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ORIGINAL PAPER

Integrals involving the Legendre Chi function

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Abstract
In this paper we investigate the representation of integrals involving the Legendre Chi function. We will show that in many cases these integrals take an explicit form involving the Riemann zeta function, the Dirichlet Eta function, Dirichlet lambda function and many other special functions. Some examples illustrating the theorems will be detailed.

Keywords Legendre Chi function · Polylogarithm function · Euler sums · Dirichlet lambda function · Zeta functions · Dirichlet beta functions

Mathematics Subject Classification 11M06 · 11M35 · 11Y60 · 33B15 · 41A58 · 65B10

1 Introduction preliminaries and notation

In this paper we investigate the representations of integrals of the type

$$\int_0^1 \chi_p(x)\ f(x)\ dx, \tag{1.1}$$

in terms of special functions such as Zeta functions, Dirichlet Eta functions, Polylogarithmic functions, Beta functions and others. In its most general form

$$f(x) = \left\{ x^a \chi_q(x) \ln^m(x),\ x^a \mathrm{Li}_q(\delta x^b) \right\}$$

where $\chi_q(x)$ is the Legendre-Chi function (LCF), $\mathrm{Li}_q(\delta x^b)$ is the polylogarithmic function, $a \in \mathbb{R} \geq -1, b \in \mathbb{R}^+,\ p \in \mathbb{N}, q \in \mathbb{N}, m \in \mathbb{N}$ and $\delta \in [-1, 1]$, for the set of natural numbers $\mathbb{N}$, the set of real numbers $\mathbb{R}$ and the set of positive real numbers, $\mathbb{R}^+$. We shall also investigate some representations of integrals involving the product of LCFs. The following notation and results will be useful in the subsequent sections of this paper. The generalized p-order harmonic numbers, $H_n^{(p)}(\alpha, \beta)$ are defined as the partial sums of the modified Hurwitz

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zeta function

$$\zeta(p,\alpha,\beta)=\sum_{n\geq 0}\frac{1}{(\alpha n+\beta)^p}.$$

The classical Hurwitz zeta function

$$\zeta(p,a)=\sum_{n\geq 0}\frac{1}{(n+a)^p}$$

for $\mathrm{Re}(p)>1$ and by analytic continuation to other values of $p\neq 1$, where any term of the form $(n+a)=0$ is excluded. Therefore

$$H_n^{(p)}(\alpha,\beta)=\sum_{j=1}^{n}\frac{1}{(\alpha j+\beta)^p}$$

and the "ordinary" p-order harmonic numbers $H_n^{(p)}=H_n^{(p)}(1,0)$. Many functions can be expanded through the generalized p-order harmonic numbers, such as the Dirichlet Beta cases

$$\beta(1)=\sum_{n\geq 0}\frac{1}{2^{n+1}\binom{n+\frac{1}{2}}{\frac{1}{2}}}=\sum_{n\geq 1}\frac{1}{2^n\binom{n-\frac{1}{2}}{\frac{1}{2}}}$$

and

$$\beta(2)=\sum_{n\geq 0}\frac{1+H_n(2,1)}{2^{n+1}\binom{n+\frac{1}{2}}{\frac{1}{2}}}=\sum_{n\geq 1}\frac{h_n}{2^n\binom{n-\frac{1}{2}}{\frac{1}{2}}},$$

here $\beta(2)$ is Catalan's constant and $h_n=H_{2n}-\frac{1}{2}H_n$. Two special cases of the Legendre-Chi function are

$$\chi_1(x)=\sum_{n\geq 0}\frac{x(-x^2)^n}{(1-x^2)^{n+1}\binom{n+\frac{1}{2}}{\frac{1}{2}}}$$

and

$$\chi_2(x)=\sum_{n\geq 0}\frac{1}{\binom{n+\frac{1}{2}}{\frac{1}{2}}}\left(1+\frac{x\left(-x^2\right)^n H_n(2,1)}{(1-x^2)^{n+1}}\right).$$

The Catalan constant

$$G=\beta(2)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\approx 0.91597$$

is a special case of the Dirichlet Beta function

$$\begin{aligned}\beta(z)&=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^z},\quad \text{for } Re(z)>0\\&=\frac{1}{(-2)^{2z}(z-1)!}\left(\psi^{(z-1)}\left(\frac{1}{4}\right)-\psi^{(z-1)}\left(\frac{3}{4}\right)\right),\end{aligned}\tag{1.2}$$

with functional equation

$$\beta(1-z)=\left(\frac{\pi}{2}\right)^{z}\sin\left(\frac{\pi z}{2}\right)\Gamma(z)\,\beta(z)$$

extending the Dirichlet Beta function to the left hand side of the complex plane $\mathrm{Re}(z)\leq 0$. The Lerch transcendent,

$$\Phi(z,t,a)=\sum_{m=0}^{\infty}\frac{z^m}{(m+a)^t}$$

is defined for $|z|<1$ and $\mathrm{Re}(a)>0$ and satisfies the recurrence:

$$\Phi(z,t,a)=z\,\Phi(z,t,a+1)+a^{-t}.$$

The Lerch transcendent generalizes the Hurwitz zeta function at $z=1$,

$$\Phi(1,t,a)=\zeta(t,a)=\sum_{m=0}^{\infty}\frac{1}{(m+a)^t} \tag{1.3}$$

and the polylogarithm, or de-Jonquière's function, when $a=1$,

$$\mathrm{Li}_t(z):=\sum_{m=1}^{\infty}\frac{z^m}{m^t},\;\; t\in\mathbb{C}\;\text{ when }\;|z|<1;\;\mathrm{Re}(t)>1\;\text{ when }\;|z|=1.$$

The polylogarithm of negative integer order arises in the sums of the form

$$\sum_{j\geq 1} j^n z^j=\mathrm{Li}_{-n}(z)=\frac{1}{(1-z)^{n+1}}\sum_{i=0}^{n-1}\left\langle {n \atop i}\right\rangle z^{n-i}$$

where the Eulerian number $\left\langle {n \atop i}\right\rangle=\sum_{j=0}^{i+1}(-1)^j\binom{n+1}{j}(i-j+1)^n$. The Legendre-Chi function is a special case of the Lerch transcendent

$$\chi_p(x)=2^{-p}\,x\,\Phi\left(x^2,p,\frac{1}{2}\right)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)^p}$$

and is related to the Polylogarithm by

$$\chi_p(x)=\frac{1}{2}\left(\mathrm{Li}_p(x)-\mathrm{Li}_p(-x)\right)=\mathrm{Li}_p(x)-2^{-p}\mathrm{Li}_p(x^2).$$

There are many special values of the LCF, from Lewin [14]

$$\chi_2(x)=\frac{1}{2}\int_0^x\ln\left(\frac{1+t}{1-t}\right)\frac{dt}{t}$$

and

$$\chi_2\left(\frac{1-x}{1+x}\right)+\chi_2(x)=\frac{3}{4}\zeta(2)+\frac{1}{2}\ln x\ln\left(\frac{1+x}{1-x}\right),$$

hence,

$$\chi_2\left(\sqrt{5}-2\right)=\frac{1}{4}\zeta(2)-\frac{3}{4}\ln^2(\phi)$$

where the golden ratio $\phi = \frac{1}{2}(1+\sqrt{5})$. The Dirichlet lambda function, $\lambda(p)$ is

$$\lambda(p) = 2\chi_p(1) = \zeta(p) + \eta(p) \tag{1.4}$$

where

$$\eta(p) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n^p} = (1-2^{1-p})\zeta(p)$$

is the alternating zeta function and $\eta(1) = \ln 2$. In the case of the summation of Harmonic numbers, we know that the famous Euler identity states, for $m \in \mathbb{N} \geq 2$,

$$EU(m) = \sum_{n=1}^{\infty} \frac{H_n}{n^m} = \frac{1}{2}(m+2)\zeta(m+1) - \frac{1}{2}\sum_{j=1}^{m-2} \zeta(m-j)\zeta(j+1), \tag{1.5}$$

for odd powers of the denominator, Georghiou and Philippou [13] established the identity:

$$\sum_{n=1}^{\infty} \frac{H_n}{n^{2m+1}} = \frac{1}{2}\sum_{r=2}^{2m} (-1)^r \zeta(r)\zeta(2m+2-r), \quad m \geq 1.$$

We know that for $n \geq 1$, $\psi(n+1) - \psi(1) = H_n$ with $\psi(1) = -\gamma$, where γ is the Euler Mascheroni constant and $\psi(n)$ is the digamma function. For real values of x, $\psi(x)$ is the digamma (or psi) function defined by

$$\psi(x) := \frac{d}{dz}\{\log\Gamma(x)\} = \frac{\Gamma'(x)}{\Gamma(x)},$$

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n}\right) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n-1}\right),$$

leading to the telescoping sum:

$$\psi(1+x) - \psi(x) = \sum_{n=1}^{\infty}\left(\frac{1}{x+n-1} - \frac{1}{x+n}\right) = \frac{1}{x}.$$

The polygamma function

$$\psi^{(k)}(z) = \frac{d^k}{dz^k}\{\psi(z)\} = (-1)^{k+1} k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}}$$

and has the recurrence

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k k!}{z^{k+1}}.$$

The connection of the polygamma function with harmonic numbers is,

$$H_z^{(\alpha+1)} = \zeta(\alpha+1) + \frac{(-1)^{\alpha}}{\alpha!}\psi^{(\alpha)}(z+1), \quad z \neq \{-1,-2,-3,\ldots\}. \tag{1.6}$$

and the multiplication formula is

$$\psi^{(k)}(pz) = \delta_{k,0}\ln p + \frac{1}{p^{k+1}}\sum_{j=0}^{p-1}\psi^{(k)}\left(z+\frac{j}{p}\right) \tag{1.7}$$

for p a positive integer and $\delta_{p,k}$ is the Kronecker delta. Since $H_n^{(p)}$ denotes harmonic numbers of order p, let

$$S(p,q) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ H_n^{(p)}}{n^q},$$

then in the case where p and q are both positive integers and $p+q$ is an odd integer, Flajolet and Salvy [11] gave the identity:

$$2S(p,q) = \left(1-(-1)^p\right)\zeta(p)\,\eta(q) + 2(-1)^p \sum_{i+2k=q} \binom{p+i-1}{p-1} \zeta(p+i)\ \eta(2k)$$
$$+\eta(p+q) - 2\sum_{j+2k=p} \binom{q+j-1}{q-1} (-1)^j\ \eta(q+j)\ \eta(2k), \tag{1.8}$$

where $\eta(0) = \frac{1}{2},\ \eta(1) = \ln 2,\ \zeta(1) = 0$, and $\zeta(0) = -\frac{1}{2}$ in accordance with the analytic continuation of the Riemann zeta function. In particular

$$2S(1,q) = (1+q)\,\eta(1+q) - \zeta(1+q) - 2\sum_{j=1}^{\frac{q}{2}-1} \eta(2j)\ \zeta(1+q-2j). \tag{1.9}$$

It is interesting to note that recently [1], established that for $p \in \mathbb{N}\setminus\{1\}$

$$S(p,1) = \frac{1}{2}p\zeta(p+1) - \frac{1}{2}\sum_{j=1}^{p} \eta(j)\ \eta(p-j+1).$$

We also know, from the work of [5] that for odd weight $(p+q)$ we have

$$BW(p.q) = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} = (-1)^p \sum_{j=1}^{\left[\frac{p}{2}\right]} \binom{p+q-2j-1}{p-1} \zeta(p+q-2j)\,\zeta(2j)$$
$$+\frac{1}{2}\left(1+(-1)^{p+1}\right)\zeta(p)\,\zeta(q)$$
$$+(-1)^p \sum_{j=1}^{\left[\frac{p}{2}\right]} \binom{p+q-2j-1}{q-1} \zeta(p+q-2j)\,\zeta(2j)$$
$$+\frac{\zeta(p+q)}{2}\left(1+(-1)^{p+1}\binom{p+q-1}{p}\right.$$
$$\left.+(-1)^{p+1}\binom{p+q-1}{q}\right), \tag{1.10}$$

where $[z]$ is the integer part of z. The next lemma relates the sum of the double argument of harmonic numbers in closed form and will be useful in the following section.

Lemma 1 *Let* $m \in \mathbb{N} \geq 2,\ p \in \mathbb{N}$ *then*

$$HE(p,m) = \sum_{n=1}^{\infty} \frac{H_{2n}^{(p)}}{(2n-1)^m} = (-1)^{m+1}\binom{p+m-2}{p-1}\ln 2$$
$$+\frac{1}{2}\left(BW(p,m) + S(p,m)\right) + \sum_{r=2}^{p} \frac{(-1)^m}{2^r}\binom{p+m-1-r}{p-r}\ \zeta(r)$$

$$+\sum_{k=2}^{m}\frac{(-1)^{m-k}}{2}\binom{p+m-1-k}{p-1}\lambda(k)$$

where $BW(p,m)$ is the Borwein identity (1.10), $S(p,m)$ is evaluated from (1.8) and $\lambda(j)$ is defined by (1.4).

Proof

$$\begin{aligned}HE(p,m)&=\sum_{n=1}^{\infty}\frac{H_{2n}^{(p)}}{(2n-1)^m}=\frac{1}{2}\left(\sum_{n=2}^{\infty}\frac{H_n^{(p)}}{(n-1)^m}-\sum_{n=2}^{\infty}\frac{(-1)^{n+1}H_n^{(p)}}{(n-1)^m}\right)\\&=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{\frac{1}{(n+1)^p}+H_n^{(p)}}{n^m}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{1}{(n+1)^p}+H_n^{(p)}\right)}{n^m}\right)\\&=\frac{1}{2}\left(BW(p,m)+S(p,m)\right)+\sum_{n=1}^{\infty}\frac{1}{(2n)^p(2n-1)^m}.\end{aligned}$$

Expanding in partial fraction form gives us

$$\begin{aligned}HE(p,m)&=\frac{1}{2}\left(BW(p,m)+S(p,m)\right)+\sum_{n=1}^{\infty}\frac{(-1)^{m+1}}{2n(2n-1)}\binom{p+m-2}{p-1}\\&\times\sum_{n=1}^{\infty}\left(\sum_{r=2}^{p}\frac{(-1)^m}{2^r n^r}\binom{p+m-1-r}{p-r}\right.\\&\left.+\sum_{k=2}^{m}\frac{(-1)^{m-k}}{(2n-1)^k}\binom{p+m-1-k}{p-1}\right)\end{aligned}$$

and the result follows. □

The following partial fraction decomposition holds.

Lemma 2 *For $y\in\mathbb{R}$, $m,p\in\mathbb{N}\setminus\{0\}$ we have*

$$\begin{aligned}\frac{1}{(2n-1)^p(2n+y)^{m+1}}&=\frac{(-1)^{p+1}}{(1+y)^{m+p-1}}\binom{m+p-1}{m}\frac{1}{(2n-1)(2n+y)}\\&+(-1)^p\sum_{r=2}^{m+1}\frac{\binom{m+p-r}{m+1-r}}{(1+y)^{m+p+1-r}\ (2n+y)^r}+(-1)^p\sum_{k=2}^{p}\frac{(-1)^k\binom{m+p-k}{m}}{(1+y)^{m+p+1-k}\ (2n-1)^k}.\end{aligned}$$

Proof Follows simply by expansion. □

The following lemma will also be useful in the evaluation of integrals of the type (1.1).

Lemma 3 *Let $m\in\mathbb{N}\geq 2$. then*

$$\begin{aligned}HM(m)=\sum_{n=1}^{\infty}\frac{H_n}{(2n-1)^m}&=\frac{1}{2}m\lambda(m+1)+(2(-1)^{m+1}-\lambda(m))\ln 2\\&+\sum_{j=2}^{m}(-1)^{m+j}\lambda(j)\end{aligned}$$

$$-\frac{1}{2(m-1)}\sum_{k=1}^{m-2}(m-k-1)\lambda(k+1)\lambda(m-k) \tag{1.11}$$

where $\lambda(m)$ *is given by* (1.4).

Proof We have, from [20, theorem 1], for x a real number, $x \neq -1, -2, -3, \ldots$

$$(-1)^m\,(m-1)!\sum_{n=1}^{\infty}\frac{H_n}{(n+x)^m} = (\psi(x)+\gamma)\,\psi^{(m-1)}(x) - \frac{1}{2}\psi^{(m)}(x) + \sum_{k=1}^{m-2}\binom{m-2}{k}\psi^{(m)}(x)\psi^{(m-k-1)}(x). \tag{1.12}$$

Choosing $x = \frac{1}{2}$, we have $\psi(\frac{1}{2}) + \gamma = -2\ln 2$ and from (1.6), $(-1)^m\,\psi^{(m)}(\frac{1}{2}) = -2^m m!\lambda\,(m+1)$, therefore substituting into the above equation and simplifying leads to

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^m} = \frac{1}{2}m\lambda\,(m+1) - \lambda\,(m)\ln 2 - \frac{1}{2\,(m-1)}\sum_{k=1}^{m-2}(m-k-1)\,\lambda\,(k+1)\,\lambda\,(m-k)\,. \tag{1.13}$$

Identity (1.13) corrects a minor error in the paper [15], now reordering the counter in (1.13) we obtain the identity (1.11). □

Since the Legendre-Chi function can be expressed as the difference of two polylogarithmic functions then we expect that integrals of the type (1.1) may be represented as Euler sums and therefore in terms of special functions such as the Riemann zeta function. A search of the current literature has not found many examples for the representation of the integral (1.1) and certainly not a systematic study of (1.1). Many papers, [10,12,16] examined some polylogarithmic integrals in terms of Euler sums. Some other important sources of information on Legendre-Chi functions are the works of [3,6–9] and the excellent books [14,23] and [24]. Other useful references related to the representation of Euler sums in terms of special functions include [1,2,17–21]. Some examples are highlighted, most of which are not amenable to a computer mathematical package.

2 Main results

Theorem 1 *Let* $a \in \mathbb{R} \geq -1,\ m \in \mathbb{N},\ p \in \mathbb{N}$, *the integral of the product of the Legendre-Chi and log functions,*

$$I\,(a,m,p) = \int_0^1 x^a\ \chi_p\,(x)\ ln^m\,(x)\,dx = \frac{(-1)^m\,m!}{2}\lambda\,(m+p+1)\,,\, for\; a=-1$$

$$= \frac{(-1)^m m!}{2} \left(\begin{array}{c} \frac{(-1)^{p+1}\binom{m+p-1}{m}}{(1+a)^{m+p}} \left(H_{\frac{a}{2}} + 2\ln 2\right) \\ + \sum_{k=2}^{p} \frac{(-1)^{p+k}\binom{m+p-k}{m}}{(1+a)^{m+p+1-k}} \lambda(k) \\ + \sum_{r=2}^{m+1} \frac{(-1)^{p}\binom{m+p-r}{m+1-r}}{2^{r-1}(1+a)^{m+p+1-r}} \zeta\left(r, 1+\frac{a}{2}\right) \end{array} \right), \quad \text{for } a \neq -1,$$

where the terms $EU(\cdot)$, $\lambda(\cdot)$ *are obtained from* (1.5) *and* (1.4) *respectively and* $\zeta(\cdot,\cdot)$ *is the Hurwitz zeta function,* (1.3).

Proof By the definition of the LCF function we have

$$I(a,m,p) = \int_0^1 x^a \chi_p(x) \ln^m(x)\, dx = \sum_{n\geq 1} \frac{1}{(2n-1)^p} \int_0^1 x^{2n+a-1} \ln^m(x)\, dx$$
$$= (-1)^m m! \sum_{n\geq 1} \frac{1}{(2n-1)^p (2n+a)^{m+1}}.$$

Considering the case $a=-1$, we have

$$I(-1,m,p) = (-1)^m m! \sum_{n\geq 1} \frac{1}{(2n-1)^{m+p+1}} = \frac{(-1)^m m!}{2} \lambda(m+p+1).$$

From Lemma 2, we have

$$I(a,m,p) = (-1)^m m! \sum_{n\geq 1} \frac{1}{(2n-1)^p (2n+a)^{m+1}}$$
$$= (-1)^m m! \sum_{n\geq 1} \left(\begin{array}{c} \frac{(-1)^{p+1}}{(1+a)^{m+p-1}} \binom{m+p-1}{m} \frac{1}{(2n-1)(2n+a)} \\ + (-1)^p \sum_{r=2}^{m+1} \frac{\binom{m+p-r}{m+1-r}}{(1+a)^{m+p+1-r}\,(2n+a)^r} \\ + (-1)^p \sum_{k=2}^{p} \frac{(-1)^k \binom{m+p-k}{m}}{(1+a)^{m+p+1-k}\,(2n-1)^k} \end{array} \right)$$

$$= \frac{(-1)^m\, m!}{2} \left(\begin{array}{c} \frac{(-1)^{p+1}\binom{m+p-1}{m}}{(1+a)^{m+p}} \left(H_{\frac{a}{2}} + 2\ln 2 \right) \\ + \sum_{k=2}^{p} \frac{(-1)^{p+k}\binom{m+p-k}{m}}{(1+a)^{m+p+1-k}} \lambda(k) \\ + \sum_{r=2}^{m+1} \frac{(-1)^{p}\binom{m+p-r}{m+1-r}}{2^{r-1}(1+a)^{m+p+1-r}} \zeta\left(r, 1+\frac{a}{2}\right) \end{array} \right), \quad \text{for } a \neq -1,$$

and Theorem 1 follows. □

Many special case of Theorem 1 can be examined separately and the next Corollary deals with a few cases.

Corollary 1 *Let the conditions of Theorem 1 hold then, for $m = 0$ and $a \neq -1$*

$$I(a, 0, p) = \int_0^1 x^a \, \chi_p(x) \, dx$$
$$= \frac{(-1)^{p+1}}{2(1+a)^p} \left(H_{\frac{a}{2}} + 2\ln 2 \right) + \sum_{k=2}^{p} \frac{(-1)^{p+k}}{2(1+a)^{p+1-k}} \lambda(k).$$

For $a = 0$,

$$\frac{(-1)^m}{m!} I(0, m, p) = \frac{(-1)^m}{m!} \int_0^1 \chi_p(x) \, ln^m(x)\, dx = (-1)^{p+1} \binom{m+p-1}{m} \ln 2$$
$$+ \sum_{k=2}^{p} \frac{(-1)^{p+k}\binom{m+p-k}{m}}{2} \lambda(k) + \sum_{r=2}^{m+1} \frac{(-1)^p \binom{m+p-r}{m+1-r}}{2^r} \zeta(r).$$

For $p = 0$,

$$I(a, m, 0) = \int_0^1 \frac{x^{a+1}}{1-x^2} \, ln^m(x)\, dx = \frac{(-1)^m\, m!}{2^{m+1}} \zeta\left(m+1, 1+\frac{a}{2}\right).$$

Some examples illustrating Theorem 1 follow.

Example 1 We offer the following examples.

$$I\left(-\frac{1}{2}, 2, 3\right) = \int_0^1 x^{-\frac{1}{2}} \, \chi_2(x) \, \ln^3(x)\, dx = 192G + 96\pi$$
$$+ 2\pi^3 - 216\zeta(2) - 192\ln 2 - 42\zeta(3),$$

where G is the Catalan constant. Let $a = 2\alpha$, where the silver ratio $\alpha = \frac{\sqrt{5}-1}{2}$, then

$$I(2\alpha, 2, 2) = \frac{\sqrt{5}}{10}\zeta(2) + \frac{1}{20}\zeta(3) - \frac{6}{25}\ln 2$$

$$
\begin{aligned}
& -\frac{3}{25} H_\alpha - \frac{\sqrt{5}}{25} H_\alpha^{(2)} - \frac{1}{20} H_\alpha^{(3)}. \\
I(4,4,2) &= \frac{3}{100}\zeta(5) + \frac{3}{125}\zeta(4) + \frac{9}{6625}\zeta(3) \\
& + \frac{42}{3225}\zeta(2) - \frac{24}{3125}\ln 2 - \frac{35199}{400000}. \\
I\left(-\frac{1}{3},2,3\right) &= \frac{729\sqrt{3}}{16}\pi - \frac{729}{32}\zeta(2) + \frac{81\sqrt{3}}{8}\pi^3 - \frac{2187}{16}\ln 3 \\
& - \frac{729}{32}\psi\left(\frac{5}{6}\right) - \frac{9}{64}\psi'''\left(\frac{5}{6}\right) - \frac{7371}{16}\zeta(3). \\
I\left(\frac{1}{2},3,3\right) &= \frac{256}{27}G - \frac{20992}{243} + \frac{320}{243}\pi + \frac{256}{27}\zeta(2) \\
& + \frac{4}{9}\pi^3 + \frac{40}{3}\zeta(4) + \frac{640}{243}\ln 2 + \frac{128}{9}\beta(4) + \frac{308}{27}\zeta(3). \\
I\left(\frac{1}{4},2,3\right) &= \frac{98304}{3125} - \frac{6144\sqrt{3+2\sqrt{2}}}{3125}\pi - \frac{1152}{625}\zeta(2) \\
& + \frac{12288\sqrt{2}}{3125}\ln\left(2-\sqrt{2}\right) - \frac{6144\left(6+\sqrt{2}\right)}{3125}\ln 2 \\
& - \frac{1152}{625}\psi\left(\frac{9}{8}\right) + \frac{48}{125}\psi''\left(\frac{9}{8}\right) - \frac{1}{25}\psi'''\left(\frac{9}{8}\right) \\
I\left(\frac{3}{2},5,4\right) &= \frac{86016}{15625}G + \frac{517390336}{94921875} + \frac{172032}{390625}\pi - \frac{419328}{78125}\zeta(2) \\
& + \frac{1536}{3125}\pi^3 - \frac{23328}{625}\zeta(4) + \frac{32}{125}\pi^5 - \frac{12096}{125}\zeta(6) - \frac{47616}{625}\zeta(5) \\
& + \frac{24576}{625}\beta(4) + \frac{12288}{125}\beta(6) - \frac{344064}{390625}\ln 2 - \frac{198912}{15625}\zeta(3).
\end{aligned}
$$

where $\beta(4)$ and $\beta(6)$ are the Dirichlet Beta functions (1.2). Let $a = 2\phi$, where the golden ratio $\phi = \frac{\sqrt{5}+1}{2}$, then

$$
\begin{aligned}
I(2\phi,2,3) &= \frac{12}{(1+2\phi)^5}\ln 2 - \frac{6}{(1+2\phi)^4}\zeta(2) + \frac{3}{2(1+2\phi)^3}\zeta(3) \\
& + \frac{6}{(1+2\phi)^5}H_\phi + \frac{3}{2(1+2\phi)^4}H_\phi^{(2)} + \frac{1}{4(1+2\phi)^3}H_\phi^{(3)}.
\end{aligned}
$$

The next theorem investigates the integral of the product of the Legendre Chi and polylogarithmic functions.

Theorem 2 *Let* $a \in \mathbb{R} \geq -1,\ b \in \mathbb{R}^+,\ p \in \mathbb{N},\ q \in \mathbb{N}$ *and* $\delta \in [-1,1)$, *then*

$$
\begin{aligned}
J(\delta,a,b,p,q) &= \int_0^1 x^a\ \chi_p(x)\ Li_q\left(\delta x^b\right) dx \\
&= \sum_{n\geq 1} \frac{(-1)^q\, b^{q-1}\ln(1-\delta)}{(2n-1)^p\,(2n+a)^q} + \delta(-1)^q\, b^{q-1} \sum_{n\geq 1} \frac{\Phi\left(\delta,1,\frac{2n+a}{b}+1\right)}{(2n-1)^p\,(2n+a)^q}
\end{aligned}
$$

$$+\sum_{r=2}^{q}\sum_{n\geq 1}\frac{(-1)^{q+r}\, b^{q-r}\, Li_r(\delta)}{(2n-1)^p\,(2n+a)^{q+1-r}}$$

where $Li_r(\delta)$ is the polylogarithmic function and $\Phi(\cdot,\cdot,\cdot)$ is the Lerch transcendent.

Proof By the definition of the Legendre Chi function and the polylogarithmic function we can write

$$\begin{aligned}
J(\delta,a,b,p,q) &= \int_0^1 x^a\,\chi_p(x)\,\mathrm{Li}_q\left(\delta x^b\right)dx \\
&= \int_0^1 \sum_{n\geq 1}\frac{x^{a+2n-1}}{(2n-1)^p}\sum_{j\geq 1}\frac{\delta^j x^{bj}}{j^q}dx = \sum_{n\geq 1}\frac{1}{(2n-1)^p}\sum_{j\geq 1}\frac{\delta^j}{j^q\,(2n+bj+a)} \\
&= \sum_{n\geq 1}\frac{1}{(2n-1)^p}\sum_{j\geq 1}\delta^j\left(\frac{(-b)^{q-1}}{j\,(2n+bj+a)\,(2n+a)^{q-1}}+\sum_{r=2}^{q}\frac{(-b)^{q-r}}{j^r\,(2n+a)^{q+1-r}}\right) \\
&= \sum_{n\geq 1}\frac{1}{(2n-1)^p}\left(\begin{array}{c}\frac{(-b)^{q-1}}{(2n+a)^q}\left(-\ln(1-\delta)-\delta\Phi\left(\delta,1,\frac{2n+a}{b}+1\right)\right) \\ +\sum_{r=2}^{q}\frac{(-b)^{q-r}}{j^r\,(2n+a)^{q+1-r}}\end{array}\right) \\
&= \sum_{n\geq 1}\frac{(-1)^q\, b^{q-1}\ln(1-\delta)}{(2n-1)^p\,(2n+a)^q}+\delta\,(-1)^q\, b^{q-1}\sum_{n\geq 1}\frac{\Phi\left(\delta,1,\frac{2n+a}{b}+1\right)}{(2n-1)^p\,(2n+a)^q} \\
&\quad+\sum_{r=2}^{q}\sum_{n\geq 1}\frac{(-1)^{q+r}\, b^{q-r}\,\mathrm{Li}_r(\delta)}{(2n-1)^p\,(2n+a)^{q+1-r}}
\end{aligned}$$

and the proof of the theorem is finished. □

The next corollary deals with the three particular cases for the values of $\delta=\pm 1$ and $a=-1$.

Corollary 2 *Let $a\in\mathbb{R}\geq -1,\ b\in\mathbb{R}^+,\ p\in\mathbb{N},\ q\in\mathbb{N}$ and $\delta=1$, then*

$$\begin{aligned}
J(1,a,b,p,q) &= \int_0^1 x^a\,\chi_p(x)\,Li_q\left(x^b\right)dx \\
&= \sum_{n\geq 1}\frac{(-b)^{q-1}\,H_{\frac{2n+a}{b}}}{(2n-1)^p\,(2n+a)^q}+\sum_{r=2}^{q}\sum_{n\geq 1}\frac{(-b)^{q-r}\;\zeta(r)}{(2n-1)^p\,(2n+a)^{q+1-r}},
\end{aligned}$$

where $H_{\frac{2n+a}{b}}$ are the Harmonic numbers and $\zeta(r)$ is the Riemann zeta function.

Let $a\in\mathbb{R}\geq -1,\ b\in\mathbb{R}^+,\ p\in\mathbb{N},\ q\in\mathbb{N}$ and $\delta=-1$, then

$$\begin{aligned}
J(-1,a,b,p,q) &= \int_0^1 x^a\,\chi_p(x)\,\mathrm{Li}_q\left(-x^b\right)dx \\
&= \sum_{n\geq 1}\frac{(-b)^{q-1}\left(H_{\frac{2n+a}{2b}}-H_{\frac{2n+a}{b}}\right)}{(2n-1)^p\,(2n+a)^q}-\sum_{r=2}^{q}\sum_{n\geq 1}\frac{(-b)^{q-r}\;\eta(r)}{(2n-1)^p\,(2n+a)^{q+1-r}},
\end{aligned}$$

where $\eta(r)$ is the Dirichlet Eta function, (or the alternating zeta function).

Let $a=-1,\ b\in\mathbb{R}^{+},\ p\in\mathbb{N},\ q\in\mathbb{N}$ and $\delta\in[-1,1)$, then

$$J(\delta,-1,b,p,q)=\int_{0}^{1}\frac{1}{x}\chi_{p}(x)\ \mathrm{Li}_{q}\left(\delta x^{b}\right)dx=\frac{(-1)^{q}b^{q-1}\ln(1-\delta)}{2}\lambda(p+q)$$
$$+\sum_{r=2}^{q}\frac{(-b)^{q-r}\ \mathrm{Li}_{r}(\delta)}{2}\lambda(p+q+1-r)-\sum_{n\geq1}\frac{\delta(-b)^{q-1}\Phi\left(\delta,1,\frac{2n-1}{b}+1\right)}{(2n-1)^{p+q}},$$

where $\lambda(\cdot)=\zeta(\cdot)+\eta(\cdot)$ is the Dirichlet lambda function.

Proof For $a\in\mathbb{R}\geq-1,\ b\in\mathbb{R}^{+},\ p\in\mathbb{N},\ q\in\mathbb{N}$ and $\delta=1$, then

$$\begin{aligned}J(1,a,b,p,q)&=\int_{0}^{1}x^{a}\ \chi_{p}(x)\ \mathrm{Li}_{q}\left(x^{b}\right)dx\\&=\sum_{n\geq1}\frac{1}{(2n-1)^{p}}\sum_{j\geq1}\left(\frac{(-b)^{q-1}}{j(2n+bj+a)(2n+a)^{q-1}}+\sum_{r=2}^{q}\frac{(-b)^{q-r}}{j^{r}(2n+a)^{q+1-r}}\right)\\&=\sum_{n\geq1}\frac{(-b)^{q-1}H_{\frac{2n+a}{b}}}{(2n-1)^{p}(2n+a)^{q}}+\sum_{r=2}^{q}\sum_{n\geq1}\frac{(-b)^{q-r}\ \zeta(r)}{(2n-1)^{p}(2n+a)^{q+1-r}}.\end{aligned}$$

For $a\in\mathbb{R}\geq-1,\ b\in\mathbb{R}^{+},\ p\in\mathbb{N},\ q\in\mathbb{N}$ and $\delta=-1$, then

$$\begin{aligned}J(-1,a,b,p,q)&=\int_{0}^{1}x^{a}\ \chi_{p}(x)\ \mathrm{Li}_{q}\left(-x^{b}\right)dx\\&=\sum_{n\geq1}\frac{1}{(2n-1)^{p}}\sum_{j\geq1}\left(\frac{(-1)^{j}(-b)^{q-1}}{j(2n+bj+a)(2n+a)^{q-1}}+\sum_{r=2}^{q}\frac{(-1)^{j}(-b)^{q-r}}{j^{r}(2n+a)^{q+1-r}}\right)\\&=\sum_{n\geq1}\frac{(-b)^{q-1}\left(\frac{1}{2}H_{\frac{2n+a}{2b}}-\frac{1}{2}H_{\frac{2n+a}{2b}-\frac{1}{2}}-\ln2\right)}{(2n-1)^{p}(2n+a)^{q}}-\sum_{r=2}^{q}\sum_{n\geq1}\frac{(-b)^{q-r}\ \eta(r)}{(2n-1)^{p}(2n+a)^{q+1-r}},\end{aligned}$$

from the multiple argument of the polygamma relation, (1.7), we have that

$$H_{2w}=\ln2+\frac{1}{2}H_{w}+\frac{1}{2}H_{w-\frac{1}{2}},\ \text{where } w=\frac{2n+a}{2b}$$

hence

$$J(-1,a,b,p,q)=\sum_{n\geq1}\frac{(-b)^{q-1}\left(H_{\frac{2n+a}{2b}}-H_{\frac{2n+a}{b}}\right)}{(2n-1)^{p}(2n+a)^{q}}-\sum_{r=2}^{q}\sum_{n\geq1}\frac{(-b)^{q-r}\ \eta(r)}{(2n-1)^{p}(2n+a)^{q+1-r}},$$

here we can use the partial fraction decomposition of Lemma 2 to reduce the summations for the counter n. For $a=-1,\ b\in\mathbb{R}^{+},\ p\in\mathbb{N},\ q\in\mathbb{N}$ and $\delta\in[-1,1)$, then

$$J(\delta,-1,b,p,q)=\int_{0}^{1}\frac{1}{x}\chi_{p}(x)\ \mathrm{Li}_{q}\left(\delta x^{b}\right)dx$$

$$= \sum_{n\geq 1} \frac{(-1)^q\, b^{q-1} \ln(1-\delta)}{(2n-1)^{p+q}} + \sum_{n\geq 1} \frac{\delta\,(-1)^q\, b^{q-1} \Phi\left(\delta, 1, \frac{2n-1}{b}+1\right)}{(2n-1)^{p+q}}$$
$$+ \sum_{r=2}^{q} \sum_{n\geq 1} \frac{(-1)^{q+r}\, b^{b-r}\, \mathrm{Li}_r(\delta)}{(2n-1)^{p+q+1-r}} \frac{(-1)^q\, b^{q-1} \ln(1-\delta)}{2} \lambda(p+q)$$
$$- \sum_{n\geq 1} \frac{\delta\,(-b)^{q-1}\, \Phi\left(\delta, 1, \frac{2n-1}{b}+1\right)}{(2n-1)^{p+q}}$$
$$+ \sum_{r=2}^{q} \frac{(-b)^{q-r}\, \mathrm{Li}_r(\delta)}{2} \lambda(p+q+1-r).$$

□

Some examples illustrating Theorem 2 follow.

Example 2 Consider

$$J(1,0,2,p,q) = \int_0^1 \chi_p(x)\, \mathrm{Li}_q\left(x^2\right) dx$$
$$= \sum_{n\geq 1} \frac{(-1)^{q-1} H_n}{2\,(2n-1)^p\, n^q} + \sum_{r=2}^{q} \sum_{n\geq 1} \frac{(-1)^{q-r}\, \zeta(r)}{2\,(2n-1)^p\, n^{q+1-r}}$$

and can be readily evaluated for particular values of p and q.

$$J(1,0,2,3,2) = \frac{7}{8}\zeta(2)\zeta(3) + \frac{7}{2}\zeta(3)\ln 2 + \frac{9}{2}\zeta(3) - 5\zeta(2)\ln 2 - 24\ln 2$$
$$+6\ln^2 2 + 9\zeta(2) - \frac{75}{16}\zeta(4).$$
$$J(-1,0,2,p,q) = \int_0^1 \chi_p(x)\, \mathrm{Li}_q\left(-x^2\right) dx$$
$$= \sum_{n\geq 1} \frac{1}{2\,(2n-1)^p} \left(\frac{(-1)^{q-1}\left(H_{\frac{n}{2}} - H_n\right)}{n^q} - \sum_{r=2}^{q} \frac{(-1)^{q-r}\, \eta(r)}{n^{q+1-r}} \right).$$

Here we require the Euler sums of harmonic numbers at half integer values which we can obtain from [22]. For $p=1, q=4$ we require

$$\sum_{n\geq 1} \frac{H_{\frac{n}{2}}}{n\,(2n-1)} = 8\ln 2 - \frac{7}{2}\ln^2 2 - \frac{1}{2}\zeta(2) - 2G$$

so that

$$J(-1,0,2,1,4) = \int_0^1 \chi_1(x)\, \mathrm{Li}_4\left(-x^2\right) dx = 8G + 2\zeta(2) + L(3) - 16\ln 2 + 6\ln^2 2$$
$$+\frac{1}{4}\zeta(2)\zeta(3) + \frac{3}{2}\zeta(3)\ln 2 - 2\zeta(2)\ln 2 - \frac{7}{8}\zeta(4)\ln 2 - \frac{5}{4}\zeta(3) - \frac{59}{64}\zeta(5),$$

where

$$L(3) = \frac{11}{4}\zeta(4) - \frac{7}{4}\zeta(3)\ln 2 + \frac{1}{2}\zeta(2)\ln^2 2 - \frac{1}{12}\ln^4 2 - 2\mathrm{Li}_4\left(\frac{1}{2}\right). \tag{2.1}$$

$$\begin{aligned} J(1,-1,1,p,q) &= \int_0^1 \frac{1}{x}\chi_p(x)\ Li_q(x)\,dx \\ &= \sum_{n\geq 1}\frac{(-1)^{q-1}H_{2n-1}}{(2n-1)^{p+q}} + \sum_{r=2}^{q}\sum_{n\geq 1}\frac{(-1)^{q-r}\ \zeta(r)}{(2n-1)^{p+q+1-r}}. \end{aligned}$$

Here we require the Euler sums of the type $\sum_{n\geq 1}\frac{H_{2n}}{(2n-1)^m}$ which may be evaluated using the results from [15]. In particular

$$\begin{aligned} J(1,-1,1,p,q) = {} & \frac{(-1)^{q-1}}{2}\left(EU(p+q) + S(1,p+q)\right) \\ & + \sum_{r=2}^{q}(-1)^{q-r}\ \zeta(r)\lambda(p+q+1-r). \end{aligned}$$

and

$$J(1,-1,1,3,3) = \frac{889}{256}\zeta(7) - \frac{55}{32}\zeta(5)\zeta(2).$$

Consider,

$$\begin{aligned} J(-1,-1,1,p,q) &= \int_0^1 \frac{1}{x}\chi_p(x)\ Li_q(-x)\,dx \\ &= \sum_{n\geq 1}\frac{(-1)^{q-1}}{(2n-1)^{p+q}}\left(H_{2n} + \frac{1}{2n} - H_n - 2\ln 2\right) \\ &\quad - \sum_{r=2}^{q}\frac{(-1)^{q-r}\ \eta(r)}{2}\lambda(p+q+1-r), \end{aligned}$$

and

$$J(-1,-1,1,3,3) = \frac{19}{16}\zeta(5)\zeta(2) - \frac{635}{256}\zeta(7).$$

Some other particular noteworthy examples are

$$J(1,-1,4,1,1) = \int_0^1 \frac{1}{x}\chi_1(x)\ Li_1\left(x^4\right)dx = \sum_{n\geq 1}\frac{H_{\frac{n}{2}-\frac{1}{4}}}{(2n-1)^2} = \frac{7}{4}\zeta(3) - \frac{1}{2}\pi G.$$

$$\begin{aligned} J(1,4,4,1,2) &= \int_0^1 x^4\chi_1(x)\ Li_2\left(x^4\right)dx = \sum_{n\geq 1}\frac{\zeta(2)}{2(2n-1)(n+2)} \\ &= -\sum_{n\geq 1}\frac{2}{(2n-1)(n+2)^3} - \sum_{n\geq 1}\frac{H_{\frac{n}{2}}}{(2n-1)(n+2)^2} \end{aligned}$$

$$= \frac{4}{25}G + \frac{19}{100}\zeta(2) - \frac{26}{125}\ln 2 - \frac{373}{500} + \frac{1}{5}\zeta(2)\ln 2 + \frac{11}{40}\zeta(3) + \frac{7}{25}\ln^2 2,$$

here we have used, from [22]

$$\sum_{n\geq 1}\frac{H_{\frac{n}{2}}}{(2n-1)(n+2)^2} = \frac{1}{8}\zeta(3) - \frac{4}{25}G + \frac{3}{25}\zeta(2) + \frac{2}{125}\ln 2 - \frac{7}{25}\ln^2 2.$$

$$\begin{aligned} J\left(\frac{1}{2},0,1,1,2\right) &= \int_0^1 \chi_1(x)\ Li_2\left(\frac{1}{2}x\right)dx = \sum_{n\geq 1}\frac{\ln 2}{2^n n^2(n+1)} \\ &\quad + \sum_{n\geq 1}\frac{H_{\frac{n}{2}}}{2^{n+1}n^2(n+1)}, \\ &= \frac{7}{4}\zeta(2) - \ln 2 + \frac{1}{4}\zeta(2)\ln 2 - \ln^2 2 \\ &\quad -\frac{1}{6}\ln^3 2 + 3\ln 2\ln 3 - \frac{3}{2}\ln^2 3 + \frac{1}{8}\zeta(3) \\ &\quad +\frac{1}{2}Li_2\left(\frac{1}{4}\right)\ln 2 - 3Li_2\left(\frac{2}{3}\right) + \frac{1}{2}Li_3\left(\frac{1}{4}\right), \end{aligned}$$

here we have used the new Euler sum identity

$$\begin{aligned} \sum_{n\geq 1}\frac{H_{\frac{n}{2}}}{2^{n+1}n^2(n+1)} &= \frac{7}{4}\zeta(2) - \frac{1}{4}\zeta(2)\ln 2 - 2\ln^2 2 + \frac{1}{3}\ln^3 2 + 2\ln 2\ln 3 - \frac{3}{2}\ln^2 3 \\ &\quad +\frac{1}{8}\zeta(3) + \frac{1}{2}Li_2\left(\frac{1}{4}\right)\ln 2 - 3Li_2\left(\frac{2}{3}\right) + \frac{1}{2}Li_3\left(\frac{1}{4}\right). \end{aligned}$$

$$\begin{aligned} J(1,-1,2,4,4) &= \int_0^1 \frac{1}{x}\chi_4(x)\ Li_4\left(x^2\right)dx \\ &= \frac{23}{16}\zeta(5)\zeta(4) + \frac{65}{16}\zeta(7)\zeta(2) - \frac{511}{64}\zeta(9). \end{aligned}$$

$$\begin{aligned} J\left(-\frac{1}{4},1,1,1,1\right) &= \int_0^1 x\chi_1(x)\ Li_1\left(-\frac{1}{4}x\right)dx \\ &= \sum_{j\geq 1}\frac{\left(-\frac{1}{4}\right)^j}{j}\sum_{n\geq 1}\frac{1}{(2n-1)(2n+j+1)} \\ &= \sum_{j\geq 1}\frac{(-1)^j}{2^{2j+1}j(j+2)}\left(\frac{2}{j+1} + 2H_j - H_{\frac{j}{2}}\right) \\ &= \frac{3}{4} - \frac{1}{8}\zeta(2) - \frac{1}{8}\ln^2\left(\frac{5}{4}\right) \\ &\quad +\frac{1}{2}\ln^2 2 + \frac{1}{2}\ln^2 5 + 3\ln 2 - 4\ln 3\ln 2 - \frac{5}{2}\ln 5 \end{aligned}$$

$$+\ln 5 \ln 2 + 4Li_2\left(-\frac{2}{3}\right) - \frac{1}{4}Li_2\left(-\frac{3}{5}\right)$$
$$-4Li_2\left(-\frac{1}{3}\right) + 4Li_2\left(\frac{1}{5}\right).$$

Corollary 3 *Form Corollary 2 we may extract the following Euler like identity*

$$\sum_{n\geq 1}\frac{H_n}{n^p(4n+1)^q} = (-4)^{p-1}\binom{p+q-2}{p-1}\left(\frac{3\pi}{2}\ln 2 + \frac{9}{2}\ln^2 2 - \frac{3}{4}\zeta(2) - 4G\right)$$
$$+\sum_{r=2}^{q}(-4)^p\binom{p+q-1-r}{q-r}\sum_{n\geq 1}\frac{H_n}{(4n+1)^r}$$
$$+\sum_{k=2}^{p}(-4)^{p-k}\binom{p+q-1-k}{p-k}EU(k),$$

where $EU(\cdot)$ is Euler's identity and G is the Catalan constant.

Proof From the partial fraction decomposition of $\frac{1}{n^p(4n+1)^q}$ we notice that

$$\sum_{n\geq 1}H_n\left(\begin{array}{c}\frac{(-4)^{p-1}\binom{p+q-2}{p-1}}{n(4n+1)} + \sum_{r=2}^{q}\frac{(-4)^p\binom{p+q-1-r}{q-r}}{(4n+1)^r}\\ +\sum_{k=2}^{p}\frac{(-4)^{p-k}\binom{p+q-1-k}{p-k}}{n^k}\end{array}\right).$$

We can evaluate

$$\sum_{n\geq 1}\frac{H_n}{n(4n+1)} = \frac{3\pi}{2}\ln 2 + \frac{9}{2}\ln^2 2 - \frac{3}{4}\zeta(2) - 4G.$$

and hence the identity follows. The terms of the form $\sum_{n\geq 1}\frac{H_n}{(4n+1)^r}$, for $r \geq 2$ can be evaluated from (1.12) by choosing $x = \frac{1}{4}$. □

Example 3 Two examples of Corollary 3 are the following:

$$\sum_{n\geq 1}\frac{H_n}{n^3(4n+1)^2} = 16\pi G - 192G - 36\zeta(2) - 2\pi^3 + \frac{5}{4}\zeta(4) + 96G\ln 2$$
$$+72\pi\ln 2 + 72\zeta(2)\ln 2 + 216\ln^2 2 - 128\zeta(3).$$
$$\sum_{n\geq 1}\frac{H_n}{n^2(4n+1)^3} = 48G - 8G^2 - 8\pi G - 12G\zeta(2) + 9\zeta(2) + \pi^3 + \frac{45}{2}\zeta(4)$$
$$+58\zeta(3) - 48G\ln 2 - 18\pi\ln 2 - 36\zeta(2)\ln 2 - \frac{3}{4}\pi^3\ln 2$$
$$+48\beta(4) - 54\ln^2 2 - \frac{7}{2}\pi\zeta(3) - 21\zeta(3)\ln 2.$$

In the next section we briefly investigate integrals containing products of Legendre Chi functions.

3 Integrals containing products of Legendre Chi functions

The following theorem holds

Theorem 3 *Let* $a \in \mathbb{R} \geq -1,\ b \in \mathbb{R}^{+},\ \ p \in \mathbb{N}$, *and* $q \in \mathbb{N}$, *then*

$$K(a,b,p,q) = \int_0^1 x^a\ \chi_p(x)\ \chi_q\left(x^b\right) dx = (-1)^{q+1} \sum_{n\geq 1} \frac{h_n(a,b)}{(2n-1)^p (2n+a)^q} + \sum_{k=2}^{q} \frac{(-b)^{q-k}\lambda(k)}{2} \sum_{n\geq 1} \frac{1}{(2n-1)^p (2n+a)^{q+1-k}} \tag{3.1}$$

where $h_n(a,b) = H_{\frac{2n+a}{b}} - \frac{1}{2} H_{\frac{2n+a}{2b}}$ *are the Harmonic numbers and* $\lambda(k)$ *is given by* (1.4).

Proof From the definition of the Legendre Chi function

$$\int_0^1 x^a\ \chi_p(x)\ \chi_q\left(x^b\right) dx = \sum_{n\geq 1} \frac{1}{(2n-1)^p} \sum_{j\geq 1} \frac{1}{(2j-1)^q (2n+2bj-b+a)}$$

$$= \sum_{n\geq 1} \frac{1}{(2n-1)^p} \sum_{j\geq 1} \left(\begin{array}{c} \frac{(-1)^{q+1}}{(2j-1)(2n+a)^{q-1}(2n+2bj-b+a)} \\ -\sum_{k=2}^{q} \frac{(-b)^{q-k}}{(2j-1)^k (2n+a)^{q+1-k}} \end{array} \right)$$

$$= \sum_{n\geq 1} \frac{1}{(2n-1)^p} \left(\frac{(-1)^{q+1}\left(\ln 2 + \frac{1}{2} H_{\frac{2n+a}{2b} - \frac{1}{2}}\right)}{(2n+a)^q} - \sum_{k=2}^{q} \frac{(-b)^{q-k}\lambda(k)}{2(2n+a)^{q+1-k}} \right).$$

Now we apply the identity (1.7) we obtain,

$$K(a,b,p,q) = (-1)^{q+1} \sum_{n\geq 1} \frac{1}{(2n-1)^p (2n+a)^q} \left(H_{\frac{2n+a}{b}} - \frac{1}{2} H_{\frac{2n+a}{2b}} \right) + \sum_{k=2}^{q} \frac{(-b)^{q-k}\lambda(k)}{2} \sum_{n\geq 1} \frac{1}{(2n-1)^p (2n+a)^{q+1-k}}$$

and the theorem is finished. Summation over n can be performed and explicit results obtained for particular parameter values. □

The next few corollaries deals with particular cases of the parameters.

Corollary 4 *Let* $a = -1,\ b = 1,\ \ p \in \mathbb{N},\ \ q \in \mathbb{N}$, *then*

$$K(-1,1,p,q) = \int_0^1 x^{-1}\ \chi_p(x)\ \chi_q(x)\, dx = \frac{(-1)^{q+1}}{2} HM(p+q) + \frac{(-1)^{q+1}}{2}(2+\lambda(p+q))\ln 2 - (-1)^{p+1} \sum_{k=2}^{p+q} \frac{(-1)^k \lambda(k)}{2} + \sum_{k=2}^{q} \frac{(-1)^{q-k}\lambda(k)\lambda(p+q+1-k)}{4} \tag{3.2}$$

where $HM(\cdot)$ *is given by* (1.11) *and* $\lambda(k)$ *by* (1.4).

Proof From (3.1)

$$K(-1,1,p,q)=\int_0^1 x^{-1}\,\chi_p(x)\,\chi_q(x)\,dx=(-1)^{q+1}\sum_{n\ge1}\frac{H_{2n}-\frac{1}{2n}-\frac{1}{2}H_{n-\frac{1}{2}}}{(2n-1)^{p+q}}$$
$$+\sum_{k=2}^{q}\frac{(-1)^{q-k}\lambda(k)}{2}\sum_{n\ge1}\frac{1}{(2n-1)^{p+q+1-k}}.$$

Utilizing the multiplication identity (1.7) we have

$$K(-1,1,p,q)=(-1)^{q+1}\sum_{n\ge1}\frac{\frac{1}{2}H_n-\frac{1}{2n}+\ln 2}{(2n-1)^{p+q}}+\sum_{k=2}^{q}\frac{(-1)^{q-k}\lambda(k)\lambda(p+q+1-k)}{4}$$
$$=\frac{(-1)^{q+1}}{2}HM(p+q)$$
$$+\frac{(-1)^{q+1}}{2}\lambda(p+q)\ln 2-(-1)^{q+1}\sum_{n\ge1}\frac{1}{2n(2n-1)^{p+q}}$$
$$+\sum_{k=2}^{q}\frac{(-1)^{q-k}\lambda(k)\lambda(p+q+1-k)}{4},$$

expanding the term $\sum_{n\ge1}\frac{1}{2n(2n-1)^{p+q}}$ in partial fraction form we obtain (3.2). □

Remark 1 From Corollary 4, the quadratic case occurs when $p=q$, so that

$$K(-1,1,q,q)=\int_0^1 x^{-1}\left(\chi_q(x)\right)^2\,dx=\frac{(-1)^{q+1}}{2}HM(2q)$$
$$+\frac{(-1)^{q+1}}{2}(2+\lambda(2q))\ln 2-(-1)^{q+1}\sum_{k=2}^{2q}\frac{(-1)^k\lambda(k)}{2}$$
$$+\sum_{k=2}^{q}\frac{(-1)^{q-k}\lambda(k)\lambda(2q+1-k)}{4}.$$

The integral (3.2) is symmetrical with respect to p and q therefore p and q are interchangeable in identity (3.2).

Remark 2 A further remarkable simplification is obtained from Corollary 4. Consider the case $p=q-1$ (or $q=p-1$, because of symmetry), so that

$$K(-1,1,q-1,q)=\int_0^1 x^{-1}\,\chi_{q-1}(x)\,\chi_q(x)\,dx=\frac{(-1)^{q+1}}{2}HM(2q-1)$$
$$+\frac{(-1)^{q+1}}{2}(\lambda(2q-1)-2)\ln 2-\sum_{k=2}^{2q-1}\frac{(-1)^{q+k}\lambda(k)}{2}$$

$$+\sum_{k=2}^{q} \frac{(-1)^{q+k} \lambda(k) \lambda(2q-k)}{4}.$$

After some careful simplification and utilizing the identity (1.11), we have

$$K(-1,1,q-1,q) = \int_0^1 x^{-1} \chi_{q-1}(x) \chi_q(x) \, dx = \frac{1}{8}\lambda^2(q)$$

Corollary 5 *Let $a=0,\ b=2,\ p\in\mathbb{N},\ q\in\mathbb{N}$, then*

$$\begin{aligned} K(0,2,p,q) &= \int_0^1 \chi_p(x) \chi_q\left(x^2\right) dx \\ &= (-1)^{q+1} \sum_{n\geq 1} \frac{H_n - \frac{1}{2}H_{\frac{n}{2}}}{(2n-1)^p (2n)^q} \\ &\quad + \sum_{k=2}^{q} \frac{(-2)^{q-k}\lambda(k)}{2} \sum_{n\geq 1} \frac{1}{(2n-1)^p (2n)^{q+1-k}} \end{aligned} \tag{3.3}$$

Here we require the Euler sums $\sum_{n\geq 1} \frac{H_n}{n^q}$, $\sum_{n\geq 1} \frac{H_n}{(2n-1)^p}$ may be obtained from (1.11) and the half integer Euler sums of the form $\sum_{n\geq 1} \frac{H_{\frac{n}{2}}}{n^q}$ can be evaluated from the results in [22].

Example 4 We give some examples to illustrate the above corollaries.

$$K(-1,1,6,7) = K(-1,1,7,6) = \int_0^1 x^{-1} \chi_6(x) \chi_7(x) \, dx = \frac{1}{8}\lambda^2(7).$$

$$\begin{aligned} K(-1,1,4,4) &= \int_0^1 x^{-1} (\chi_4(x))^2 \, dx = \frac{465}{256}\zeta(4)\zeta(5) \\ &\quad + \frac{381}{256}\zeta(2)\zeta(7) - \frac{511}{128}\zeta(9). \\ K(0,2,1,2) &= \int_0^1 \chi_1(x) \chi_2\left(x^2\right) dx = \frac{21}{32}\zeta(3) + \frac{1}{4}\ln^2 2 \\ &\quad + \frac{3}{4}\zeta(2)\ln 2 - \frac{1}{4}\zeta(2) - G, \end{aligned}$$

here we have used the identity

$$\sum_{n\geq 1} \frac{H_{\frac{n}{2}}}{n^2(2n-1)} = 16\ln 2 - 7\ln^2 2 - \zeta(2) - 4G - \frac{11}{8}\zeta(3).$$

$$\begin{aligned} K(0,2,1,3) &= \int_0^1 \chi_1(x) \chi_3\left(x^2\right) dx = 2G + \frac{1}{4}L(3) + \frac{5}{16}\zeta(4) - \frac{1}{2}\ln^2 2 \\ &\quad + \frac{7}{8}\zeta(3)\ln 2 - \frac{21}{16}\zeta(3) - \frac{3}{2}\zeta(2)\ln 2 + \frac{1}{2}\zeta(2) \end{aligned}$$

where $L(3)$ is given by (2.1) and we have utilized

$$\sum_{n\geq 1}\frac{H_{2n}}{(2n-1)(4n-1)}=G+\frac{\pi}{2}+\frac{1}{4}\zeta(2)-2\ln 2-\frac{\pi}{2}\ln 2+\frac{3}{4}\ln^2 2.$$

$$K\left(\frac{1}{2},\frac{1}{2},2,1\right)=\int_0^1 x^{\frac{1}{2}}\chi_2(x)\ \chi_1\left(x^{\frac{1}{2}}\right)dx=\frac{2}{9}G+\frac{7}{36}\zeta(2)-\frac{13}{9}\ln 2$$
$$+\frac{1}{2}\zeta(2)\ln 2+\frac{1}{18}\ln^2 2+\frac{7}{16}\zeta(3).$$

Concluding remarks We have carried out a systematic study of Legendre-Chi function integrals in terms of Euler sums. We believe most of our results are new in the literature and given many examples which most are not amenable to a mathematical computer package.

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