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ORIGINAL PAPER

On a closed form for derangement numbers: an elementary proof

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Abstract
We provide a very short proof of a Qi-Zhao-Guo closed form for derangements numbers based on the determinants of certain Hessenberg matrices. The proof is grounded on a basic result on finite sequences and an inductive argument.

Keywords Derangement numbers · Hessenberg matrices

Mathematics Subject Classification 05A05 · 05A10 · 15A15 · 11B37 · 11C20

1 Preliminaries

The *nth derangement number* or *subfactorial of* n, $!n$, is the number of permutations of n elements, such that no element appears in its original position, i.e., is a permutation that has no fixed points. The first derangement numbers are

$$0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, 14684570$$

and the sequence is labelled in The On-Line Encyclopedia of Integer Sequences [13] as A000166.

Derangement numbers were first combinatorially studied by the French mathematician and Fellow of the Royal Society, Pierre Rémond de Montmort, in his celebrated book *Essay d'analyse sur les jeux de hazard*, published in 1708. The two best known recurrence relations are

$$!n = (n-1)(!(n-1)+!(n-2)), \quad \text{for } n \geq 2,$$

and

$$!n = n\,(!(n-1)) + (-1)^n, \quad \text{for } n \geq 1, \tag{1.1}$$

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with $!0 = 1$ and $!1 = 0$, were established and proved by Euler in *Calcul de la probabilité dans le jeu de rencontre*, in 1753. They can be written in the explicit forms

$$!n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k!$$

which coincides with permanent of the all ones matrix minus the identity matrix, all of order n [10] (for more details on this matter, the reader is referred to [14, Chapter 2]).

Other natural representations of the derangement numbers are those in terms of the determinant of certain tridiagonal matrices as we can find in [7,8].

Recently, Qi, Zhao, and Guo in [12] claimed the discovery of two new identities for the derangement numbers, namely

$$!n = \sum_{k=0}^{n-2} \binom{n}{k} (n-k-1)(!k)\,, \quad \text{for } n \geq 2, \tag{1.2}$$

and

$$n! = \sum_{k=0}^{n} \binom{n}{k} (!k)\,, \quad \text{for } n \geq 0. \tag{1.3}$$

The proofs are based on the construction of the Hessenberg matrix

$$H_n = \begin{pmatrix} 0 & -1 & & & & & \\ \binom{2}{0} & 0 & -1 & & & & \\ \binom{3}{0}2 & \binom{3}{1} & \ddots & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \\ \binom{n-2}{0}(n-3) & \binom{n-2}{1}(n-4) & \dots & \ddots & \ddots & -1 & \\ \binom{n-1}{0}(n-2) & \binom{n-1}{1}(n-3) & \dots & \dots & \binom{n-1}{n-3} & 0 & -1 \\ \binom{n}{0}(n-1) & \binom{n}{1}(n-2) & \dots & \dots & \binom{n}{n-3}2 & \binom{n}{n-2} & 0 \end{pmatrix}, \tag{1.4}$$

an exponential generating function, a higher derivatives formula, and the calculation of a specific limit, which leads to

$$!n = \det H_n. \tag{1.5}$$

Our aim here is to provide a very short proof for (1.2) using a standard result in finite differences theory. *En passant*, we show (1.5) based on another result involving Hessenberg matrices and recurrence relations.

Notice that (1.3) is well-known in the literature. See, for example, [6] for a simple combinatorial interpretation.

We should also point out that the terminology "Hessenberg determinant", as well as "tridiagonal determinant", used in many instances by the authors in [12] and elsewhere, is inaccurate as we mentioned in [1].

For relations involving Hessenberg matrices, the reader is referred, for example, to [2–5,11,17].

2 Proof

From the theory of finite differences we know that, for any polynomial $p(x)$ of degree less than n,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}p(k)=0\,. \tag{2.1}$$

See, for instances, [9, p.64] or [17]. In particular, for $n\geq 2$, we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k-1)=0$$

which implies

$$\sum_{k=0}^{n-2}(-1)^k\binom{n}{k}(n-k-1)=(-1)^n. \tag{2.2}$$

Now, from (1.1), (2.2), and an elementary inductive argument, we have

$$\begin{aligned}\sum_{k=0}^{n-2}\binom{n}{k}(n-k-1)(!k)&=\sum_{k=1}^{n-2}\binom{n}{k}k(n-k-1)(!(k-1))\\&\quad+\sum_{k=0}^{n-2}(-1)^k\binom{n}{k}(n-k-1)\\&=n\sum_{k=0}^{n-3}\binom{n-1}{k}(n-k-2)(!k)+(-1)^n\\&=n\,(!(n-1))+(-1)^n\\&=\,!n\,.\end{aligned}$$

Regarding the Hessenberg matrix (1.4), we know (cf., e.g., [15, Theorem 4.20] or [7, Proposition 6]), if $a_1,a_2,\ldots$ is a sequence such that

$$a_{n+1}=p_{n,1}a_1+\cdots+p_{n,n}a_n\,,$$

then

$$a_{n+1}=a_1\det\begin{pmatrix}p_{1,1}&-1&&&&\\p_{2,1}&p_{2,2}&-1&&&\\p_{3,1}&p_{3,2}&\ddots&\ddots&&\\\vdots&\vdots&\ddots&\ddots&\ddots&\\p_{n-1,1}&p_{n-1,2}&\cdots&p_{n-1,n-2}&p_{n-1,n-1}&-1\\p_{n,1}&p_{n,2}&\cdots&p_{n,n-2}&p_{n,n-1}&p_{n,n}\end{pmatrix}.$$

Setting

$$a_k=\,!(k-1)\quad\text{and}\quad p_{n,k}=\binom{n}{k-1}(n-k),$$

we get the desired result.

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