}.$$

If $x_n = y_n$, then STOP. Otherwise go to Step 2.
Step 2: Compute

$$p_n = \arg\min\{\zeta_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in \mathbb{E}_n\},$$

where $\omega_n \in \partial_2 f(x_n, y_n)$ satisfying $x_n - \zeta_n\omega_n - y_n \in N_{\mathbb{K}}(y_n)$ and construct a half-space

$$\mathbb{E}_n = \{z \in \mathbb{E} : \langle x_n - \zeta_n\omega_n - y_n, z - y_n\rangle \le 0\}.$$

Step 3: Compute

$$x_{n+1} = (1 - \phi_n - \varphi_n)x_n + \phi_n p_n.$$

Step 4: Compute

$$\zeta_{n+1} = \begin{cases} \min\left\{\zeta_n, \frac{\eta\|x_n - y_n\|^2 + \eta\|p_n - y_n\|^2}{2[f(x_n,p_n) - f(x_n,y_n) - f(y_n,p_n)]}\right\} \\ \quad \text{if } \ f(x_n, p_n) - f(x_n, y_n) - f(y_n, p_n) > 0, \\ \zeta_n, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad else. \end{cases}$$

Put $n := n + 1$ and go back to **Step 1**.

4 Applications

In this section we consider two mathematical applications, resolve variational inequalities and fixed point problem, and translates our methods for solving these problems.

Given a operator $\mathcal{T} : \mathbb{E} \to \mathbb{E}$, the fixed point problem is formulated as follows:

$$\text{Find } \wp^* \in \mathbb{K} \text{ such that } \mathcal{T}(\wp^*) = \wp^*. \tag{FPP}$$

Assume that the above operator $\mathcal{T}$ fulfils the following conditions. We assume that following requirements:

($\mathcal{T}$1). The operator $\mathcal{T}$ is κ-*strict pseudo-contraction* (see e.g., [5]) on $\mathbb{K}$ if

$$\|Ty_1 - Ty_2\|^2 \le \|y_1 - y_2\|^2 + \kappa\|(y_1 - Ty_1) - (y_2 - Ty_2)\|^2, \ \forall\, y_1, y_2 \in \mathbb{K};$$

($\mathcal{T}$2). The operator $\mathcal{T}$ is *weakly sequentially continuous* on $\mathbb{K}$ if

$$\mathcal{T}(y_n) \rightharpoonup \mathcal{T}(y^*) \text{ for any sequence in } \mathbb{K} \text{ satisfying } y_n \rightharpoonup y^*.$$

For such $\mathcal{T}$ we define the bifunction $f(x, y) = \langle x - \mathcal{T}x, y - x\rangle$, and it can be easily proven that f satisfies Assumptions (f1)–(f4) with $2c_1 = 2c_2 = \frac{3-2\kappa}{1-\kappa}$, see for example [39]. With this data Algorithm 1 translates to:

$$\begin{cases} y_n = \arg\min_{y\in\mathbb{K}}\{\zeta f(x_n, y) + \frac{1}{2}\|x_n - y\|^2\} = P_{\mathbb{K}}\big[x_n - \zeta(x_n - \mathcal{T}(x_n))\big], \\ p_n = \arg\min_{y\in\mathbb{E}_n}\{\zeta f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\} = P_{\mathbb{E}_n}\big[x_n - \zeta(y_n - \mathcal{T}(y_n))\big]. \end{cases} \tag{55}$$

Corollary 4.1 *Let $\mathcal{T} : \mathbb{K} \to \mathbb{K}$ be a mapping satisfying ($\mathcal{T}$1)–($\mathcal{T}$2) and $Fix(\mathcal{T}) \neq \emptyset$. Let $x_0 \in \mathbb{K}$, $0 < \zeta < \frac{1-\kappa}{3-2\kappa}$, $\{\phi_n\} \subset (a, b) \subset (0, 1 - \varphi_n)$ and $\{\varphi_n\} \subset (0, 1)$ such that*

$$\lim_{n\to\infty} \varphi_n = 0 \text{ and } \sum_{n=1}^{\infty} \varphi_n = +\infty.$$

Consider the iterative update:

$$\begin{cases} y_n = P_{\mathbb{K}}\big[x_n - \zeta(x_n - \mathcal{T}(x_n))\big], \\ p_n = P_{\mathbb{E}_n}\big[x_n - \zeta(y_n - \mathcal{T}(y_n))\big], \\ x_{n+1} = (1 - \phi_n - \varphi_n)x_n + \phi_n p_n, \end{cases}$$

where

$$\mathbb{E}_n = \{z \in \mathbb{E} : \langle (1 - \zeta)x_n + \zeta\mathcal{T}(x_n) - y_n, z - y_n\rangle \leq 0\}.$$

Then, $\{x_n\}$ converges strongly to $\wp^* \in Fix(\mathcal{T})$.

Corollary 4.2 *Let $\mathcal{T} : \mathbb{K} \to \mathbb{K}$ be a mapping satisfying ($\mathcal{T}$1)–($\mathcal{T}$2) and $Fix(\mathcal{T}) \neq \emptyset$. Let $x_0 \in \mathbb{K}$, $\eta \in (0, 1)$, $\zeta_0 > 0$, $\{\phi_n\} \subset (a, b) \subset (0, 1 - \varphi_n)$ and $\{\varphi_n\} \subset (0, 1)$ such that*

$$\lim_{n\to\infty} \varphi_n = 0 \text{ and } \sum_{n=1}^{\infty} \varphi_n = +\infty.$$

Consider the iterative update:

$$\begin{cases} y_n = P_{\mathbb{K}}\big[x_n - \zeta_n(x_n - \mathcal{T}(x_n))\big], \\ p_n = P_{\mathbb{E}_n}\big[x_n - \zeta_n(y_n - \mathcal{T}(y_n))\big], \\ x_{n+1} = (1 - \phi_n - \varphi_n)x_n + \phi_n p_n, \end{cases}$$

where

$$\mathbb{E}_n = \{z \in \mathbb{E} : \langle (1 - \zeta_n)x_n + \zeta_n\mathcal{T}(x_n) - y_n, z - y_n\rangle \leq 0\}.$$

Compute

$$\zeta_{n+1} = \begin{cases} \min\left\{\zeta_n, \frac{\eta\|x_n - y_n\|^2 + \eta\|p_n - y_n\|^2}{2\left[\left\langle (x_n - y_n) - [T(x_n) - T(y_n)], p_n - y_n\right\rangle\right]}\right\} \\ \quad \text{if } \left\langle (x_n - y_n) - [T(x_n) - T(y_n)], p_n - y_n\right\rangle > 0, \\ \zeta_n, \quad \text{otherwise}. \end{cases}$$

Then, $\{x_n\}$ converges strongly to $\wp^* \in Fix(\mathcal{T})$.

Next, we apply our results to the classical *variational inequalities* (VI) problem [18,35]. Given a set $\mathbb{K}$ and an operator $\mathcal{G} : \mathbb{E} \to \mathbb{E}$, the (VI) is formulated as follows:

$$\text{Find } \wp^* \in \mathbb{K} \text{ such that } \left\langle \mathcal{G}(\wp^*), y - \wp^* \right\rangle \geq 0, \ \ \forall y \in \mathbb{K}. \tag{VI}$$

For our purposes we assume that the following requirements are fulfilled.

($\mathcal{G}1$). The problem (VI) has solution set, represented by $VI(\mathcal{G}, \mathbb{K})$ is non-empty.

($\mathcal{G}2$). The operator $\mathcal{G}$ is *pseudo-monotone*, that is

$$\left\langle \mathcal{G}(y_1), y_2 - y_1 \right\rangle \geq 0 \Longrightarrow \left\langle \mathcal{G}(y_2), y_1 - y_2 \right\rangle \leq 0, \ \ \forall y_1, y_2 \in \mathbb{K}.$$

($\mathcal{G}3$). The operator $\mathcal{G}$ is *Lipschitz continuous*, i.e., there is $L > 0$ such that

$$\|\mathcal{G}(y_1) - \mathcal{G}(y_2)\| \leq L\|y_1 - y_2\|, \ \ \forall y_1, y_2 \in \mathbb{K};$$

($\mathcal{G}4$). $\limsup_{n\to\infty} \langle G(x_n), y - x_n \rangle \leq \langle G(p), y - p \rangle$, $\forall y \in \mathcal{C}$ and $\{x_n\} \subset \mathcal{C}$ satisfying $x_n \rightharpoonup p$.

For the above $\mathcal{G}$ we define $f(x, y) := \left\langle \mathcal{G}(x), y - x \right\rangle$ for all $x, y \in \mathbb{K}$. Thus (EP) translates to the above variational inequality with $L = 2c_1 = 2c_2$. Observe that with such bifunction f, we have

$$\begin{cases} y_n = \arg\min_{y \in \mathbb{K}} \{\zeta f(x_n, y) + \frac{1}{2}\|x_n - y\|^2\} = P_{\mathbb{K}}(x_n - \zeta \mathcal{G}(x_n)), \\ p_n = \arg\min_{y \in \mathbb{E}_n} \{\zeta f(y_n, y) + \frac{1}{2}\|x_n - y\|^2\} = P_{\mathbb{E}_n}(x_n - \zeta \mathcal{G}(y_n)). \end{cases} \tag{56}$$

Corollary 4.3 *Let the operator* $\mathcal{G} : \mathbb{K} \to \mathbb{E}$ *satisfy* ($\mathcal{G}1$)–($\mathcal{G}4$). *Let* $x_0 \in \mathbb{K}$, $0 < \zeta < \frac{1}{L}$, $\{\phi_n\} \subset (a, b) \subset (0, 1 - \varphi_n)$ *and* $\{\varphi_n\} \subset (0, 1)$ *such that*

$$\lim_{n\to\infty} \varphi_n = 0 \ \textit{and} \ \sum_{n=1}^{\infty} \varphi_n = +\infty.$$

Consider the iterative update

$$\begin{cases} y_n = P_{\mathbb{K}}(x_n - \zeta \mathcal{G}(x_n)), \\ p_n = P_{\mathbb{E}_n}(x_n - \zeta \mathcal{G}(y_n)), \\ x_{n+1} = (1 - \phi_n - \varphi_n)x_n + \phi_n p_n, \end{cases}$$

where

$$\mathbb{E}_n = \{z \in \mathbb{E} : \langle x_n - \zeta \mathcal{G}(x_n) - y_n, z - y_n \rangle \leq 0\}.$$

Then the sequence $\{x_n\}$ *converges strongly to* $\wp^* \in VI(\mathcal{G}, \mathbb{K})$.

Corollary 4.4 *Let the operator* $\mathcal{G} : \mathbb{K} \to \mathbb{E}$ *satisfy* ($\mathcal{G}1$)–($\mathcal{G}4$). *Let* $x_0 \in \mathbb{K}$, $\eta \in (0, 1)$, $\zeta_0 > 0$, $\{\phi_n\} \subset (a, b) \subset (0, 1 - \varphi_n)$ *and* $\{\varphi_n\} \subset (0, 1)$ *such that*

$$\lim_{n\to\infty} \varphi_n = 0 \ \textit{and} \ \sum_{n=1}^{\infty} \varphi_n = +\infty.$$

Consider the iterative update

$$\begin{cases} y_n = P_{\mathbb{K}}(x_n - \zeta_n \mathcal{G}(x_n)), \\ p_n = P_{\mathbb{E}_n}(x_n - \zeta_n \mathcal{G}(y_n)), \\ x_{n+1} = (1 - \phi_n - \varphi_n)x_n + \phi_n p_n, \end{cases}$$

where

$$\mathbb{E}_n = \{z \in \mathbb{E} : \langle x_n - \zeta_n \mathcal{G}(x_n) - y_n, z - y_n \rangle \le 0\}.$$

Compute

$$\zeta_{n+1} = \begin{cases} \min\left\{\zeta_n, \ \frac{\eta\|x_n - y_n\|^2 + \eta\|p_n - y_n\|^2}{2\left[\left\langle \mathcal{G}(x_n) - \mathcal{G}(y_n), p_n - y_n \right\rangle\right]}\right\} & \textit{if } \left\langle \mathcal{G}(x_n) - \mathcal{G}(y_n), p_n - y_n \right\rangle > 0, \\ \zeta_n, & \textit{else}. \end{cases}$$

Then $\{x_n\}$ *strongly converges to* $\wp^* \in VI(\mathcal{G}, \mathbb{K})$.

Remark 4.5 Note that Assumption ($\mathcal{G}$4) could be exempted in case that $\mathcal{G}$ is monotone. Indeed, this condition is a particular case of (f3) and is used to prove (37). In the absence of condition ($\mathcal{G}$4), the inequality (36) can be accomplished by imposing the monotonicity condition on $\mathcal{G}$. In that case, we get

$$\langle \mathcal{G}(y), y - y_n \rangle \ge \langle \mathcal{G}(y_n), y - y_n \rangle, \quad \forall\, y \in \mathbb{K}. \tag{57}$$

From (36), we obtain

$$\limsup_{k\to\infty} \langle \mathcal{G}(y_{n_k}), y - y_{n_k} \rangle \ge 0, \ \ \forall\, y \in \mathbb{E}_n. \tag{58}$$

By (57) and (58), we conclude that

$$\limsup_{k\to\infty} \langle \mathcal{G}(y), y - y_{n_k} \rangle \ge 0, \quad \forall\, y \in \mathbb{K}. \tag{59}$$

Let $y_s = (1-s)z + sy$, for $s \in [0, 1]$. It is clear that $y_s \in \mathbb{K}$ for every $s \in (0, 1)$. Since $y_{n_k} \rightharpoonup z \in \mathbb{K}$ and $\langle \mathcal{G}(y), y - z \rangle \ge 0$ for every $y \in \mathbb{K}$, we have

$$0 \le \langle \mathcal{G}(y_s), y_s - z \rangle = s \langle \mathcal{G}(y_s), y - z \rangle. \tag{60}$$

Therefore, $\langle \mathcal{G}(y_s), y - z \rangle \ge 0$, for $s \in (0, 1)$. Since $y_s \to z$ as $s \to 0$ and due to continuity of $\mathcal{G}$, we have $\langle \mathcal{G}(z), y - z \rangle \ge 0$, for $y \in \mathbb{K}$, gives that $z \in VI(\mathcal{G}, \mathbb{K})$.

Remark 4.6 From the above discussion, it can be concluded that Corollaries 4.3 and 4.4 still hold, even if we remove the condition ($\mathcal{G}$4) in case that the bifunction are monotone.

5 Numerical illustrations

In this section, we include three numerical test problems and illustrate the behaviour of our methods with comparisons to some related works in the literature.

Example 5.1 Consider the set (box)

$$\mathbb{K} := \{x \in \mathbb{R}^m : -5 \le x_i \le 5\},$$

and $f : \mathbb{K} \times \mathbb{K} \to \mathbb{R}$ is

$$f(x, y) = \langle Ax + By + d, y - x \rangle, \quad \forall\, x, y \in \mathbb{K},$$

where $d \in \mathbb{R}^m$ and A, B are matrices of order m. The matrix A is symmetric positive semi-definite and the matrix and a symmetric negative semi-definite matrix $B - A$ through

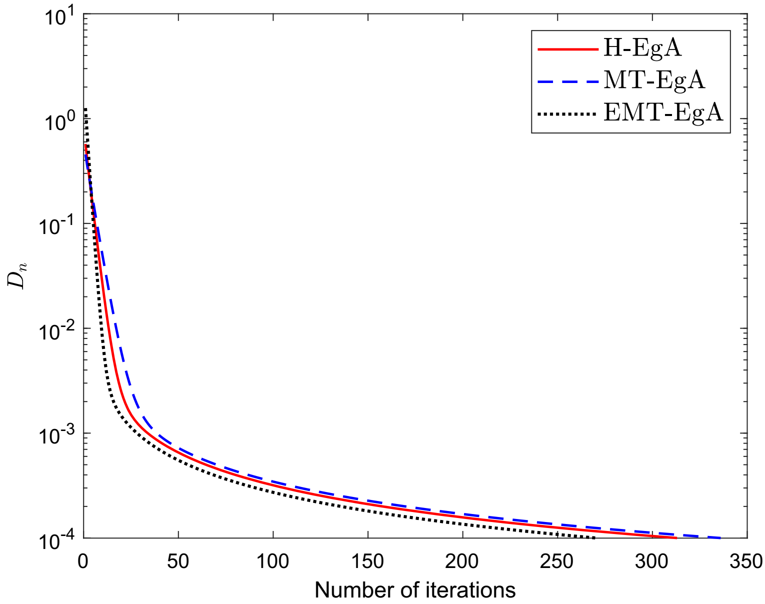

Fig. 1 Comparison of Algorithm 1 with Algorithm 2 and Algorithm 3.2 in [14] with $x_0 = (0, 0, 0, 0, 0)^T$

Lipschitz-type constants are $c_1 = c_2 = \frac{1}{2}\|A - B\|$ (see [37] for details). Two matrices A, B are define by

$$A = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad d = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}.$$

The numerical and graphical results are presented in Figs. 1, 2, 3, 4 and Table 1 by considering different initial points and $TOL = 10^{-4}$. The control parameters are taken in the following way:

(i) $\zeta = \frac{1}{3c_1}$ and $\phi_n = \frac{1}{60(n+2)}$ for Algorithm 2 (H-EgA) in [14].
(ii) $\zeta = \frac{1}{3c_1}$, $\varphi_n = \frac{1}{60(n+2)}$ and $\phi_n = \frac{7}{10}(1 - \varphi_n)$ for Algorithm 1 (MT-EgA).
(iii) $\zeta_0 = 0.7$, $\eta = 0.9$, $\varphi_n = \frac{1}{60(n+2)}$ and $\phi_n = \frac{7}{10}(1 - \varphi_n)$ for Algorithm 2 (EMT-EgA).

Example 5.2 Assume that set $\mathbb{K} \subset \mathbb{R}^5$ is defined by

$$\mathbb{K} = \{(x_1, \ldots, x_5) : x_1 \geq -1, x_i \geq 1, i = 2, \cdots, 5\}.$$

Let $f : \mathbb{K} \times \mathbb{K} \to \mathbb{R}$ is defined in the following way:

$$f(x, y) = \sum_{i=2}^{5} (y_i - x_i)\|x\|, \;\; \forall x, y \in \mathbb{R}^5.$$

Then, f is Lipschitz-type continuous with $c_1 = c_2 = 2$, and satisfy the items (f1)–(f4). The solution set of the equilibrium problem is $EP(f, \mathbb{K}) = \{(x_1, 1, 1, 1, 1) : x_1 > -1\}$ (for

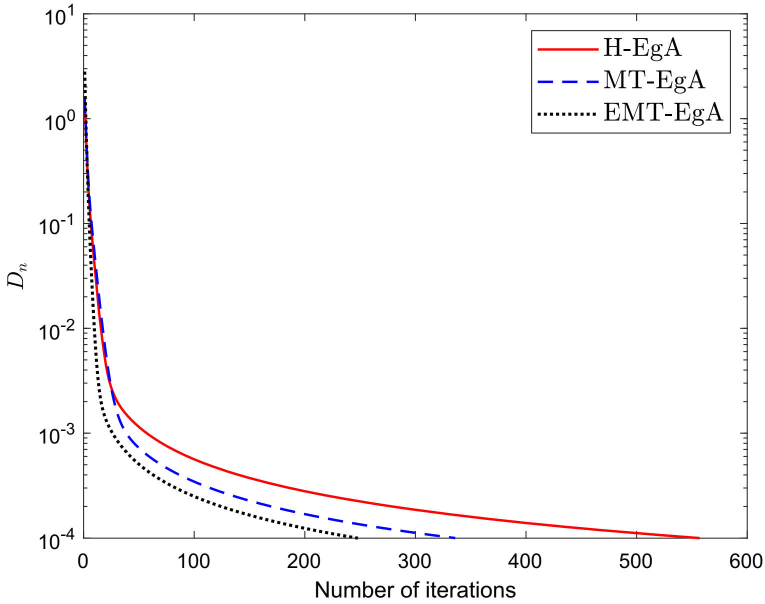

Fig. 2 Comparison of Algorithm 1 with Algorithm 2 and Algorithm 3.2 in [14] with $x_0 = (1, 1, 1, 1, 1)^T$

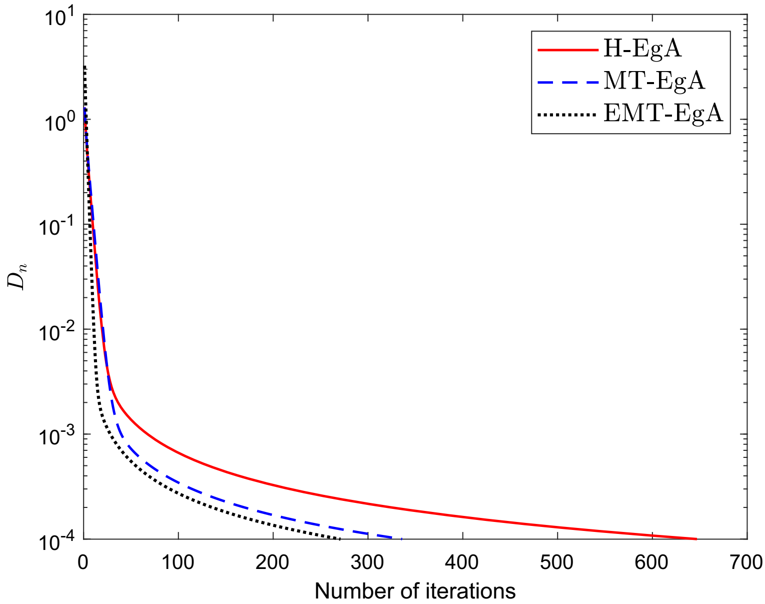

Fig. 3 Comparison of Algorithm 1 with Algorithm 2 and Algorithm 3.2 in [14] with $x_0 = (1, 0, -1, 2, 1)^T$

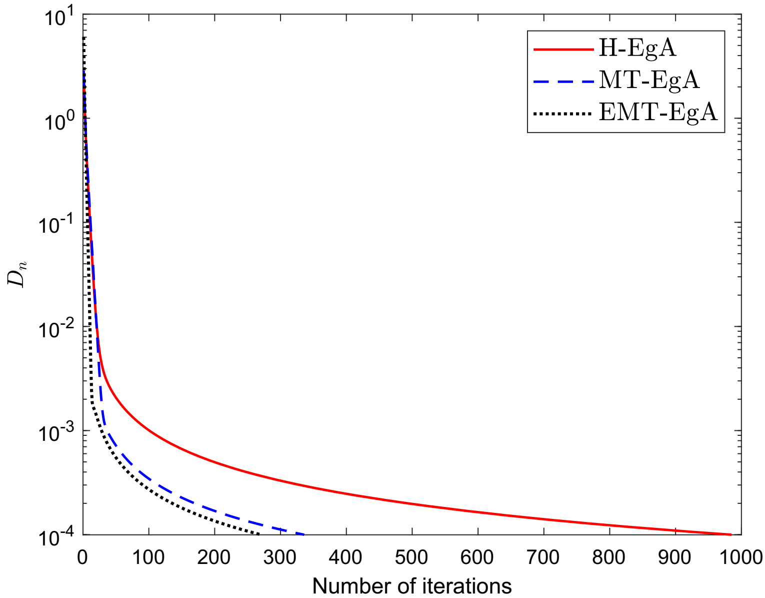

Fig. 4 Comparison of Algorithm 1 with Algorithm 2 and Algorithm 3.2 in [14] with $x_0 = (2, -1, 3, -4, 5)^T$

Table 1 Numerical data for Figs. 1, 2, 3 and 4

	Number of iterations			Execution time in seconds		
x_0	H-EgA	MT-EgA	EMT-EgA	H-EgA	MT-EgA	EMT-EgA
$(0, 0, 0, 0, 0)^T$	313	336	271	3.436302	3.616331	2.826888
$(1, 1, 1, 1, 1)^T$	557	336	248	7.431420	3.810916	2.517595
$(1, 0, -1, 2, 1)^T$	647	336	271	9.023943	3.929099	3.241147
$(2, -1, 3, -4, 5)^T$	985	336	271	14.593684	4.090390	2.899323

more details see [39]). All numerical results are reported in Figs. 5, 6 and Table 2, 3, 4, 5, 6 and 7 by assuming different initial points and $TOL = 10^{-3}$. The control parameters are taken in the following way:

(i) $\zeta = \frac{1}{4c_1}$ and $\phi_n = \frac{1}{40(n+2)}$ for Algorithm 2 (H-EgA) in [14];
(ii) $\zeta = \frac{1}{4c_1}$, $\varphi_n = \frac{1}{40(n+2)}$ and $\phi_n = \frac{6}{10}(1 - \varphi_n)$ for Algorithm 1 (MT-EgA).
(iii) $\zeta_0 = 0.7$, $\eta = 0.85$, $\varphi_n = \frac{1}{40(n+2)}$ and $\phi_n = \frac{6}{10}(1 - \varphi_n)$ for Algorithm 2 (EMT-EgA).

Example 5.3 Consider the set

$$\mathbb{K} := \{x \in L^2([0, 1]) : \|x\| \leq 1\}.$$

Let us define an operator $\mathcal{G} : \mathbb{K} \to \mathbb{E}$ such that

$$\mathcal{G}(x)(t) = \int_0^1 \big[x(t) - H(t, s) f(x(s))\big] ds + g(t),$$

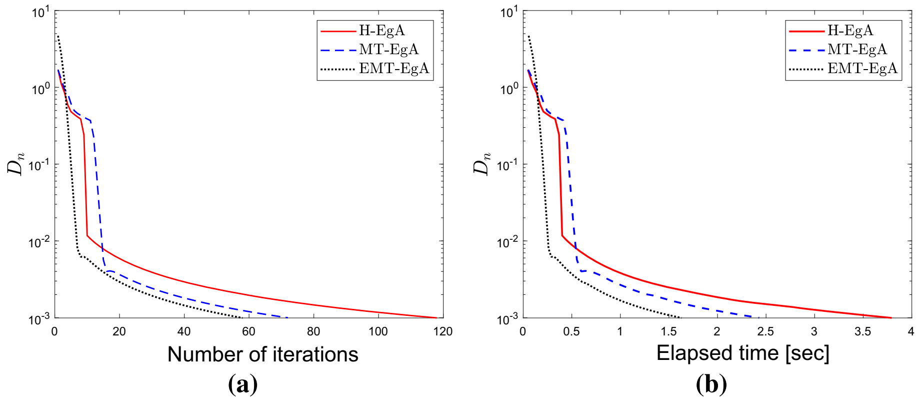

Fig. 5 Comparison of Algorithm 1 with Algorithm 2 and Algorithm 3.2 in [14] with $x_0 = (2, 3, 2, 5, 2)^T$

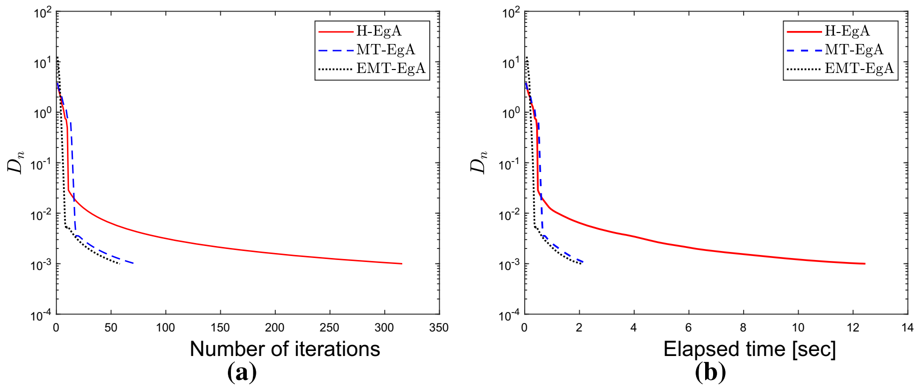

Fig. 6 Comparison of Algorithm 1 with Algorithm 2 and Algorithm 3.2 in [14] with $x_0 = (5, 6, 3, 10, 8)^T$

Table 2 Example 5.2: numerical study of Algorithm 3.2 in [14] and $x_0 = (2, 3, 2, 5, 2)^T$

Iter (n)	x_1	x_2	x_3	x_4	x_5
1	2	2.3407	1.3407	4.3407	1.3407
2	2	1.766347	1.008335	3.766347	1.008335
3	2	1.766347	1.008335	3.766347	1.008335
4	2	1.010002	1.005001	2.793095	1.005001
5	2	1.008333	1.004167	2.364959	1.004167
6	2	1.007143	1.003571	1.967146	1.003571
⋮	⋮	⋮	⋮	⋮	⋮
116	2	1.000427	1.000214	1.000855	1.000214
117	2	1.000424	1.000212	1.000848	1.000212
118	2	1.00042	1.00021	1.00084	1.00021
CPU time is seconds	3.793708				

Table 3 Example 5.2: numerical study of Algorithm 1 and $x_0 = (2, 3, 2, 5, 2)^T$

Iter (n)	x_1	x_2	x_3	x_4	x_5
1	1.975	2.50099	1.51349	4.47599	1.51349
2	1.958542	2.064089	1.144436	4.02263	1.144436
3	1.946301	1.669915	1.03681	3.616215	1.03681
4	1.936569	1.311534	1.005988	3.248101	1.005988
5	1.9285	1.088906	0.997623	2.906991	0.997623
6	1.921613	1.023005	0.995718	2.587585	0.995718
⋮	⋮	⋮	⋮	⋮	⋮
70	1.816254	0.999494	0.999494	0.999494	0.999494
71	1.815623	0.999501	0.999501	0.999501	0.999501
72	1.815001	0.999508	0.999508	0.999508	0.999508
CPU time is seconds	2.429083				

Table 4 Example 5.2: numerical study of Algorithm 2 and $x_0 = (2, 3, 2, 5, 2)^T$

Iter (n)	x_1	x_2	x_3	x_4	x_5
1	1.98	1.597948	1.287	3.577929	1.287
2	1.9668	1.171522	1.078859	2.184252	1.078859
3	1.956966	1.046199	1.01854	1.348499	1.01854
4	1.949139	1.009805	1.00154	1.100132	1.00154
5	1.942642	0.999599	0.997128	1.026607	0.997128
6	1.937092	0.997024	0.996284	1.005103	0.996284
⋮	⋮	⋮	⋮	⋮	⋮
56	1.859773	0.999496	0.999496	0.999496	0.999496
57	1.859132	0.999505	0.999505	0.999505	0.999505
58	1.858502	0.999513	0.999513	0.999513	0.999513
CPU time is seconds	1.629112				

where

$$H(t,s) = \frac{2tse^{(t+s)}}{e\sqrt{e^2-1}}, \quad f(x) = \cos(x), \quad g(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

In the above $\mathbb{E} = L^2([0,1])$ is a Hilbert space through inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)dt,\ \forall x, y \in \mathbb{E}$ and the induced norm is $\|x\| = \sqrt{\int_0^1 |x(t)|^2 dt}$. Numerical and graphical results of three methods are shown in Figs. 7, 8, 9, 10 and Table 8 by considering different initial points and $TOL = 10^{-3}$. The control parameters are taken in the following way:

(i) $\zeta = \frac{1}{5c_1}$ and $\phi_n = \frac{1}{100(n+2)}$ for Algorithm 2 (H-EgA) in [14];
(ii) $\zeta = \frac{1}{5c_1}$, $\varphi_n = \frac{1}{100(n+2)}$ and $\phi_n = \frac{3}{10}(1-\varphi_n)$ for Algorithm 1 (MT-EgA);
(iii) $\zeta_0 = 0.50$, $\eta = 0.50$, $\varphi_n = \frac{1}{100(n+2)}$ and $\phi_n = \frac{3}{10}(1-\varphi_n)$ for Algorithm 2 (EMT-EgA).

Table 5 Example 5.2: numerical study of Algorithm 3.2 in [14] and $x_0 = (5, 6, 3, 10, 8)^T$

Iter (n)	x_1	x_2	x_3	x_4	x_5
1	5	4.511721	1.511721	8.511721	6.511721
2	5	3.244471	1.016667	7.244471	5.244471
3	5	2.154032	1.0125	6.154032	4.154032
4	5	1.202343	1.01	5.20234	3.20234
5	5	1.020833	1.008333	4.347146	2.347146
6	5	1.017857	1.007143	3.569716	1.569716
⋮	⋮	⋮	⋮	⋮	⋮
314	4.99999	1.000397	1.000159	1.000714	1.000556
315	4.99999	1.000396	1.000158	1.000712	1.000554
316	4.99999	1.000394	1.000158	1.00071	1.000552
CPU time is seconds	12.450176				

Table 6 Example 5.2: numerical study of Algorithm 1 and $x_0 = (5, 6, 3, 10, 8)^T$

Iter (n)	x_1	x_2	x_3	x_4	x_5
1	4.9375	4.883205	1.920705	8.833205	6.858205
2	4.896354	3.915427	1.265576	7.832511	5.873969
3	4.865752	3.061258	1.072925	6.953859	5.007559
4	4.841423	2.299238	1.016768	6.172377	4.235807
5	4.82125	1.61268	1.000843	5.469681	3.54118
6	4.804032	1.179577	0.996681	4.826808	2.905195
⋮	⋮	⋮	⋮	⋮	⋮
70	4.540633	0.999494	0.999494	0.999494	0.999494
71	4.539056	0.999501	0.999501	0.999501	0.999501
72	4.537502	0.999508	0.999508	0.999508	0.999508
CPU time is seconds	2.293300				

6 Conclusion

This study established two techniques to figure out the problems of equilibrium. The initial method is a strong convergence through a Mann-type scheme and fixed step size, based on the Lipschitz coefficients. The second method includes a key edge over the initial method due to the self-adapting step size rule. Numerical conclusions have been mentioned to show the numerical effectiveness of proposed methods compared to other methods. Such numerical studies indicate that the Mann-type scheme normally helps in increasing the efficiency of the iterative sequence compared to the Halpern method.

Table 7 Example 5.2: numerical study of Algorithm 2 and $x_0 = (5, 6, 3, 10, 8)^T$

Iter (n)	x_1	x_2	x_3	x_4	x_5
1	4.95	3.327656	1.584	7.287656	5.307656
2	4.917	1.686976	1.167366	4.640513	2.673715
3	4.892415	1.200062	1.044959	2.081703	1.494604
4	4.872846	1.055779	1.009434	1.319213	1.143788
5	4.856603	1.013345	0.999488	1.092112	1.03966
6	4.842727	1.001135	0.99699	1.024698	1.009007
⋮	⋮	⋮	⋮	⋮	⋮
56	4.649381	0.999495	0.999495	0.999495	0.999495
57	4.647778	0.999504	0.999504	0.999504	0.999504
58	4.646203	0.999513	0.999513	0.999513	0.999513
CPU time is seconds	2.076987				

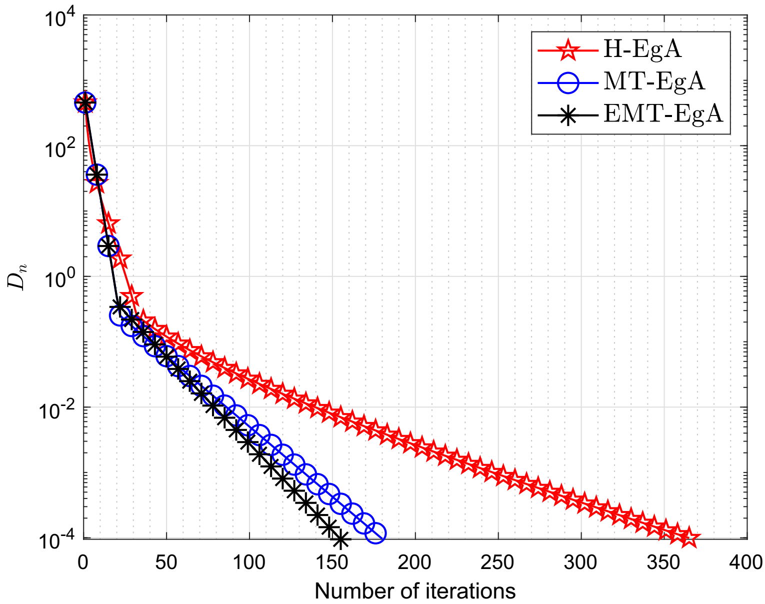

Fig. 7 Numerical inspection of Algorithm 1 with Algorithm 2 and Algorithm 3.2 in [14] when $x_0 = 2t^2$

Table 8 Numerical data for Figs. 7, 8, 9 and 10

x_0	Number of iterations			Execution time in seconds		
	H-EgA	MT-EgA	EMT-EgA	H-EgA	MT-EgA	EMT-EgA
$2t^2$	365	180	155	0.3897509000	0.1877632000	0.3815304000
$6t^3$	258	169	144	0.2712121000	0.1970006000	0.3883090000
$t^2 \cos(t)$	273	181	154	0.3081868000	0.1941321000	0.3813073000
$t^3 \sin(t)$	315	189	162	0.3325743000	0.2033284000	0.3821870000

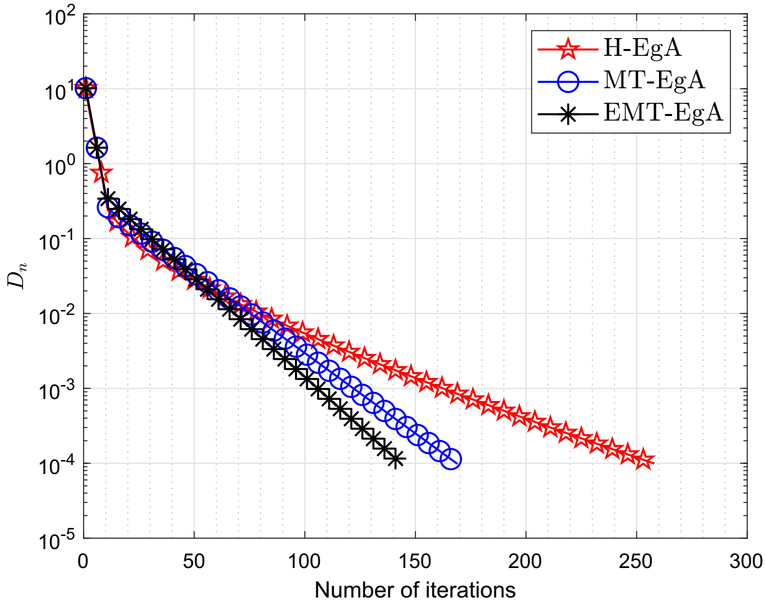

Fig. 8 Numerical inspection of Algorithm 1 with Algorithm 2 and Algorithm 3.2 in [14] when $x_0 = 6t^3$

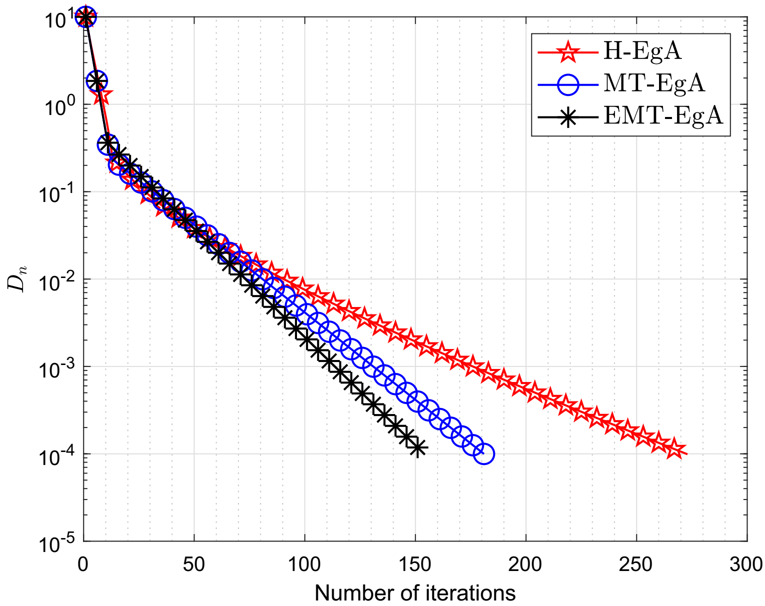

Fig. 9 Numerical inspection of Algorithm 1 with Algorithm 2 and Algorithm 3.2 in [14] when $x_0 = t^2 \cos(t)$

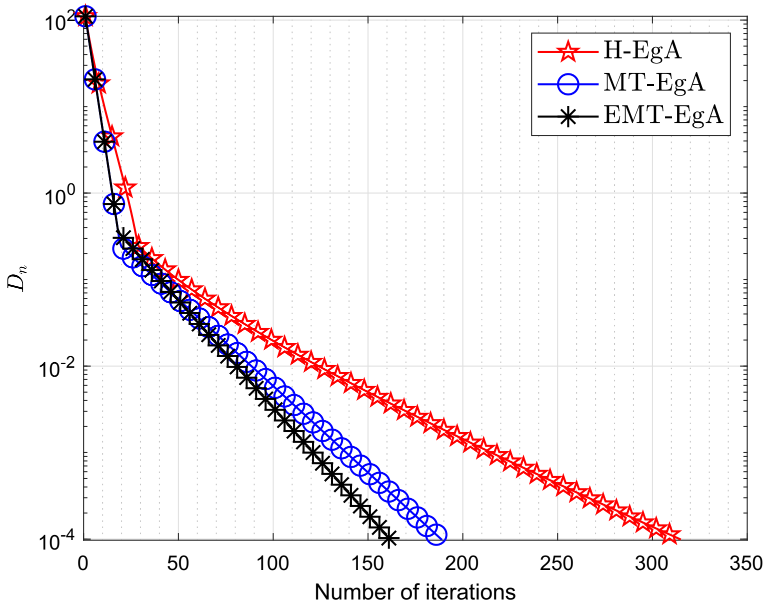

Fig. 10 Numerical inspection of Algorithm 1 with Algorithm 2 and Algorithm 3.2 in [14] when $x_0 = t^2 \sin(t)$

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conflict of interest The authors declare no competing interests.

References

1. Argyros, I.K.: Undergraduate Research at Cameron University on Iterative Procedures in Banach and Other Spaces. Nova Science Publisher, New York (2019)
2. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. CMS Books in Mathematics, 2nd edn. Springer International Publishing, New York (2017)
3. Bigi, G., Castellani, M., Pappalardo, M., Passacantando, M.: Existence and solution methods for equilibria. Eur. J. Oper. Res. **227**(1), 1–11 (2013)
4. Blum, E.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
5. Browder, F., Petryshyn, W.: Construction of fixed points of nonlinear mappings in Hilbert space. J. Math. Anal. Appl. **20**(2), 197–228 (1967)
6. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**(2), 318–335 (2010)
7. Dafermos, S.: Traffic equilibrium and variational inequalities. Transp. Sci. **14**(1), 42–54 (1980)
8. Dong, Q.L., Cho, Y.J., Zhong, L.L., Rassias, T.M.: Inertial projection and contraction algorithms for variational inequalities. J. Glob. Optim. **70**(3), 687–704 (2017)
9. Dong, Q.L., Huang, J.Z., Li, X.H., Cho, Y.J., Rassias, T.M.: MiKM: multi-step inertial Krasnosel'skiĭ–Mann algorithm and its applications. J. Glob. Optim. **73**(4), 801–824 (2018)

10. Fan, J., Liu, L., Qin, X.: A subgradient extragradient algorithm with inertial effects for solving strongly pseudomonotone variational inequalities. Optimization 1–17 (2019)
11. Fan, K.: A Minimax Inequality and Applications, Inequalities III (O. Shisha, ed.). Academic Press, New York (1972)
12. Ferris, M.C., Pang, J.S.: Engineering and economic applications of complementarity problems. SIAM Rev. **39**(4), 669–713 (1997)
13. Giannessi, F., Maugeri, A., Pardalos, P.M.: Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models, vol. 58. Springer Science & Business Media, Berlin (2006)
14. Hieu, D.V.: Halpern subgradient extragradient method extended to equilibrium problems. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales Serie A Matemáticas **111**(3), 823–840 (2016)
15. Hieu, D.V., Cho, Y.J., Bin Xiao, Y.: Modified extragradient algorithms for solving equilibrium problems. Optimization **67**(11), 2003–2029 (2018)
16. Hung, P.G., Muu, L.D.: The Tikhonov regularization extended to equilibrium problems involving pseudomonotone bifunctions. Nonlinear Anal. Theory Methods Appl. **74**(17), 6121–6129 (2011)
17. Konnov, I.: Application of the proximal point method to nonmonotone equilibrium problems. J. Optim. Theory Appl. **119**(2), 317–333 (2003)
18. Konnov, I.V.: On systems of variational inequalities. Russ Math C/C Izvestiia-Vysshie Uchebnye Zavedeniia Matematika **41**, 77–86 (1997)
19. Konnov, I.V.: Partial proximal point method for nonmonotone equilibrium problems. Optim. Methods Softw. **21**(3), 373–384 (2006)
20. Konnov, I.V.: Regularization method for nonmonotone equilibrium problems. J. Nonlinear Convex Anal. **10**(1), 93–101 (2009)
21. Korpelevich, G.: The extragradient method for finding saddle points and other problems. Matecon **12**, 747–756 (1976)
22. Kreyszig, E.: Introductory Functional Analysis with Applications. Wiley Classics Library, 1st edn. Wiley, New York (1989)
23. Maingé, P.-E.: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. **16**(7–8), 899–912 (2008)
24. Mann, W.R.: Mean value methods in iteration. Proc. Am. Math. Soc. **4**(3), 506 (1953)
25. Mastroeni, G.: On auxiliary principle for equilibrium problems. Nonconvex Optimization and its Applications, pp. 289–298. Springer US, New York (2003)
26. Moudafi, A.: Proximal point algorithm extended to equilibrium problems. J. Nat. Geom. **15**(1–2), 91–100 (1999)
27. Muu, L., Oettli, W.: Convergence of an adaptive penalty scheme for finding constrained equilibria. Nonlinear Anal. Theory Methods Appl. **18**(12), 1159–1166 (1992)
28. Oliveira, P., Santos, P., Silva, A.: A Tikhonov-type regularization for equilibrium problems in Hilbert spaces. J. Math. Anal. Appl. **401**(1), 336–342 (2013)
29. Regmi, S.: Optimized Iterative Methods with Applications in Diverse Disciplines. Nova Science Publisher, New York (2021)
30. Rehman, H.u., Kumam, P., Abubakar, A.B., Cho, Y.J.: The extragradient algorithm with inertial effects extended to equilibrium problems. Comput. Appl. Math. **39**(2), 1–26 (2020a)
31. Rehman, H.u., Kumam, P., Cho, Y.J., Yordsorn, P.: Weak convergence of explicit extragradient algorithms for solving equilibirum problems. J. Inequal. Appl. **2019**(1), 1–25 (2019)
32. Rehman, H.u., Kumam, P., Dong, Q.-L., Cho, Y.J.: A modified self-adaptive extragradient method for pseudomonotone equilibrium problem in a real Hilbert space with applications. Math. Methods Appl. Sci. **44**(5), 3527–3547 (2021)
33. Rehman, H.u., Kumam, P., Je Cho, Y., Suleiman, Y.I., Kumam, W.: Modified Popov's explicit iterative algorithms for solving pseudomonotone equilibrium problems. Optim. Methods Soft. **36**(1), 82–113 (2021)
34. Rehman, H.u., Pakkaranang, N., Kumam, P., Cho, Y.J.: Modified subgradient extragradient method for a family of pseudomonotone equilibrium problems in real a Hilbert space. J. Nonlinear Convex Anal. **21**(9), 2011–2025 (2020)
35. Stampacchia, G.: Formes bilinéaires coercitives sur les ensembles convexes. Comptes Rendus Hebdomadaires Des Seances De L Academie Des Sciences **258**(18), 4413 (1964)
36. Tiel, J.V.: Convex Analysis: an Introductory Text, 1st edn. Wiley, New York (1984)
37. Tran, D.Q., Dung, M.L., Nguyen, V.H.: Extragradient algorithms extended to equilibrium problems. Optimization **57**(6), 749–776 (2008)
38. Vinh, N.T., Muu, L.D.: Inertial extragradient algorithms for solving equilibrium problems. Acta Math. Vietnam. **44**(3), 639–663 (2019)

39. Wang, S., Zhang, Y., Ping, P., Cho, Y., Guo, H.: New extragradient methods with non-convex combination for pseudomonotone equilibrium problems with applications in Hilbert spaces. Filomat **33**(6), 1677–1693 (2019)
40. Xu, H.-K.: Another control condition in an iterative method for nonexpansive mappings. Bull. Aust. Math. Soc. **65**(1), 109–113 (2002)

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