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ORIGINAL PAPER

On a Brunn–Minkowski inequality for measures with quasi-convex densities

Jesús Yepes Nicolás[1]

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Abstract
In this paper we prove that the classical Brunn–Minkowski inequality holds for product measures on the Euclidean space with quasi-convex densities when considering certain classes of sets that contain, among others, the complements (within a centered box) of unconditional sets. As a consequence, we derive an isoperimetric type inequality.

Keywords Brunn–Minkowski inequality · Quasi-convex density · Product measure · Isoperimetric inequality

Mathematics Subject Classification Primary 52A40 · 28A35; Secondary 26B25

1 Introduction

As usual, we write $\mathbb{R}^n$ to represent the n-dimensional Euclidean space, and we denote by e_i the i-th canonical unit vector. For $i = 1, \dots, n$, we represent by $H_i = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0\}$ the i-th coordinate hyperplane. The n-dimensional volume of a measurable set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$ (when integrating, as usual, $\mathrm{d}x$ will stand for $\mathrm{dvol}(x)$). We write $M(t) = \{x \in \mathbb{R}^{n-1} : (x, t) \in M\}$ for the $(n-1)$-dimensional section at height $t \in \mathbb{R}$ (in the direction of e_n), whereas the orthogonal projection of M onto an i-dimensional linear subspace H is denoted by $M|H$. Moreover, $H^{\perp}$ represents the orthogonal complement of H and, for any $x \in M|H_i$, we set $M_i(x) = \{t \in \mathbb{R} : x + t\mathrm{e}_i \in M\}$ to denote the one-dimensional section of M through the point x in the direction of e_i. Finally, given $r > 0$, rM stands for the set $\{rm : m \in M\}$.

The Minkowski sum of two non-empty sets $A, B \subset \mathbb{R}^n$ is the classical vector addition of them: $A + B = \{a + b : a \in A,\ b \in B\}$. It is natural to wonder about the possibility of bounding the volume of the Minkowski sum of two sets in terms of their volumes; this is the statement of the *Brunn–Minkowski inequality* (for extensive and beautiful surveys on

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✉ Jesús Yepes Nicolás
jesus.yepes@um.es

1 Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain

this inequality we refer the reader to [1,7]). One form of it asserts that if $\lambda \in (0, 1)$ and A and B are non-empty measurable subsets of $\mathbb{R}^n$ such that $(1-\lambda)A + \lambda B$ is also measurable then

$$\mathrm{vol}\big((1-\lambda)A + \lambda B\big)^{1/n} \geq (1-\lambda)\mathrm{vol}(A)^{1/n} + \lambda \mathrm{vol}(B)^{1/n}. \tag{1.1}$$

The Brunn–Minkowski inequality was generalized to different types of measures, including the cases of log-concave measures [10,15] and of p-concave measures (see e.g. [3,4]). It is interesting to note that it was proved by Borell [2,3] that such generalizations would require a p-concavity assumption on the density of the underlying measure (see (2.1) below for the precise definition). As a consequence of this approach (see also [21]), when dealing with arbitrary measurable sets and a Radon measure on $\mathbb{R}^n$, the $(1/n)$-form of the Brunn–Minkowski inequality (1.1) is only true, in general, for the volume (up to a constant). However, when considering some special families of sets (e.g. that of *unconditional sets*), the $(1/n)$-Brunn–Minkowski inequality holds for some types of measures, such as the standard Gaussian measure, which is given by

$$\mathrm{d}\gamma_n(x) = \frac{1}{(2\pi)^{n/2}} e^{\frac{-|x|^2}{2}} \mathrm{d}x$$

(see e.g. [8,11,12,14,16]). Furthermore, for the family of *C-coconvex sets* (complements of closed convex sets, of positive and finite volume, within a pointed closed convex cone with non-empty interior C), a "complemented" version of the Brunn–Minkowski inequality (1.1) holds for the volume (see [9,19]), namely

$$\mathrm{vol}\big(C\backslash((1-\lambda)K + \lambda L)\big)^{1/n} \leq (1-\lambda)\mathrm{vol}(C\backslash K)^{1/n} + \lambda \mathrm{vol}(C\backslash L)^{1/n}$$

for all $\lambda \in (0, 1)$. And again, this (complemented) Brunn–Minkowski inequality can be also generalized for certain general measures (see [13]).

To complete the picture, one may ask about possible p-convexity conditions on the density of the underlying measure. Among others, what can be said about the measure ν_n on $\mathbb{R}^n$ given by

$$\mathrm{d}\nu_n(x) = e^{|x|^2} \mathrm{d}x,$$

whose density is log-convex? In [13], when dealing with measures involving certain log-convex functions as part of their densities, the authors showed another type of complemented Brunn–Minkowski inequality. Nevertheless, not much more seems to be known regarding Brunn–Minkowski inequalities for log-convex densities or, more generally, quasi-convex densities (see (2.2) below for the precise definition).

To this regard, and inspired by the above-mentioned (complemented) Brunn–Minkowski inequalities, it is natural to wonder whether one may find certain classes of sets for which a measure on $\mathbb{R}^n$ of the kind of ν_n satisfies the $(1/n)$-form of the Brunn–Minkowski inequality. Here we give a positive answer to this question, by showing that it is enough to consider *congruous sets* (see Definition 2.1): a family that contains, among others, the complements of unconditional sets within a centered box (cf. Example 2.1). This is the content of the following result, in the more general setting of product measures with quasi-convex densities (with minimum at the origin).

Theorem 1.1 *Let* $\mu = \mu_1 \otimes \cdots \otimes \mu_n$ *be a product measure on* $\mathbb{R}^n$ *such that* μ_i *is the measure given by* $\mathrm{d}\mu_i(x) = \phi_i(x)\,\mathrm{d}x$*, where* $\phi_i : \mathbb{R} \longrightarrow [0, \infty)$ *is quasi-convex with* $\phi_i(0) = \min_{x\in\mathbb{R}} \phi_i(x)$*, for all* $i = 1, \ldots, n$*.*

Let $\lambda \in (0,1)$ *and let* $A, B \subset \mathbb{R}^n$ *be non-empty measurable congruous sets such that* $(1-\lambda)A + \lambda B$ *is also measurable. Then*

$$\mu\big((1-\lambda)A + \lambda B\big)^{1/n} \geq (1-\lambda)\mu(A)^{1/n} + \lambda\mu(B)^{1/n}. \tag{1.2}$$

Section 2 is mainly devoted to showing this result. Finally, in Sect. 3, we derive an isoperimetric type inequality as a consequence of (1.2).

2 Proof of the main result

2.1 Background

We recall that a function $\phi : \mathbb{R}^n \longrightarrow [0,\infty)$ is p-concave, for $p \in \mathbb{R} \cup \{\pm\infty\}$, if

$$\phi\big((1-\lambda)x + \lambda y\big) \geq M_p\big(\phi(x), \phi(y), \lambda\big) \tag{2.1}$$

for all $x, y \in \mathbb{R}^n$ such that $\phi(x)\phi(y) > 0$ and any $\lambda \in (0,1)$. Here M_p denotes the *p-mean* of two non-negative numbers a, b:

$$M_p(a,b,\lambda) = \begin{cases} \big((1-\lambda)a^p + \lambda b^p\big)^{1/p}, & \text{if } p \neq 0, \pm\infty, \\ a^{1-\lambda}b^{\lambda} & \text{if } p = 0, \\ \max\{a,b\} & \text{if } p = \infty, \\ \min\{a,b\} & \text{if } p = -\infty. \end{cases}$$

A 0-concave function is usually called *log-concave* whereas a $(-\infty)$-concave function is called *quasi-concave*. Quasi-concavity is equivalent to the fact that the superlevel sets $\{x \in \mathbb{R}^n : \phi(x) \geq t\}$ are convex for all $t \in [0,1]$.

On the other side of the coin, one is led to p-convex functions, where $p \in \mathbb{R} \cup \{\pm\infty\}$, i.e., those functions satisfying

$$\phi\big((1-\lambda)x + \lambda y\big) \leq M_p\big(\phi(x), \phi(y), \lambda\big) \tag{2.2}$$

for all $x, y \in \mathbb{R}^n$ and all $\lambda \in (0,1)$. Again, 0-convex functions are referred to as *log-convex* whereas ∞-convex functions are called *quasi-convex*.

Now we define a new class of (pairs of) sets that will play a relevant role throughout this paper.

Definition 2.1 Let $A, B \subset \mathbb{R}^n$ be non-empty bounded sets. For $n = 1$, we say that A and B are *congruous* if one of the following assertions holds.

(i) $A \cap (-\infty, 0), B \cap (-\infty, 0) = \emptyset$ and $\max(A) = \max(B)$.
(ii) $A \cap (0, \infty), B \cap (0, \infty) = \emptyset$ and $\min(A) = \min(B)$.
(iii) $A \cap (0, \infty), B \cap (0, \infty), A \cap (-\infty, 0), B \cap (-\infty, 0) \neq \emptyset$, $\min(A) = \min(B)$ and $\max(A) = \max(B)$.

For $n \geq 2$, we say that A and B are *congruous* if, for any $i = 1, \ldots, n$, the sets $A_i(x)$ and $B_i(y)$ are congruous for all $x \in A|H_i$ and all $y \in B|H_i$.

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We notice that the fact that, for any $i = 1, \ldots, n$, the sets $A_i(x)$ and $B_i(y)$ are congruous (for all $x \in A|H_i$ and all $y \in B|H_i$) does not mean that the same condition in Definition 2.1 holds for all i (see Fig. 1; there $A_2(x)$, $B_2(x')$ satisfy condition (iii) of Definition 2.1, for all

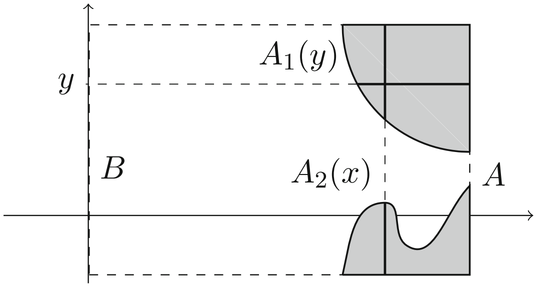

Fig. 1 The congruous sets A (in gray) and B (the box), with the sections $A_2(x)$, $A_1(y)$ for given $x \in A|H_2$, $y \in A|H_1$

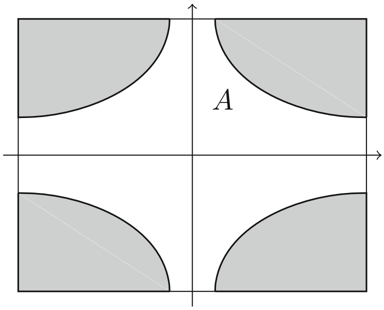

Fig. 2 A set A (in gray) contained in a centered box P such that $P\backslash A$ is unconditional

$x \in A|H_2$ and all $x' \in B|H_2$, whereas $A_1(y)$, $B_1(y')$ fulfil condition (i), for any $y \in A|H_1$ and any $y' \in B|H_1$).

Unconditional convex sets are of particular interest in convexity, also regarding Brunn–Minkowski type inequalities (see e.g. [11,18]). A subset $A \subset \mathbb{R}^n$ is said to be unconditional (not necessarily convex) if for every $(x_1, \dots, x_n) \in A$ and every $(\epsilon_1, \dots, \epsilon_n) \in [-1, 1]^n$ one has $(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in A$. As announced before, the family of congruous sets contains certain complements of unconditional sets:

Example 2.1 Let $P = \prod_{i=1}^n [-\alpha_i, \alpha_i]$, $\alpha_i > 0$ for $i = 1, \dots, n$, be a centered orthogonal compact box and let $A, B \subset P$ be non-empty compact sets such that $P\backslash A$, $P\backslash B$ are unconditional. Then A and B are congruous. Indeed, from the unconditionality of $P\backslash A$ and $P\backslash B$, we have that $\max\bigl(A_i(x)\bigr) = \max\bigl(B_i(y)\bigr) = \alpha_i$ and $\min\bigl(A_i(x)\bigr) = \min\bigl(B_i(y)\bigr) = -\alpha_i$, for all $x \in A|H_i$ and all $y \in B|H_i$; thus $A_i(x)$ and $B_i(y)$ are congruous for any $i = 1, \dots, n$ since they satisfy condition (iii) in Definition 2.1 (see Fig. 2).

The following result is well-known in the literature (see e.g. the one-dimensional case of [6, Theorem 4.1] and the references therein. Regarding its statement, and following the notation used in [6], we notice that for a quasi-concave function $\phi : \mathbb{R} \longrightarrow [0, \infty)$ we have $(1-\lambda)\phi\chi_{_A} \star_{-\infty} \lambda\phi\chi_{_B} = \phi\chi_{_{(1-\lambda)A+\lambda B}}$, where $\chi_{_M}$ denotes the characteristic function of the set $M \subset \mathbb{R}$).

Lemma 2.1 *Let μ be the measure on $\mathbb{R}$ given by $\mathrm{d}\mu(x) = \phi(x)\mathrm{d}x$, where $\phi : \mathbb{R} \longrightarrow [0, \infty)$ is quasi-concave with $\phi(0) = \max_{x\in\mathbb{R}} \phi(x)$. Let $\lambda \in (0, 1)$ and let $A, B \subset \mathbb{R}$ be measurable sets with $0 \in A \cap B$. Then*

$$\mu(C) \geq (1-\lambda)\mu(A) + \lambda\mu(B)$$

for any measurable set C such that $C \supset (1-\lambda)A + \lambda B$.

As a consequence of such a Brunn–Minkowski inequality for quasi-concave densities on $\mathbb{R}$, we will obtain the one-dimensional Brunn–Minkowski inequality for measures associated to quasi-convex functions when working with congruous sets. This is the content of Lemma 2.2.

2.2 Proof

We start this subsection by showing the one-dimensional case of our main result, Theorem 1.1.

Lemma 2.2 *Let μ be the measure on $\mathbb{R}$ given by $\mathrm{d}\mu(x) = \phi(x)\mathrm{d}x$, where $\phi : \mathbb{R} \longrightarrow [0, \infty)$ is quasi-convex with $\phi(0) = \min_{x\in\mathbb{R}} \phi(x)$. Let $\lambda \in (0, 1)$ and let $A, B \subset \mathbb{R}$ be non-empty measurable congruous sets. Then*

$$\mu(C) \geq (1-\lambda)\mu(A) + \lambda\mu(B)$$

for any non-empty measurable set C such that $C \supset (1-\lambda)A + \lambda B$.

Proof Let A and B satisfy condition (iii) in Definition 2.1. Assuming that the result is true if either (i) or (ii) (of Definition 2.1) holds, it is enough to consider $A^+, A^-, B^+, B^-, C^+, C^-$ where, for any $M \subset \mathbb{R}$, the sets M^+ and M^- stand for $M^+ = M \cap (0, \infty)$ and $M^- = M \cap (-\infty, 0)$. Indeed, applying the result to the sets A^+, B^+, C^+ and A^-, B^-, C^-, respectively, we have

$$\begin{aligned}(1-\lambda)\mu(A) + \lambda\mu(B) &= (1-\lambda)\mu(A^+) + \lambda\mu(B^+) + (1-\lambda)\mu(A^-) + \lambda\mu(B^-)\\ &\leq \mu(C^+) + \mu(C^-) = \mu(C).\end{aligned}$$

Moreover, we note that the function $\bar{\phi} : \mathbb{R} \longrightarrow [0, \infty)$ given by $\bar{\phi}(x) = \phi(-x)$ is quasi-convex (and, clearly, $\bar{\phi}(0) = \min_{x\in\mathbb{R}} \bar{\phi}(x)$). Thus, considering if necessary $\bar{A} = -A$, $\bar{B} = -B$, $\bar{C} = -C$, and the measure $\bar{\mu}$ with density $\bar{\phi}$, it is enough to prove the result for congruous sets satisfying (i). Now, the quasi-convexity of ϕ implies that $\phi(x) \leq \max\{\phi(0), \phi(y)\} = \phi(y)$ for any $0 < x < y$. This shows that ϕ is increasing on $(0, \infty)$ and then $\phi \cdot \chi_{(0,\infty)}$ is quasi-concave. Thus, setting $x_0 = \max(A) = \max(B)$, the result follows from applying Lemma 2.1 to the function $\psi : \mathbb{R} \longrightarrow [0, \infty)$ given by $\psi(x) = \phi(x + x_0) \cdot \chi_{(-\infty,0]}(x)$ and the sets $A - x_0$, $B - x_0$, $C - x_0$. □

As stated in Theorem 1.1, the above result extends to dimension n. The approach we follow here is based on the underlying idea of [16, Theorem 1.3], and it goes back to some classical proofs of functional versions of the Brunn–Minkowski inequality (such as the *Prékopa-Leindler inequality*) and other related results.

Proof of Theorem 1.1 For the sake of brevity we write $C = (1-\lambda)A + \lambda B$ and, given $t_1, t_2 \in \mathbb{R}$, $t_\lambda = (1-\lambda)t_1 + \lambda t_2$. We also set $\bar{\mu} = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_{n-1}$ (i.e., $\mu = \bar{\mu} \otimes \mu_n$).

Since μ is inner regular (i.e., $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ compact}\}$ for any measurable set A), we may assume, without loss of generality, that A and B are compact. Indeed, given sequences of compact sets $(K_n)_{n\in\mathbb{N}}$, $(L_n)_{n\in\mathbb{N}}$ that approximate from inside the congruous sets A and B, respectively, one may clearly consider certain sequences of congruous compact sets $(K'_n)_{n\in\mathbb{N}}$, $(L'_n)_{n\in\mathbb{N}}$ such that $\mu(K'_n) = \mu(K_n)$ and $\mu(L'_n) = \mu(L_n)$, for all $n \in \mathbb{N}$. In fact, it is enough to add to K_n and L_n, respectively, the projections $\big(A|H_i\big)$ and $\big(B|H_i\big)$, located at the appropriate height(s) in the direction of e_i, for $i = 1, \ldots, n$.

Moreover, we observe that we may assume that $\mu(A)\mu(B) > 0$. Indeed, the case in which one of the sets, say B, has measure zero whereas the other one, A, has positive measure can be obtained (cf. [16, Proposition 2.7]) by applying the positive measures case to A and the following set: let P be an orthogonal compact box congruous with B (and so, with A) and let C_m be a decreasing sequence of (unions of) boxes, which are congruous with B, that shrinks (as $m \to \infty$) to the subset of vertices of P that belong to B; then we take $B_m = B \cup C_m$, which is also congruous with A for all $m \in \mathbb{N}$. We note that this congruence ensures that the points in the limit case belong to B, and hence $\bigcap_{m\in\mathbb{N}}\bigl((1-\lambda)A+\lambda B_m\bigr) = (1-\lambda)A+\lambda B$. Taking into account that

$$\mu\left(\bigcap_{m\in\mathbb{N}}\bigl((1-\lambda)A+\lambda B_m\bigr)\right) = \lim_m \mu\bigl((1-\lambda)A+\lambda B_m\bigr),$$

we get (1.2).

We then show the result by (finite) induction on the dimension n. The case $n = 1$ is just Lemma 2.2. So, we suppose that $n \geq 2$ and that the inequality is true for dimension $n-1$. The sets $A(t_1)$, $B(t_2)$, for $t_1, t_2 \in \mathbb{R}$ such that $t_1\mathrm{e}_n \in A|H_n^{\perp}$, $t_2\mathrm{e}_n \in B|H_n^{\perp}$, are clearly congruous and thus, applying the induction hypothesis (i.e., (1.2) in $\mathbb{R}^{n-1}$ for $\bar{\mu}$) together with the fact that $C(t_\lambda) \supset (1-\lambda)A(t_1)+\lambda B(t_2)$, we have

$$\bar{\mu}\bigl(C(t_\lambda)\bigr) \geq \left((1-\lambda)\bar{\mu}\bigl(A(t_1)\bigr)^{1/(n-1)} + \lambda\bar{\mu}\bigl(B(t_2)\bigr)^{1/(n-1)}\right)^{n-1}. \tag{2.3}$$

Now, we take the non-negative functions $f, g, h : \mathbb{R} \longrightarrow [0,\infty)$ given by

$$f(t) = \frac{\bar{\mu}(A(t))}{|\bar{\mu}(A(\cdot))|_\infty},\ g(t) = \frac{\bar{\mu}(B(t))}{|\bar{\mu}(B(\cdot))|_\infty},\ h(t) = \frac{\bar{\mu}(C(t))}{c},$$

where

$$c = \left((1-\lambda)\,|\bar{\mu}(A(\cdot))|_\infty^{1/(n-1)} + \lambda\,|\bar{\mu}(B(\cdot))|_\infty^{1/(n-1)}\right)^{n-1}.$$

We notice that the above functions are well-defined: denominators are positive since $\mu(A)\mu(B) > 0$, and they are finite because $A|H_{n-1}$ and $B|H_{n-1}$ are compact sets and $\bar{\mu}$ is locally finite. Furthermore, $\sup_{t\in\mathbb{R}} f(t) = \sup_{t\in\mathbb{R}} g(t) = 1$.

Using (2.3), and setting $\theta = \dfrac{\lambda\,|\bar{\mu}(B(\cdot))|_\infty^{1/(n-1)}}{c^{1/(n-1)}} \in (0,1)$, we get

$$\begin{aligned}\bar{\mu}\bigl(C(t_\lambda)\bigr) &\geq \left((1-\lambda)\bar{\mu}\bigl(A(t_1)\bigr)^{1/(n-1)} + \lambda\bar{\mu}\bigl(B(t_2)\bigr)^{1/(n-1)}\right)^{n-1}\\ &= c\left((1-\theta)f(t_1)^{1/(n-1)} + \theta g(t_2)^{1/(n-1)}\right)^{n-1}\\ &\geq c\,\min\bigl\{f(t_1), g(t_2)\bigr\}.\end{aligned}$$

This shows that $h((1-\lambda)t_1+\lambda t_2) \geq \min\{f(t_1), g(t_2)\}$ for any $t_1, t_2 \in \mathbb{R}$, which clearly implies that

$$\{t\in\mathbb{R} : h(t) \geq s\} \supset (1-\lambda)\{t\in\mathbb{R} : f(t)\geq s\} + \lambda\{t\in\mathbb{R} : g(t)\geq s\} \tag{2.4}$$

for all $s \in [0,1)$. Moreover, since $A_n(x)$ and $B_n(y)$ are congruous for all $x \in A|H_n$ and all $y \in B|H_n$ then the superlevel sets $\{t\in\mathbb{R} : f(t)\geq s\}$ and $\{t\in\mathbb{R} : g(t)\geq s\}$ are also congruous for any $s \in [0,1)$. Indeed, assuming without loss of generality that $A_n(x)$, $B_n(y)$ satisfy condition (i) of Definition 2.1, for all $x \in A|H_n$ and all $y \in B|H_n$, then there exists $s_0 > 0$ such that $\bigl(A|H_n\bigr)+s_0\mathrm{e}_n \subset A$, $\bigl(B|H_n\bigr)+s_0\mathrm{e}_n \subset B$ and $A, B \subset [0, s_0\mathrm{e}_n]+H_n$. Hence, both f and g attain their maximum at s_0 and vanish on $(-\infty,0)\cup(s_0,\infty)$, which implies that their superlevel sets satisfy condition (i) of Definition 2.1 and thus are congruous.

Therefore, we may apply Lemma 2.2 to get

$$\mu_n\big(\{t \in \mathbb{R} : h(t) \geq s\}\big) \geq (1-\lambda)\mu_n\big(\{t \in \mathbb{R} : f(t) \geq s\}\big) + \lambda\mu_n\big(\{t \in \mathbb{R} : g(t) \geq s\}\big)$$

for any $s \in [0, 1)$. This, together with Fubini's theorem and the Cavalieri Principle

$$\int_{\mathbb{R}} \psi(x)\,\mathrm{d}\mu_n(x) = \int_0^{|\psi|_\infty} \mu_n\big(\{t \in \mathbb{R} : \psi(t) \geq s\}\big)\,\mathrm{d}s$$

for $\psi = f, g, h$, jointly with the fact that $|h|_\infty \geq 1 = |f|_\infty = |g|_\infty$ (cf. (2.4)), allows us to obtain

$$\begin{aligned}\mu\big((1-\lambda)A + \lambda B\big) &= c\int_{\mathbb{R}} h(x)\,\mathrm{d}\mu_n(x)\\ &\geq c\left((1-\lambda)\int_{\mathbb{R}} f(x)\,\mathrm{d}\mu_n(x) + \lambda\int_{\mathbb{R}} g(x)\,\mathrm{d}\mu_n(x)\right)\\ &= c\left((1-\lambda)\frac{\mu(A)}{|\bar{\mu}(A(\cdot))|_\infty} + \lambda\frac{\mu(B)}{|\bar{\mu}(B(\cdot))|_\infty}\right).\end{aligned}$$

And then, applying the (reverse) Hölder inequality (see e.g. [5, Theorem 1, page 178]),

$$a_1b_1 + a_2b_2 \geq \left(a_1^p + a_2^p\right)^{1/p}\left(b_1^q + b_2^q\right)^{1/q},$$

with parameters $p = 1/n$ and $q = -1/(n-1)$, and taking $a_1 = (1-\lambda)^{1/p}\mu(A)$, $a_2 = \lambda^{1/p}\mu(B)$, $b_1 = (1-\lambda)^{1/q}\,|\bar{\mu}(A(\cdot))|_\infty^{-1}$ and $b_2 = \lambda^{1/q}\,|\bar{\mu}(B(\cdot))|_\infty^{-1}$, we conclude that

$$\mu\big((1-\lambda)A + \lambda B\big) \geq \left((1-\lambda)\mu(A)^{1/n} + \lambda\mu(B)^{1/n}\right)^n,$$

as desired. □

3 A remark on an isoperimetric inequality

Given a set $M \subset \mathbb{R}^n$, let $\operatorname{pos} M$ and $\operatorname{int} M$ denote, respectively, the positive hull and interior of M. Moreover, let $\varepsilon_1, \ldots, \varepsilon_{2^n}$ denote the elements of $\{-1, 1\}^n$. Then, setting $\varepsilon_j = (\varepsilon_1^j, \ldots, \varepsilon_n^j)$ for any $j = 1, \ldots, 2^n$, we write

$$O_j = \operatorname{pos}\{\varepsilon_1^j \mathrm{e}_1, \ldots, \varepsilon_n^j \mathrm{e}_n\}$$

for the corresponding orthant of $\mathbb{R}^n$.

Along this section, we deal with certain sets contained in an orthogonal compact box (which, for the sake of simplicity, will be assumed to be centered): fixing a box $P = \prod_{i=1}^n [-\alpha_i, \alpha_i]$, with $\alpha_i > 0$ for all i, we consider unions of orthants of unconditional compact convex sets 'embedded' in the corners of P. More precisely, such a set A satisfies that, for all $j = 1, \ldots, 2^n$,

$$A \cap O_j = x_j + (K_j \cap (-O_j)) \tag{3.1}$$

for some unconditional compact convex set $K_j \subset \operatorname{int} P$ (cf. Fig. 3), where $x_j = (\varepsilon_1^j\alpha_1, \ldots, \varepsilon_n^j\alpha_n)$ is the corresponding vertex of P. In the following, for the sake of brevity, we will write $A_j = A \cap O_j$.

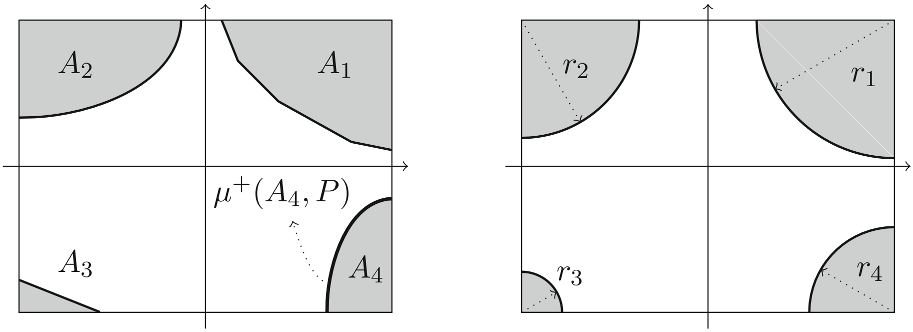

Fig. 3 Union of orthants A_j of unconditional compact convex sets (left) and the corresponding orthants of balls $x_j + r_j(B_n \cap (-O_j))$ of the same measure (right)

As in the Euclidean setting, we will obtain an isoperimetric type inequality as a consequence of (1.2). To this aim, we introduce some notation. Let

$$\mathrm{W}_1^{\mu}(A; B) = \frac{1}{n} \liminf_{t \to 0^+} \frac{\mu(A + tB) - \mu(A)}{t}$$

be the first *quermassintegral* of A with respect to the set B associated to the measure μ. Here we assume that A and B are measurable sets such that $A + tB$ is also measurable for all $t \geq 0$.

In a similar way, and denoting by B_n the n-dimensional Euclidean (closed) unit ball, we may define

$$\mu^+(A) = \liminf_{t \to 0^+} \frac{\mu(A + tB_n) - \mu(A)}{t},$$

the surface area measure associated to μ, i.e., its (lower) Minkowski content. Clearly, $\mu^+(A) = n\mathrm{W}_1^{\mu}(A; B_n)$. The relative Minkowski content of a set $A \subset \mathbb{R}^n$ with respect to a second set $\Omega \subset \mathbb{R}^n$ is defined by

$$\mu^+(A, \Omega) = \liminf_{t \to 0^+} \frac{\mu\big((A + tB_n) \cap \Omega\big) - \mu(A \cap \Omega)}{t}.$$

Moreover, given $x \in \mathbb{R}^n$, we set

$$M^{\mu}(x, A) = n\mu(x + A) - \left.\frac{\mathrm{d}^-}{\mathrm{d}t}\right|_{t=1} \mu(x + tA),$$

provided that $((x, A), \mu)$ is so that the above (left) derivative exists. When dealing with a set $A \subset \mathbb{R}^n$ satisfying (3.1) for all $j = 1, \ldots, 2^n$, we also write $M^{\mu}(A) = \sum_{j=1}^{2^n} M_j^{\mu}(A_j)$, where $M_j^{\mu}(A_j) = M^{\mu}(x_j, K_j \cap (-O_j))$. We notice that, from the convexity of $K_j \cap (-O_j)$ and using Theorem 1.1, the function $t \mapsto \mu(x_j + t(K_j \cap (-O_j)))^{1/n}$ is (increasing and) concave on $(0, 1]$ for any product measure μ in the conditions of the latter result. This implies that the left derivative of $\mu(x_j + t(K_j \cap (-O_j)))$ at $t = 1$ (possibly infinite) exists (cf. [17, Theorem 23.1]) and hence, for all $j = 1, \ldots, 2^n$, $M_j^{\mu}(A_j)$ (and so $M^{\mu}(A)$) is well-defined. Clearly, $M^{\mathrm{vol}}(A) = 0$ for such a set A and thus this functional does not appear in the classical isoperimetric inequality. For more information about this functional, we refer the reader to [11,16] and the references therein.

Now we show an isoperimetric type inequality for unions of orthants of unconditional compact convex sets embedded in the corners of a fixed orthogonal box, in the setting of product measures with quasi-convex densities. This a straightforward consequence of the following result for (such) a sole orthant.

Theorem 3.1 *Let $\mu = \mu_1 \otimes \cdots \otimes \mu_n$ be a product measure on $\mathbb{R}^n$ such that μ_i is the measure given by $\mathrm{d}\mu_i(x) = \phi_i(x)\,\mathrm{d}x$, where $\phi_i : \mathbb{R} \longrightarrow [0,\infty)$ is quasi-convex with $\phi_i(0) = \min_{x\in\mathbb{R}} \phi_i(x)$, for all $i = 1,\dots,n$.*

Let $P = \prod_{i=1}^n [-\alpha_i, \alpha_i]$, with $\alpha_i > 0$ for all i and let $K \subset \operatorname{int} P$ be a non-empty unconditional compact convex set. Let $A = x_1 + (K \cap (-O_1))$, where $x_1 = (\alpha_1,\dots,\alpha_n)$ and $O_1 = \mathrm{pos}\{\mathrm{e}_1,\dots,\mathrm{e}_n\}$. Then, for any $r > 0$ such that $rB_n \subset \operatorname{int} P$,

$$r\mu^+(A,P) + M^\mu(x_1, K_1 \cap (-O_1)) \geq n\mu(A)^{1-1/n}\mu\bigl(x_1 + (rB_n \cap (-O_1))\bigr)^{1/n},$$

with equality if $A = x_1 + (rB_n \cap (-O_1))$.

Following the same argument for any orthant A_j of a non-empty set $A \subset P$ satisfying (3.1) for all $j = 1,\dots,2^n$, we get that, for any $r_1,\dots,r_{2^n} > 0$ such that $r_jB_n \subset \operatorname{int} P$ for all j, we have

$$\sum_{j=1}^{2^n}\Bigl(r_j\mu^+(A_j,P) + M_j^\mu(A_j)\Bigr) \geq n\sum_{j=1}^{2^n}\mu(A_j)^{1-1/n}\mu\bigl(x_j + (r_jB_n \cap (-O_j))\bigr)^{1/n},$$

with equality if $A_j = x_j + (r_jB_n \cap (-O_j))$ for all $j = 1,\dots,2^n$.

The particular case $r_1 = \cdots = r_{2^n} (=: r)$ of the latter inequality shows that

$$r\mu^+(A,P) + M^\mu(A) \geq n\sum_{j=1}^{2^n}\mu(A_j)^{1-1/n}\mu\bigl(x_j + (rB_n \cap (-O_j))\bigr)^{1/n}.$$

In other words: among all unions A of orthants of unconditional compact convex sets embedded in the corners of a fixed centered orthogonal box P (i.e., satisfying (3.1) for all $j = 1,\dots,2^n$) *with predetermined measure* $\mu(A_j) = \mu(x_j + (rB_n \cap (-O_j)))$, (union of orthants embedded in the corners of P of) *Euclidean balls rB_n minimize the functional* $r\mu^+(A,P) + M^\mu(A)$.

The main idea of the proof we present here goes back to the classical proof of the Minkowski first inequality that can be found in [20, Theorem 7.2.1]. We refer also the reader to [16, Sect. 4] and the references therein.

Proof We consider $L = rB_n$ and we denote by $B = x_1 + L^-$, where $L^- = L \cap (-O_1)$. In the same way, we will write $K^- = K \cap (-O_1)$.

Notice that, for any $\epsilon > 0$ such that $K^- + \epsilon L^- \subset P$, we have that $x_1 + K^- + t_1L^-$ and $x_1 + K^- + t_2L^-$ are congruous for all $t_1, t_2 \in [0,\epsilon]$ (since each one-dimensional section of them in the direction of e_i, $i = 1,\dots,n$, satisfies condition (i) in Definition 2.1, with maximum equal to α_i). Then, from the convexity of L^- (and K^-) and using Theorem 1.1, the function $t \mapsto \mu(A + tL^-)^{1/n}$ is concave on $[0,\epsilon]$. This implies that the right derivative of $\mu(A + tL^-)$ at $t = 0$ (possibly infinite) exists (cf. [17, Theorem 23.1]). Similarly, the left derivative of $\mu(x_1 + tK^-)$ at $t = 1$ exists.

Now, we consider the function $f : [0,1] \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$f(t) = \mu\bigl((1-t)A + tB\bigr)^{1/n} - \bigl((1-t)\mu(A)^{1/n} + t\mu(B)^{1/n}\bigr).$$

By Theorem 1.1 (and from the convexity of both K^- and L^-) f is concave (we notice that the fact of being an unconditional set is closed under convex combinations) and, moreover, $f(0) = f(1) = 0$. Thus, the right derivative of f at $t = 0$ exists and furthermore

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} f(t) \geq 0 \tag{3.2}$$

with equality if and only if $f(t) = 0$ for all $t \in [0, 1]$, i.e., if and only if (1.2) holds with equality for all $t \in [0, 1]$.

Now, since

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} f(t) = \frac{1}{n}\mu(A)^{(1/n)-1}\frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} \mu\big((1-t)A + tB\big) + \mu(A)^{1/n} - \mu(B)^{1/n},$$

we just must compute the right derivative at 0 of $\mu\big((1-t)A + tB\big)$. Writing $g(r, s) = \mu\big(x_1 + r(K^- + sL^-)\big)$, we have

$$\begin{aligned}\frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} \mu\big((1-t)A + tB\big) &= \frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} g\left(1-t, \frac{t}{1-t}\right)\\ &= -\frac{\mathrm{d}^-}{\mathrm{d}t}\bigg|_{t=1} \mu(x_1 + tK^-) + \frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} \mu(A + tL^-)\\ &= M^\mu(x_1, K^-) - n\mu(A) + n\mathrm{W}_1^\mu(A; L^-),\end{aligned}$$

and thus

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} f(t) = \frac{1}{n}\mu(A)^{(1/n)-1}\big(M^\mu(x_1, K^-) + n\mathrm{W}_1^\mu(A; L^-)\big) - \mu(B)^{1/n}.$$

Hence, the latter identity, together with (3.2), gives

$$\mathrm{W}_1^\mu(A; L^-) + \frac{1}{n}M^\mu(x_1, K^-) \geq \mu(A)^{1-1/n}\mu(B)^{1/n},$$

with equality if $A = B$.

Finally, from the unconditionality of K^- we clearly have that $\big((A + tL) \cap P\big) = A + tL^-$, which yields $n\mathrm{W}_1^\mu(A; L^-) = r\mu^+(A, P)$. Then, we have

$$r\mu^+(A, P) + M^\mu(x_1, K_1 \cap (-O_1)) \geq n\mu(A)^{1-1/n}\mu\big(x_1 + (rB_n)^-\big)^{1/n},$$

with equality if $A = x_1 + (rB_n \cap (-O_1))$. This concludes the proof. □

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