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Metrizable quotients of free topological groups

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Abstract
We consider the following problem: For which Tychonoff topological spaces X do the free topological group $F(X)$ and the free abelian topological group $A(X)$ admit a nontrivial (i.e. not finitely generated) metrizable quotient group? First, we give a complete solution of this problem for the key class of compact spaces X. Then, relying on this result, we resolve the problem for several more general important classes of spaces X, including the class of σ-compact spaces, the class of pseudocompact spaces, the class of ω-bounded spaces, the class of Čech-complete spaces and the class of K-analytic spaces. Also we describe all absolutely analytic metric spaces X (in particular, completely metrizable spaces) such that the free topological group $F(X)$ and the free abelian topological group $A(X)$ admit a nontrivial metrizable quotient group. Our results are based on an extensive use of topological properties of non-scattered spaces.

Keywords Free topological group · Quotient group · Metrizable space · Scattered space

Mathematics Subject Classification Primary 22A05, 54C10, 54E35 · Secondary 54D30, 54G12

1 Introduction

Our research is motivated by the well-known, still unsolved, Banach–Mazur Separable Quotient Problem about the existence of a separable infinite-dimensional quotient for any infinite-dimensional Banach space. We refer to [10] (and references therein) which surveys the main advances in this area. The original Separable Quotient Problem concerns infinite-dimensional quotients only because in a locally convex linear space L, every finite-dimensional subspace is complemented and, hence, is a quotient of L [24, Lemma 4.21]. For locally convex spaces, analogous questions have been studied in two different directions:

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Find classes of infinite-dimensional locally convex spaces admitting an infinite-dimensional locally convex quotient space which is either *separable* or *metrizable* (or both). In the recent articles by Banakh, Kąkol and Śliwa [3], [4], [13], these questions are considered for the spaces of continuous real-valued functions with the pointwise convergence topology, $C_p(X)$.

The following similar problems have been thoroughly studied in the class of general topological groups [17]:

Problem 1.1 *Characterize topological groups which have an infinite separable topological quotient group.*

Problem 1.2 *Characterize topological groups which have an infinite (separable) metrizable topological quotient group.*

Among other results, it has been shown in [17] that every infinite topological group G admits an infinite separable metrizable quotient group provided G belongs to one of the following important classes:

(a) compact groups;
(b) locally compact abelian groups;
(c) σ-compact locally compact groups;
(d) abelian pro-Lie groups;
(e) σ-compact pro-Lie groups;
(f) pseudocompact groups.

However, there exists an uncountable precompact topological abelian group G such that every quotient group of G is either the one-point group or non-separable [17, Theorem 3.5]. Also, there exists a countable precompact topological abelian group H such that every quotient group of H is either the one-point group or non-metrizable [17, Proposition 4.4].

Free topological groups constitute a very important subclass of topological groups. It suffices to mention that every topological group is a quotient group of a free topological group and that every Tychonoff space X embeds as a *closed* subspace to the free topological group $F(X)$ on X [20]. Hence the space of the topological group $F(X)$ is not normal provided the space X is not normal. This conclusion was one of Markov's main motivations for introducing free topological groups in [19].

The topology of the free topological group $F(X)$ is the finest topological group topology on the group that induces on X its original topology. This "extremal" property of $F(X)$ guarantees that $F(X)$, for a 'big' space X, has many nontrivial quotient groups. For example, according to Proposition 2.1, one can take a quotient mapping $f\colon X \to Y$ onto a Tychonoff space Y and extend it to a continuous homomorphism $f^*\colon F(X) \to F(Y)$. Then f^* is an open and surjective homomorphism, i.e. $F(Y)$ is a quotient group of $F(X)$. This approach has been systematically applied in [18], where the authors study the Separable Quotient Problem for the free (abelian) topological groups.

The fact that in a locally convex space L, every finite-dimensional linear subspace is a quotient of L, has its analogue in free topological groups: Each of the topological groups $F(X)$ and $A(X)$ admits an open continuous homomorphism onto the discrete group of integers, $\mathbb{Z}$, provided that $X \neq \emptyset$.

Indeed, for every nonempty Tychonoff space X, we can take the constant mapping f from X to $\mathbb{Z}$ defined by $f(x) = 1$ for each $x \in X$. Then f admits an extension to a continuous homomorphism $f^*\colon A(X) \to \mathbb{Z}$ which is open by [1, Corollary 7.1.10] since the mapping f is evidently quotient. Thus, the discrete group $\mathbb{Z}$ is always a quotient of $A(X)$. Since $A(X)$ is a quotient group of $F(X)$ with respect to the commutator subgroup of $F(X)$ [20], the group $\mathbb{Z}$ is also a quotient of $F(X)$.

To exclude this 'ever present' quotient from our considerations, we reserve the term *nontrivial group* for the groups which are not finitely generated.

Our aim here is to consider the "Metrizable Quotient Problem" for the free (abelian) topological groups:

Problem 1.3 *Characterize the infinite Tychonoff spaces X such that the topological groups $F(X)$ and/or $A(X)$ admit a nontrivial metrizable (and separable) quotient group.*

Since $A(X)$ is a quotient of $F(X)$, it suffices for our purposes to find a nontrivial metrizable quotient group of the free abelian topological group $A(X)$. To shorten arguments, we frequently denote the groups $F(X)$ or $A(X)$ by $G(X)$.

The main obstacle to this study is the fact that the free topological group $F(Y)$ and the free abelian topological group $A(Y)$ are seldom metrizable. In fact, these groups are metrizable if and only if the space Y is discrete [1, Theorem 7.1.20]. Therefore, if a given space X is connected or has finitely many connected components, the method explained earlier (and based on Proposition 2.1 below) produces only trivial metrizable quotients of $F(X)$ and $A(X)$, that are all finitely generated and discrete, hence, countable. To overcome this difficulty, we employ the following strategy: For a given Tychonoff space X, we try to find a continuous quotient mapping φ of X onto a topological (abelian) group G and then extend φ to an open continuous homomorphism φ^* of $F(X)$ (resp., $A(X)$) onto G (see Proposition 2.2).

Throughout the article, all topological spaces under consideration are assumed to be Tychonoff and infinite. We denote by $\mathbb{I}$ the closed unit interval $[0, 1]$ and by $\mathbb{T}$ the circle group, both carrying their usual compact topologies.

Let us recall that a space X is said to be *scattered* if every nonempty subset S of X has an isolated point in S. The following classical theorem of Pełczyński and Semadeni is extensively used in this article.

Theorem 1.4 ([25, Theorem 8.5.4]) *For a compact space X, the following conditions are equivalent*:

(1) *There is no continuous mapping of X onto $\mathbb{I}$.*
(2) *X is scattered.*

Since $\mathbb{I}$ admits a closed continuous mapping onto $\mathbb{T}$ we obtain that for every non-scattered compact X, there is a closed continuous (hence, quotient) mapping of X onto the circle group $\mathbb{T}$.

With the help of this simple observation, in Theorem 3.1 we resolve Problem 1.3 completely for compact spaces X. In essence, Theorem 3.1 states that for a compact space X, the groups $F(X)$ and $A(X)$ admit a nontrivial metrizable quotient group if and only if X is not scattered. As a corollary, assuming that X is compact and first-countable, the groups $F(X)$ and $A(X)$ admit a nontrivial metrizable quotient group if and only if X is uncountable.

Then, we solve Problem 1.3 for the wider class of pseudocompact spaces (Theorem 4.5). However, we need to warn the reader that our characterizations in the compact and pseudocompact cases are somewhat different. This is explained in Remark 4.9.

The property of being a non-scattered space is closely tied to Problem 1.3, and we study possible extensions of the aforementioned Pełczyński–Semadeni theorem to non-compact spaces. This allows us to show that Theorem 3.1 holds true for the broader class of ω-bounded spaces (Theorem 4.7).

We explore several important classes of topological spaces which either contain a homeomorphic copy of the Cantor set $\mathbb{C}$ or admit a quotient mapping onto the countable discrete space $\mathbb{N}$.

This approach provides us with a complete description of the Tychonoff spaces X for which the groups $F(X)$ and $A(X)$ admit a nontrivial metrizable quotient group for the following classes of spaces: Čech-complete (in particular, locally compact) spaces (Theorem 5.2), σ-compact spaces (Theorem 5.6), K-analytic spaces (Theorem 5.8), absolutely analytic metric (in particular, completely metrizable) spaces (Theorem 6.2).

Also, we solve Problem 1.3 in the class of connected and locally connected spaces in Proposition 2.6. It is worth noting that the latter class has no relations with the compactness type properties mentioned so far.

Several results presented here have been announced (without proofs) in the survey paper [16].

2 Sufficient conditions on spaces X for which $G(X)$ admits a nontrivial metrizable quotient group

A continuous onto mapping $\varphi\colon X \to Y$ is said to be *quotient* if for a subset U of Y, the preimage $\varphi^{-1}(U)$ is open in X if and only if U is open in Y. All closed continuous mappings and all open continuous mappings are quotient (we assume that the mappings are surjective).

Below we formulate fundamental properties of the free (abelian) topological groups which will be widely used in the article.

Proposition 2.1 (see Corollary 7.1.9 in [1]) *Let $\varphi\colon X \to Y$ be a continuous mapping of Tychonoff spaces. Then φ admits an extension to a continuous homomorphism $\varphi^*\colon G(X) \to G(Y)$, where $G(X)$ denotes $F(X)$ or $A(X)$. If, in addition, φ is quotient then the homomorphism φ^* is open and surjective.*

Proposition 2.2 (see Corollary 7.1.10 in [1]) *If a Tychonoff space X admits a quotient mapping onto a topological (abelian) group G, then $F(X)$ (resp., $A(X)$) admits an open continuous homomorphism onto G.*

Here we present several simple applications of Propositions 2.1 and 2.2 .

Proposition 2.3 *If a Tychonoff space X contains a compact subset K which is not scattered, then the topological group $G(X)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$.*

Proof By Theorem 1.4, there is a continuous closed mapping f of K onto the closed interval $[0, 1]$. Since X is Tychonoff f extends to a continuous mapping g of X onto $[0, 1]$. Let $F \subset [0, 1]$ be a subset such that $g^{-1}(F)$ is closed in X. Then $g^{-1}(F) \cap K$ is a compact set and $g(g^{-1}(F) \cap K) = F \cap g(K) = F$ is also compact and hence closed in $[0, 1]$. We conclude that g is a quotient mapping, so there exists a quotient mapping of X onto $\mathbb{T}$ and hence $G(X)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$, by Proposition 2.2. □

Proposition 2.4 *If a space X contains an infinite discrete family of nonempty clopen subsets, then the group $G(X)$ admits an open continuous homomorphism onto the nontrivial discrete group $G(\mathbb{N})$.*

Proof We claim that X admits a continuous mapping onto the discrete space of natural numbers, $\mathbb{N}$. Indeed, let $\{U_n : n \in \omega\}$ be an infinite discrete family of nonempty clopen sets in X. Then the set $O = \bigcup_{n\in\omega} U_n$ is clopen in X. Let $W = X\setminus O$. We define a mapping f of X onto $\mathbb{N}$ by letting $f(x) = 0$ if $x \in W$ and $f(y) = n+1$ if $y \in U_n$ for some $n \in \omega$. Clearly f is continuous and open. Hence the extension of f to a continuous homomorphism of $G(X)$ onto $G(\mathbb{N})$ is also open, by Proposition 2.1. Notice that the group $G(\mathbb{N})$ is countable, discrete and not finitely generated. □

We recall that a Tychonoff space X is called *pseudocompact* if every continuous real-valued function on X is bounded. Since every non-pseudocompact Tychonoff space X contains an infinite discrete family of nonempty open subsets, the following corollary is immediate.

Corollary 2.5 *Let X be a non-pseudocompact zero-dimensional space. Then the group $G(X)$ admits an open continuous homomorphism onto the nontrivial discrete group $G(\mathbb{N})$.*

Below we apply Proposition 2.2 to solve Problem 1.3 for connected locally connected spaces.

Proposition 2.6 *Let X be a connected and locally connected infinite space. Then the group $G(X)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$.*

Proof Take a non-constant continuous function $f : X \to \mathbb{I}$. Since X is connected, the image $Y = f(X)$ is a connected nontrivial subset of the closed interval $[0, 1]$. Thus, Y is an interval and then there exists a quotient mapping g from Y onto the circle $\mathbb{T}$. By [18, Lemma 2.15], the mapping f is quotient, so the composition $h = g \circ f : X \to \mathbb{T}$ is also quotient. Then h extends to an open continuous homomorphism $h^* : G(X) \to \mathbb{T}$, by Proposition 2.2. □

3 Compact spaces

Here we resolve completely Problem 1.3 for compact spaces. Our proof of Theorem 3.1 is split into two parts. A major part of implications between conditions (a)–(d) of the theorem is proved directly. The only exception is the implication (b) ⇒ (d) which is postponed till Proposition 3.5. Needless to say, our proof of Proposition 3.5 does not depend on Theorem 3.1.

Theorem 3.1 *Let X be a compact space. Then the following conditions are equivalent*:

(a) *The group $G(X)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$.*
(b) *The group $G(X)$ admits a nontrivial metrizable quotient.*
(c) *The group $G(X)$ admits a nontrivial separable and metrizable quotient.*
(d) *X is not scattered.*

Proof We observe first that items (b) and (c) are equivalent. Indeed, let φ be a continuous homomorphism of $G(X)$ onto a metrizable topological group M. Then the image $\varphi(X)$ is a compact subset of M, so $\varphi(X)$ is separable. Since $\varphi(X)$ generates the group M algebraically, M is separable as well.

(a) ⇒ (b) is trivial. The implication (d) ⇒ (a) is an immediate consequence of Proposition 2.3. The remaining implication (b) ⇒ (d) follows from Proposition 3.5 that will be established afterward. □

We now collect several properties of scattered spaces that will be used in the sequel.

Lemma 3.2

(a) *The product of finitely many scattered spaces is scattered.*
(b) *The union of finitely many scattered subspaces is scattered.*
(c) *An image of a scattered space under an open continuous mapping is scattered.*
(d) *A continuous Hausdorff image of a compact scattered space is scattered.*

Proof Items (a), (b) and (c) are evident. Item (d) is Proposition 8.5.3 in [25]. □

We recall that a Hausdorff space X is said to be a k_ω*-space* if X is the union of an increasing sequence $\{X_n : n \in \omega\}$ of compact subspaces, with the property that a subset $A \subset X$ is closed in X if and only if $A \cap X_n$ is a closed subspace of X_n, for each $n \in \omega$. For instance, every countable CW-complex is a k_ω-space. Clearly, every compact space is a k_ω-space and every k_ω-space is normal.

Proposition 3.3 *Let X be a k_ω-space. If a metrizable topological group M is a quotient of the free topological group $F(X)$ (or $A(X)$), then M is locally compact and second countable.*

Proof Let φ be an open continuous homomorphism from $F(X)$ onto M. By [1, Theorem 7.4.1.], $F(X)$ (and $A(X)$) is a k_ω-space, and so is its open image M according to Fact 11 on page 116 of [9]. However, every first countable k_ω-space is locally compact (see Fact 5 on page 113 of [9]). Therefore, the group M is locally compact.

It remains to verify that M has a countable base. Since M is metrizable, the image $\varphi(X)$ is a countable union of metrizable compact subspaces. Notice that X generates $F(X)$ algebraically, so $C = \varphi(X)$ generates M algebraically as well. This implies that M is the union of a countable family of its subspaces each of which is a continuous image of a finite power of the compact metrizable space C. Hence each element of this family has a countable base and M has a countable network. Therefore, $w(M) \le nw(M)\chi(M) \le \omega$, by [1, Proposition 5.7.14]. This completes the proof. □

We need one more auxiliary fact about the structure of the free (abelian) topological groups on a scattered space. As usual, we denote by $F_n(X)$ ($A_n(X)$) the subset of $F(X)$ (resp., $A(X)$) consisting of all words of reduced length $\le n$ with respect to the free basis X. All $F_n(X)$ ($A_n(X)$) are closed in $F(X)$ (resp., $A(X)$) [1, Theorem 7.1.13].

Lemma 3.4 *If X is a scattered Tychonoff space, then so are $F_n(X)$ and $A_n(X)$, for each integer $n \ge 0$.*

Proof First, we prove the lemma for the group $A(X)$. Let us apply induction on n. If $n = 0$, then $A_0(X)$ contains a single element, the identity e of $A(X)$. Denote by Y the topological sum of X, its copy $-X$ and the point e, $Y = X \oplus \{e\} \oplus (-X)$. Notice that $Y \cong A_1(X)$ is scattered. For $n \ge 1$, let j_n be the canonical mapping of Y^n onto $A_n(X)$,

$$j_n(y_1, y_2, \ldots, y_n) = y_1 + y_2 + \cdots + y_n.$$

Clearly j_n is continuous.

Assume that $A_n(X)$ is scattered for some $n \ge 1$. We claim that the subspace $C_{n+1}(X) = A_{n+1}(X) \backslash A_n(X)$ of $A(X)$ is scattered. Indeed, consider the subspace $C^*_{n+1}(X) = j^{-1}_{n+1}(C_{n+1}(X))$ of Y^{n+1}. It follows from [1, Proposition 7.1.14] that the restriction of j_{n+1} to $C^*_{n+1}(X)$ is a perfect and open mapping of $C^*_{n+1}(X)$ onto $C_{n+1}(X)$. Hence $C_{n+1}(X)$ is scattered, by item (c) of Lemma 3.2. Therefore, $A_{n+1}(X)$ is the union of two scattered subspaces, $A_n(X)$ and $C_{n+1}(X)$, so item (b) of Lemma 3.2 implies that $A_{n+1}(X)$ is scattered as well.

In the case of the group $F(X)$, one considers the multiplication mapping i_n of Y^n onto $F_n(X)$, and then applies [1, Theorem 7.1.13] instead of [1, Proposition 7.1.14]. □

Proposition 3.5 *Let X be a scattered compact space. Then every quotient group of $F(X)$ and $A(X)$ is either discrete and finitely generated or non-metrizable.*

Proof It suffices to prove the statement for $F(X)$. Let φ be an open continuous homomorphism of $F(X)$ onto a metrizable topological group M. By Proposition 3.3, M is a locally compact and second countable topological space, therefore M is a Polish locally compact group. For each $n \in \omega$, the subspace $F_n(X)$ of $F(X)$ is a continuous image of the compact space Y^n, where Y is the topological sum of X, its copy X^{-1}, and the singleton $\{e\}$ (here e is the identity of $F(X)$). Thus, all subspaces $F_n(X)$ are compact and, hence, $\varphi(F_n(X))$ are compact subsets of M.

Applying the Baire property of $M = \bigcup_{n\in\omega} \varphi(F_n(X))$, we see that there is an integer $m \geq 1$ such that $\varphi(F_m(X))$ has a nonempty interior in M. Since the space X is scattered, Lemma 3.4 implies that so is $F_m(X)$ and therefore its continuous image $K = \varphi(F_m(X))$ is a scattered subspace of M.

Take a nonempty open subset U of M with $U \subset K$. Since K is scattered, U contains an isolated point. This implies that the group M is discrete because M is a homogeneous space. Evidently, the compact set $\varphi(X)$ algebraically generates the group M. However, any compact set in a discrete space is finite, so M is a finitely generated discrete group. □

Our last Proposition 3.5 in fact completes the proof of the remaining implication (b) ⇒ (d) in Theorem 3.1.

Corollary 3.6 *The following conditions are equivalent for a first-countable (in particular, metrizable) compact space X:*

(a) *The group $G(X)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$.*
(b) *The group $G(X)$ admits a nontrivial metrizable quotient.*
(c) *The group $G(X)$ admits a nontrivial metrizable and separable quotient.*
(d) *X is uncountable.*

Proof In view of Theorem 3.1, it suffices to remark that by the classical theorem of Mazurkiewicz and Sierpiński [25, Theorem 8.6.10], a first-countable compact space is scattered if and only if it is countable. □

4 Pseudocompact spaces and ω-bounded spaces

Our next aim is to extend Theorem 3.1 to pseudocompact spaces. We need to do some preparatory work. First, we examine under which conditions the existence of a nontrivial metrizable quotient for a dense subgroup of a topological group G implies the existence of a nontrivial metrizable quotient for G.

We say that a subset Y of a space X is G_δ*-dense* in X if Y meets every nonempty G_δ-set in X.

Lemma 4.1 *Let H be a subgroup of a topological group G.*

(a) *If H is dense in G and H has a nontrivial* commutative *metrizable quotient group, then so does G.*

(b) *If H is G_δ-dense in G and H admits an open continuous homomorphism onto a nontrivial metrizable group, then G admits an open continuous homomorphism onto the same nontrivial metrizable group.*

Proof Let N be a closed normal subgroup of H such that the quotient group H/N is nontrivial. We denote by K the closure of N in G. It is clear that K is a closed normal subgroup of G. Therefore, we can consider the quotient homomorphism $p\colon G \to G/K$. Since $N = H \cap K$ is dense in K, one can identify the group H/N, algebraically and topologically, with the subgroup $p(H)$ of G/K.
(a) Assume that $H/N \cong p(H)$ is a nontrivial commutative subgroup of the quotient group G/K. Then G/K is also commutative and nontrivial since $p(H)$ is dense in G/K and a subgroup of a finitely generated commutative group is finitely generated. If the dense subgroup $p(H)$ of G/K is metrizable, the group G/K is metrizable as well according to [1, Proposition 3.6.20].
(b) We now assume that H is G_δ-dense in G and that the group $H/N \cong p(H)$ is metrizable and nontrivial. Then the group G/K is also metrizable and $p(H)$ is G_δ-dense in G/K. Hence $G/K = p(H)$, so G/K is nontrivial. □

Remark 4.2 It is worth noting that the existence of a nontrivial metrizable quotient of a topological group G does not imply the same property for a dense subgroup H of G, even if the group G is compact and abelian. This follows from [17, Theorem 4.5], where H is a dense subgroup of the compact group $G = \mathbb{T}^{\mathfrak{c}}$ with the property that every nontrivial quotient of H is not separable. Here $\mathfrak{c}$ stands for the power of the continuum. Clearly, the group H is precompact as well as every quotient of H. Notice that every precompact metrizable group is separable (this follows e.g. from [1, Proposition 3.4.5]. Hence H does not have nontrivial metrizable quotients, while the group $G = \mathbb{T}^{\mathfrak{c}}$ trivially has a separable metrizable quotient $\mathbb{T}$.

In the following proposition, we establish that having a nontrivial metrizable quotient is inherited by G_δ-*dense* subgroups of topological groups. This shows that the converse of Lemma 4.1 (b) is valid.

Proposition 4.3 *Let H be a G_δ-dense subgroup of a topological group G. Then G admits an open continuous homomorphism onto a nontrivial metrizable group M if and only H admits an open continuous homomorphism onto the same nontrivial metrizable group M.*

Proof Let $p\colon G \to M$ be an open continuous homomorphism onto a nontrivial metrizable topological group M. Then the kernel of p, say, K is a closed normal subgroup of type G_δ in G. Hence $N = K \cap H$ is a closed normal subgroup of H and N is dense in K. It follows from [1, Theorem 1.5.16] that the restriction of p to H is an open continuous homomorphism of H onto the subgroup $p(H)$ of M. Since $p(H)$ meets every nonempty G_δ-set in M, we see that $p(H) = M$. Hence the quotient group $H/N \cong p(H) = M$ is metrizable, so H admits an open continuous homomorphism onto the nontrivial metrizable group M.

The inverse implication is immediate from Lemma 4.1(b). □

Below βX denotes the Stone-Čech compactification of X.

Proposition 4.4 *Let X be a pseudocompact space. Then the free topological group $G(\beta X)$ admits an open continuous homomorphism onto a metrizable group if and only if so does $G(X)$.*

Proof According to [1, Corollary 7.7.5], the group $G(X)$ is topologically isomorphic to the subgroup of $G(\beta X)$ generated by the subspace X of βX. Since X is pseudocompact, it meets every nonempty G_δ-set in βX. Hence we can apply [1, Lemma 7.7.6] to conclude that $G(X)$ meets every nonempty G_δ-set in $G(\beta X)$. Now we apply Proposition 4.3 with $G = G(\beta X)$ and $H = G(X)$. □

Combining Theorem 3.1 and Proposition 4.4 we obtain the following solution to Problem 1.3 in the class of pseudocompact spaces.

Theorem 4.5 *Let X be a pseudocompact space. Then the following conditions are equivalent*:

(a) *The group $G(X)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$.*
(b) *The group $G(X)$ admits a nontrivial metrizable quotient.*
(c) *The group $G(X)$ admits a nontrivial separable metrizable quotient.*
(d) *The Stone-Čech compactification βX of X is not scattered.*

A space X is called *ω-bounded* if the closure of every countable subset of X is compact. Clearly, the following implications hold:

$$(*)\ \text{compact} \Rightarrow \omega\text{-bounded} \Rightarrow \text{countably compact} \Rightarrow \text{pseudocompact}$$

Examples of ω-bounded non-compact spaces include the spaces of ordinals $[0, \alpha)$ with the order topology, where α is any ordinal of uncountable cofinality, and Σ-products of uncountably many compact spaces. In the class of metrizable spaces, all of the four notions in $(*)$ coincide. It is also known that every normal pseudocompact space is countably compact.

Proposition 4.6 *Let X be a scattered ω-bounded Tychonoff space. Then the Stone-Čech compactification βX of X is also scattered.*

Proof By contradiction, assume that βX is not scattered and let f be a continuous mapping from βX onto the interval $\mathbb{I}$. Take a countable subset S of X such that $f(S)$ is dense in $\mathbb{I}$. Denote by K the closure of S in X. Then K is a compact subset of X, hence the image $f(K)$ is a compact dense subset of $\mathbb{I}$, which means that $f(K) = \mathbb{I}$. So, the restriction $f \restriction K$ is a continuous mapping from the compact scattered space K onto the segment $\mathbb{I}$, which is impossible. This contradiction completes the proof. □

Now we are ready to prove that Theorem 3.1 holds true under the weaker assumption that X is ω-bounded.

Theorem 4.7 *Let X be an ω-bounded space. Then the following conditions are equivalent*:

(a) *The group $G(X)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$.*
(b) *The group $G(X)$ admits a nontrivial metrizable quotient.*
(c) *The group $G(X)$ admits a nontrivial separable metrizable quotient.*
(d) *X is not scattered.*

Proof Since X is a pseudocompact space we can apply Theorem 4.5 to conclude that items (a), (b), (c) are mutually equivalent and also they are equivalent to the claim that the Stone-Čech compactification βX of X is not scattered. The latter property is equivalent to item (d) of Theorem 4.7, by Proposition 4.6. □

As a straightforward consequence of Theorem 4.7 we have the following

Corollary 4.8 *Let X be either*

(a) *the compact space of ordinals* $[0, \alpha]$ *with the order topology; or*
(b) *the space of ordinals* $[0, \alpha)$ *with the order topology, where* α *is any ordinal of uncountable cofinality; or*
(c) *the one-point compactification of an arbitrary discrete space (in particular, a convergent sequence with its limit point).*

Then every metrizable quotient group of $F(X)$ *or* $A(X)$ *is discrete and finitely generated.*

Remark 4.9 The study of scattered spaces has attracted much attention of many researchers in 70s of the previous century. Among other results, it was shown that there were scattered Tychonoff spaces with no scattered compactification. Some of the most simple examples of such kind can be found e.g. in [6,27].

It is natural to try finding some subclasses of the Tychonoff spaces X for which the following conditions are equivalent:

(1) The Stone-Čech compactification βX is not scattered;
(2) X is not scattered.

Obviously, (2) always implies (1). Thus, if X is a countably compact Tychonoff space which is not scattered, then βX is not scattered either and therefore the group $G(X)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$, by Theorem 4.5.

Note that the existence of a countably compact scattered Tychonoff space that maps continuously onto the closed interval [0, 1] from just ZFC is a major unsolved problem going back to 70s of the previous century. For instance, according to [14], under the Continuum Hypothesis (CH), there exists a countably compact, scattered and locally compact Tychonoff space which admits a closed continuous map onto the closed interval [0, 1]. Moreover, Ostaszewski [23] with the help of the Jensen's diamond principle ($\diamondsuit$) constructed a countably compact, scattered and locally compact space X admitting a closed continuous map onto [0, 1], which is, in addition, perfectly normal and hereditarily separable.

As a consequence, for a countably compact and normal space X, the existence of an open continuous homomorphism from the free abelian topological group $A(X)$ onto the circle group $\mathbb{T}$ does not imply that X is not scattered. This means that Theorem 3.1 fails to be true for normal countably compact spaces without extra set-theoretic assumptions.

An "absolute" example of a countably compact scattered space that can be continuously mapped onto [0, 1] has not been known till now. A bit later in Remark 5.3 we show that a pseudocompact non-normal scattered space which can be continuously mapped onto [0, 1] does exist in ZFC.

5 Čech-complete spaces, σ-compact spaces and K-analytic spaces

Now we turn our attention to the class of Čech-complete topological spaces. We recall that a Tychonoff space X is Čech-complete if X is a G_δ-set in some (equivalently, any) compactification of X, see [8, 3.9.1]. It is well known [8, 4.3.26] that each completely metrizable space is Čech-complete. The following probably folklore result plays a crucial role. Its detailed proof can be found in [15,26]; for some further generalizations see [2].

Proposition 5.1 *A Čech-complete space* X *is scattered if and only if every compact subset of* X *is scattered.*

Theorem 5.2 *For a Čech-complete (in particular, locally compact) space* X *the following holds*:

(1) *If X is not scattered, then the topological group $G(X)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$.*
(2) *If X is scattered and not pseudocompact, then the topological group $G(X)$ admits an open continuous homomorphism onto the nontrivial discrete group $G(\mathbb{N})$.*
(3) *If X is scattered and pseudocompact, then the topological group $G(X)$ admits a non-trivial metrizable quotient iff $G(X)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$ iff the Stone-Čech compactification βX of X is not scattered.*

Proof (1) By Proposition 5.1, X contains a non-scattered compact subset K, and the required conclusion follows from Proposition 2.3.

(2) A non-pseudocompact Tychonoff space X contains an infnite discrete family of nonempty open subsets $\{U_n : n \in \omega\}$. Since X is scattered we can pick an isolated point x_n in each U_n. Now the claim follows from Proposition 2.4.

(3) It suffices to apply Theorem 4.5 which deals with pseudocompact spaces. □

Remark 5.3 A very interesting question arises whether there exists in ZFC a space X which is pseudocompact, locally compact and scattered and such that its Stone-Čech compactification βX is not scattered. Below we outline an example providing the positive answer to this question.

We recall the construction and basic properties of the Isbell–Mrówka space $\Psi(\mathcal{M})$ described in [11, 5I]. Let $\mathcal{M}$ be an infinite maximal almost disjoint (MAD) family of subsets of the set of natural numbers $\mathbb{N}$ and let $\Psi(\mathcal{M})$ be the set $\mathbb{N} \cup \mathcal{M}$ equipped with the topology defined as follows. For each $n \in \mathbb{N}$, the singleton $\{n\}$ is open, and for each $A \in \mathcal{M}$, a base of neighborhoods of A is the collection of all sets of the form $\{A\} \cup B$, where $|A \backslash B| < \omega$. The space $\Psi(\mathcal{M})$ is then a first-countable pseudocompact locally compact Tychonoff space which is not normal and thus is not countably compact [11, 5I]. (Readers are advised to consult [12, Chapter 8] which surveys various topological properties of these spaces).

The following fact has been remarked for the first time in [22]: There exists in ZFC a MAD family $\mathcal{M}$ on $\mathbb{N}$ such that the corresponding Isbell–Mrówka space $\Psi(\mathcal{M})$ admits a continuous mapping onto the closed interval $[0, 1]$. Detailed constructions of such MAD families $\mathcal{M}$ can be found in [5,29]. Summarizing the above discussion, a pseudocompact locally compact and non-compact space X satisfying the conditions of Theorem 4.5 (item (d)) and Theorem 5.2 (item (3)) does exist in ZFC.

In view of these facts we are interested in a closely related question: Under which assumptions (1) implies (2), where

(1) there is no continuous mapping of X onto the closed interval $[0, 1]$;
(2) X is scattered.

First we prove the following apparently well-known simple result.

Proposition 5.4 *If a Tychonoff space X does not admit any continuous mapping onto* $[0, 1]$*, then X is zero-dimensional, i.e. X has a base consisting of clopen sets.*

Proof Let $U \subset X$, $U \neq X$ be an open set and $x \in U$. Fix a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$, $f(X \backslash U) = 0$. By assumption, there is a number $t \in (0, 1) \backslash f(X)$. Then $V = f^{-1}(t, 1]$ is a clopen set in X and $x \in V \subset U$. □

In the next result we try to extend the validity of the implication (1) ⇒ (2) to pseudocompact spaces. The cost of our extension is that we replace (1) with a somewhat stronger condition which is, however, equivalent to (1) for normal spaces.

Proposition 5.5 *Let X be a pseudocompact space such that no closed subspace of X admits a continuous mapping onto* $[0, 1]$. *Then X is scattered.*

Proof Striving to a contradiction we assume that X is not scattered. Let $\mathcal{P} = \bigcup_{n\in\omega} 2^n$ be the usual binary tree of height ω, where $2 = \{0, 1\}$. Let us explain the standard relevant notation used below. If $f \in 2^n$, then $f^\frown 0$ and $f^\frown 1$ denote two finite sequences in 2^{n+1} extending f. If $h \in 2^\omega$, then $h{\restriction}n \in 2^n$ denotes the first n elements of the infinite binary sequence h.

Assume that Y is a nonempty closed subset of X without isolated points. We can define by induction a family $\{U_f : f \in \mathcal{P}\}$ of nonempty open subsets of X satisfying the following conditions for all $f \in \mathcal{P}$:

(i) $U_f \cap Y \neq \emptyset$;
(ii) the closures $\overline{U_{f^\frown 0}}$ and $\overline{U_{f^\frown 1}}$ are disjoint;
(iii) $\overline{U_{f^\frown 0}} \cup \overline{U_{f^\frown 1}} \subset U_f$.

We claim that the set $K = \bigcap_{n\in\omega} \bigcup_{f\in 2^n} \overline{U_f}$ admits a continuous mapping onto $[0, 1]$.

Identifying the Cantor set $\mathbb{C}$ with 2^ω, one defines a mapping $p\colon K \to \mathbb{C}$ by letting $p(x) = h$ for each $x \in \bigcap_{n\in\omega} \overline{U_{h\restriction n}}$, where $h \in 2^\omega$. Clearly (ii) implies that for every $x \in K$, there exists a unique element $h \in 2^\omega$ with $x \in \bigcap_{n\in\omega} \overline{U_{h\restriction n}}$. Since X is pseudocompact, it follows from (iii) that the set $\bigcap_{n\in\omega} \overline{U_{h\restriction n}}$ is nonempty for each $h \in 2^\omega$. Hence $p(K) = \mathbb{C}$. We omit a straightforward verification of the continuity of p which follows from (ii) and (iii). It suffices to observe that the closed interval $[0, 1]$ is a continuous image of the Cantor set $\mathbb{C}$. Therefore, the closed set $K \subset X$ admits a continuous mapping onto $[0, 1]$, which contradicts our assumptions. □

In the next theorem we resolve completely Problem 1.3 for σ-compact spaces.

Theorem 5.6 *Let X be a Tychonoff σ-compact space and $X = \bigcup_{n\in\omega} X_n$, where each X_n is compact. Then the following hold*:

(1) *If there is X_n which is not scattered, then the topological group $G(X)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$.*
(2) *If each X_n is scattered and X is not compact, then the topological group $G(X)$ admits an open continuous homomorphism onto the nontrivial discrete group $G(\mathbb{N})$.*
(3) *If each X_n is scattered and X is compact, then the topological group $G(X)$ does not admit a nontrivial metrizable quotient.*

Proof (1) It suffices to apply Proposition 2.3.
(2) X is a Lindelöf space, therefore it is normal. Every scattered compact space X_n is zero-dimensional by [25, 8.5.4], so a normal space X is a countable union of zero-dimensional closed subspaces X_n. Hence $\dim(X) = 0$ by the Countable Sum Theorem [8, 7.2.1]. Finally, X has a base consisting of clopen sets, since X is Lindelöf. Now we observe that X is not pseudocompact, because X is normal and not countably compact. All conditions of Corollary 2.5 are satisfied and hence $G(X)$ admits an open continuous homomorphism onto the nontrivial discrete group $G(\mathbb{N})$.
(3) It is a folklore result stating that a compact σ-scattered space is scattered (apply the Baire property of X). So the required conclusion follows from Theorem 3.1. □

It follows from Proposition 2.2 that there exists an open continuous homomorphism of the free group $G(\mathbb{Q})$ onto the metrizable group of rationals $\mathbb{Q}$. Clearly, $\mathbb{Q}$ is a σ-scattered σ-compact non-discrete topological group. Using the arguments in the proof of Proposition 3.3 one can show that $G(X)$ cannot have a non-discrete metrizable quotient group provided that X is a σ-scattered k_ω-space.

Proposition 5.7 *Let X be a σ-scattered k_ω-space. If a metrizable topological group M is a quotient of $F(X)$ or $A(X)$, then M is discrete and countable.*

Recall that a topological space X is called *K-analytic* if X is a continuous image of a $K_{\sigma\delta}$-subset of a compact space, or, equivalently, X is a continuous image of a Lindelöf Čech-complete space. Evidently, the class of K-analytic spaces contains all σ-compact spaces as a proper subclass.

A topological space X is called k-scattered if every compact subspace of X is scattered. The set $\mathbb{Q}$ of rational numbers is an example of k-scattered but not scattered space.

Theorem 5.8 *For a K-analytic space X the following holds*:

(1) *If X is not k-scattered, then the topological group $G(X)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$.*
(2) *If X is k-scattered and not compact, then the topological group $G(X)$ admits an open continuous homomorphism onto the nontrivial discrete group $G(\mathbb{N})$.*
(3) *If X is scattered and compact, then the topological group $G(X)$ does not admit a nontrivial metrizable quotient.*

Proof (1) It suffices to apply Proposition 2.3.
(2) X is a Lindelöf non-compact space, therefore it is not pseudocompact. It has been shown in [2] that for a K-analytic space X the following conditions are equivalent: (a) X is k-scattered; (b) no continuous map $f : X \to [0, 1]$ is surjective. Hence, by Proposition 5.4, X is zero-dimensional and the required conclusion follows from Corollary 2.5.
(3) Again we make use of Theorem 3.1. □

The class of K-analytic spaces is a proper subclass of Lindelöf Σ-spaces. We recall that the class of Lindelöf Σ-spaces is a minimal class containing all compact and all separable metrizable spaces, and which is closed under taking closed subspaces, countable products and continuous images. Our last question is a special case of the general Problem 1.3.

Problem 5.9 *Characterize Lindelöf Σ-spaces (spaces with a countable network, separable metrizable spaces) X such that the free topological group $F(X)$ and the free abelian topological group $A(X)$ admit a nontrivial metrizable (and necessarily separable) quotient group.*

6 Metrizable spaces

Recall that a subset A of a Polish space is called an *analytic set* if A is a continuous image of a Polish space, or equivalently, a continuous image of the Baire space ω^ω.

Theorem 6.1 *For an analytic set A (in particular, Polish space) the following hold*:

(1) *If A is uncountable, then the topological group $G(A)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$.*
(2) *If A is countable and not compact, then the topological group $G(A)$ admits an open continuous homomorphism onto the nontrivial discrete group $G(\mathbb{N})$.*
(3) *If A is countable and compact, then the topological group $G(A)$ does not admit a nontrivial metrizable quotient.*

Proof By the classical Alexandroff-Hausdorff theorem, every uncountable analytic set A contains a homeomorphic copy of the Cantor set $\mathbb{C}$. Therefore, the uncountable analytic set A admits a quotient continuous mapping onto $\mathbb{I}$. This proves (1), by Proposition 2.3. If a metrizable set A is countable then it is zero-dimensional. For metrizable spaces, compactness and pseudocompactness coincide and we apply Corollary 2.5 implying (2). Item (3) follows from the fact that every countable and compact space is scattered. □

It turns out that a similar trichotomy result is valid for a more general class of absolutely analytic metrizable topological spaces. A metrizable space A is called an *absolutely analytic* if A is homeomorphic to a Souslin subspace of a complete metric space X (of an arbitrary weight), i.e. A is expressible as

$$A = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} A_{\sigma|n},$$

where each $A_{\sigma|n}$ is a closed subset of X. Every Borel subspace of a complete metric space is an absolutely analytic space.

We say that a space A is *strongly σ-discrete* if A is a countable union of its closed discrete subspaces.

Theorem 6.2 *For an absolutely analytic metrizable space A the following hold*:

(1) *If A is not strongly σ-discrete, then the topological group $G(A)$ admits an open continuous homomorphism onto the circle group $\mathbb{T}$.*
(2) *If A is strongly σ-discrete and not compact, then the topological group $G(A)$ admits an open continuous homomorphism onto the nontrivial discrete group $G(\mathbb{N})$.*
(3) *If A is strongly σ-discrete and compact, then the topological group $G(A)$ does not admit a nontrivial metrizable quotient.*

Proof Every absolutely analytic metrizable space A either contains a homeomorphic copy of the Cantor set $\mathbb{C}$ or is strongly σ-discrete, by the main result proved in [7]. It is easy to see that every strongly σ-discrete metrizable space is zero-dimensional. Further, a strongly σ-discrete compact space is scattered and we finish the proof applying a standard scheme. □

Probably one can generalize Theorem 6.2 to non-metrizable spaces with the help of results obtained in [15]. Also notice that for a completely metrizable space A, 'strongly σ-discrete' assumption in Theorem 6.2 can be replaced by 'scattered', as it follows from Theorem 5.2.

7 Concluding remarks

Remark 7.1 In view of results in Sect. 2 one might conjecture that every connected Tychonoff space admits a quotient mapping onto a nontrivial metrizable space. This conjecture is not true since there exists a connected Tychonoff space X such that any quotient mapping from X onto a metrizable space is constant. Indeed, let X be the real line $\mathbb{R}$ equipped with the *density topology*. The density topology τ_d on $\mathbb{R}$ consists of the empty set and the family of all subsets $E \subset \mathbb{R}$ with the property that every $x \in E$ has density 1 with respect to the Lebesgue measure m, that is

$$\lim_{h \to 0} \frac{m(E \cap [x-h, x+h])}{2h} = 1.$$

Every $E \in \tau_d$ is Lebesgue measurable. A set $E \subset \mathbb{R}$ is τ_d-nowhere dense iff it is closed and discrete iff it has Lebesque measure zero. The density topology τ_d is connected, Tychonoff but not normal. Yet, X satisfies the countable chain condition (see, e.g. [28]).

Fact 7.2 *Let $X = (\mathbb{R}, \tau_d)$ and $f: X \to Y$ be a quotient mapping onto a metrizable space Y. Then Y is a singleton.*

Proof Striving to a contradiction, assume that Y contains at least two points. Since X is connected, its continuous image Y is also connected and has the power of the continuum. Y satisfies the countable chain condition, because so does X. But Y is metrizable, so it has a countable base. For every $y \in Y$, let $E_y = f^{-1}(y)$. Then E_y is closed in X and, therefore, measurable. We have a partition of $\mathbb{R}$ into continuum many disjoint measurable sets E_y. Only countably many sets E_y can have a positive measure, so there is a subset $Y' \subset Y$ such that $|Y'| = \mathfrak{c}$ and $m(E_y) = 0$ for each $y \in Y'$. We claim that the subspace Y' of Y is not separable and therefore Y cannot have a countable base. Indeed, if A is a countable subset of Y', then $f^{-1}(A) = \bigcup_{y \in A} E_y$ has Lebesgue measure zero and hence $f^{-1}(A)$ is closed in X. Since f is quotient, the set A is closed in Y. This contradiction completes the proof. □

We do not know, however, whether the group $G(X)$ admits an open continuous homomorphism onto a nontrivial metrizable (necessarily separable) group for $X = (\mathbb{R}, \tau_d)$. Since X maps continuously onto $\mathbb{R}$, evidently $G(X)$ admits a continuous homomorphism onto the group of reals $\mathbb{R}$. One can also show that $G(X)$ has a nontrivial separable quotient group.

Remark 7.3 Clearly, there exists a continuous mapping g of $\mathbb{I}$ onto the countable power of the circle group, $\mathbb{T}^\omega$. This mapping is evidently closed. Therefore, once we have a quotient mapping f from X onto $\mathbb{I}$, we can take a composition $h = g \circ f: X \to \mathbb{T}^\omega$ which is also quotient. Then h extends to an open continuous homomorphism $h^*: G(X) \to \mathbb{T}^\omega$, by Proposition 2.2. By this reason, in all our results stating that the group $G(X)$ admits an open continuous homomorphism onto the group $\mathbb{T}$, one can add the equivalent clause saying that the group $G(X)$ admits an open continuous homomorphism onto the group $\mathbb{T}^\omega$.

We also note that in the recently published article [21] the following result is proved: For every connected compact Hausdorff space X, the topological groups $F(X)$ and $A(X)$ have $\mathbb{T}^\omega$ as a quotient group. In fact this is a very special case of our Theorem 3.1.

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