Bohr's phenomenon for the classes of Quasi-subordination and K-quasiregular harmonic mappings

Ming-Sheng Liu[1] · Saminathan Ponnusamy[2] · Jun Wang[3]

Abstract

In this paper, we investigate the Bohr radius for K-quasiregular sense-preserving harmonic mappings $f = h + \overline{g}$ in the unit disk $\mathbb{D}$ such that the translated analytic part $h(z) - h(0)$ is quasi-subordinate to some analytic function. The main aim of this article is to extend and to establish sharp versions of four recent theorems by Liu and Ponnusamy (Bull Malays Math Sci Soc 42:2151–2168, 2019) and, in particular, we settle affirmatively the two conjectures proposed by them. Furthermore, we establish two refined versions of Bohr's inequalities and determine the Bohr radius for the derivatives of analytic functions associated with quasi-subordination.

1 Introduction

Throughout we let $\mathcal{B}$ denote the class of all analytic functions ω in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $|\omega(z)| \leq 1$ for all $z \in \mathbb{D}$, and let $\mathcal{B}_0 = \{\omega \in \mathcal{B} : \omega(0) = 0\}$. Bohr's inequality says that if $f \in \mathcal{B}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then, for the majorant series $M_f(r) = \sum_{k=0}^{\infty} |a_k| r^k$ of f, we have

✉ Saminathan Ponnusamy
  samy@iitm.ac.in

  Ming-Sheng Liu
  liumsh65@163.com

  Jun Wang
  majwang@fudan.edu.cn

[1] School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China

[2] Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India

[3] School of Mathematical Sciences, Fudan University, Shanghai 200433, China

$$M_{f_0}(r) = \sum_{k=1}^{\infty} |a_k| r^k \le 1 - |a_0| = \text{dist}(f(0), \partial\mathbb{D})$$

for all $z \in \mathbb{D}$ with $|z| = r \le \frac{1}{3}$, where $f_0(z) = f(z) - f(0)$. This inequality was discovered by Bohr [12]. Bohr actually obtained the inequality for $|z| \le \frac{1}{6}$. Later, Wiener, Riesz and Schur, independently established the inequality for $|z| \le \frac{1}{3}$ and showed that $1/3$ is sharp. See [5], [19, Chapter 8] and the references therein. Few other proofs are also available in the literature. The Bohr radius has been discussed for certain power series in $\mathbb{D}$, as well as for analytic functions from $\mathbb{D}$ into specific domains, such as convex domains, simply connected domains, the punctured unit disk, the exterior of the closed unit disk, and concave wedge-domains. The analogous Bohr radius has also been studied for harmonic and starlike log-harmonic mappings in $\mathbb{D}$. In particular, in [22], the authors settled the conjecture of Ali et al. [6] about the Bohr radius for odd functions from $\mathcal{B}$. In the year 2000, powered Bohr inequality was initiated by Djakov and Ramanujan [17] and a conjecture related to their work was settled in [24] affirmatively. Bohr's idea naturally extends to functions of several complex variables [2,4,5,11,17]. Several other aspects of Bohr's inequality may be obtained from [3,5,7,9,13–16,19,26,29,31–33] and the references therein.

In this article, we shall consider Bohr's radius for complex-valued K-quasiregular harmonic mappings of the unit disk $\mathbb{D}$. In order to state our main results, we recall the following notions and notations (see [7,36]).

Definition 1 Let $f(z)$ and $g(z)$ be analytic in $\mathbb{D}$. We say that

1. $f(z)$ is subordinate to $g(z)$ in $\mathbb{D}$, written by $f(z) \prec g(z)$ or $f \prec g$, if there exists an $\omega \in \mathcal{B}_0$ such that $f(z) = g(\omega(z))$ for $z \in \mathbb{D}$. Furthermore, if $g(z)$ is univalent in $\mathbb{D}$, then we have the following relation

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0), \ f(\mathbb{D}) \subset g(\mathbb{D}).$$

2. $f(z)$ is majorized by $g(z)$ in $\mathbb{D}$, denoted by $f(z) \ll g(z)$ or $f \ll g$, if $|f(z)| \le |g(z)|$ for $z \in \mathbb{D}$.
3. f is quasi-subordinate to g (relative to Φ), denoted by $f(z) \prec_q g(z)$ in $\mathbb{D}$, if there exists a $\Phi \in \mathcal{B}$ and an $\omega \in \mathcal{B}_0$ such that $f(z) = \Phi(z)g(\omega(z))$ for $z \in \mathbb{D}$.

Evidently if either $f \prec g$ or $|f(z)| \le |g(z)|$ in $\mathbb{D}$, then $f(z) \prec_q g(z)$ in $\mathbb{D}$. Thus, the notion of quasi-subordination generalizes both the concept of subordination and the principle of majorization.

A harmonic mapping in $\mathbb{D}$ is a complex-valued function $f = u + iv$ of $z = x + iy$ in $\mathbb{D}$, which satisfies the Laplace equation $\triangle f = 4f_{z\bar{z}} = 0$. It follows that every harmonic mapping f admits a representation of the form $f = h + \bar{g}$, where h and g are analytic in D. This representation is unique up to an additive constant. It is convenient to assume that $f(0) = g(0)$. The Jacobian J_f of f is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$.

We say that f is sense-preserving in $\mathbb{D}$ if $J_f(z) > 0$ in $\mathbb{D}$. Consequently, a harmonic mapping f is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $J_f(z) > 0$ in $\mathbb{D}$; or equivalently if $h' \ne 0$ in $\mathbb{D}$ and the dilatation $\omega_f := \frac{g'}{h'}$ of f has the property that $|\omega_f| < 1$ in $\mathbb{D}$ [27].

If a locally univalent and sense-preserving harmonic mapping $f = h + \bar{g}$ satisfies the condition $|\omega_f(z)| \le k < 1$, then f is called K-quasiregular harmonic mapping on $\mathbb{D}$, where $K = \frac{1+k}{1-k} \ge 1$ (cf. [21,30]). Obviously $k \to 1$ corresponds to the limiting case $K \to \infty$. Note that when $k = 1$, the condition on the dilatation of f becomes $|\omega_f(z)| \le 1$ in which

case the Jacobian could be zero at some point. Thus, it is worth pointing out that our results below cover this case as well as the case where f is sense-preserving. A harmonic extension of the classical Bohr theorem was established in [1,23,25].

Definition 2 We say that $f = h + \overline{g} \in H_{K,\,h \prec_q \varphi}(\mathbb{D})$ if it is a K-quasiregular sense-preserving harmonic mapping of $\mathbb{D}$ and has the power series form

$$f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \quad z \in \mathbb{D},$$

together with an additional condition that $h \prec_q \varphi$, where $k = \frac{K-1}{K+1}$. If $h \prec \varphi$, then we simply write $H_{K,\,h \prec \varphi}(\mathbb{D})$ instead of $H_{K,\,h \prec_q \varphi}(\mathbb{D})$ by suppressing the subscript 'q'.

Similarly, we can define $H_{K,\,h \ll \varphi}(\mathbb{D})$ by replacing quasi-subordination condition $h \prec_q \varphi$ by the majorization condition $h \ll \varphi$.

Recently, Liu and Ponnusamy [28] have investigated the class $H_{K,\,h \prec \varphi}(\mathbb{D})$ and obtained the following results.

Theorem A [28] *For $f = h + \overline{g} \in H_{K,\,h \prec \varphi}(\mathbb{D})$ with $k = (K-1)/(K+1)$, we have the following:*

(1) *If φ is analytic and univalent in $\mathbb{D}$, then*

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leq \mathrm{dist}\,(\varphi(0), \partial\varphi(\mathbb{D})) \tag{1.1}$$

for $|z| = r \leq r_u$, where $r_u = r_u(k)$ is the root of the equation

$$(1-r)^2 - 4r(1 + k\sqrt{1+r}) = 0$$

in the interval $(0, 1)$.

(2) *If φ is univalent and convex in $\mathbb{D}$, then (1.1) holds for $|z| = r \leq \frac{K+1}{5K+1}$. The result is sharp.*

(3) *If φ is univalent and convex in $\mathbb{D}$ and $b_1 = g'(0) = 0$, then (1.1) holds for $|z| = r \leq r_c$, where $r_c = r_c(k)$ is the positive root of the equation*

$$\frac{r}{1-r} + \frac{kr^2}{1-r^2}\sqrt{\left(\frac{1+r^2}{1-r^2}\right)\left(\frac{\pi^2}{6} - 1\right)} = \frac{1}{2}.$$

The number $r_c(k)$ cannot be replaced by a number larger than $\rho := \rho_c(k)$, where ρ is the positive root of the equation

$$\frac{2(1+k)\rho}{1-\rho} + 2k\ln(1-\rho) = 1. \tag{1.2}$$

(4) *If φ is analytic and univalent in $\mathbb{D}$, $h(0) = 0$, and $b_1 = g'(0) = 0$, then (1.1) holds for $|z| = r \leq r_s$, where $r_s = r_s(k)$ is the positive real root of the equation*

$$\frac{r}{(1-r)^2} + \frac{kr^2}{(1-r^2)^2}\sqrt{\left(\frac{r^6 + 11r^4 + 11r^2 + 1}{1-r^2}\right)\left(\frac{\pi^2}{6} - 1\right)} = \frac{1}{4}$$

in the interval $(0, 1)$. The number $r_s(k)$ cannot be replaced by a number larger than $\rho = \rho_s(k)$, where ρ is the positive root of the equation

$$\frac{\rho(1 - k + 2k\rho)}{(1-\rho)^2} - k\ln(1-\rho) = \frac{1}{4}. \tag{1.3}$$

One of the important special cases is when $K \to \infty$, i.e. $k \to 1$. Thus, the authors in [28] proposed the following two conjectures.

Conjecture B *Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} b_n z^n}$ is a sense-preserving harmonic mapping in $\mathbb{D}$ and $h \prec \varphi$.*

(a) *If φ is univalent and convex in $\mathbb{D}$, then*

$$\sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=2}^{\infty} |b_n| r^n \leq \text{dist}(\varphi(0), \partial\varphi(\mathbb{D})) \tag{1.4}$$

for $|z| = r \leq \rho_c = 0.299823\cdots$, where ρ_c is the positive root of the equation

$$\frac{4r}{1-r} + 2\ln(1-r) = 1,$$

(compare with (1.2) with $k = 1$).

(b) *If φ is univalent in $\mathbb{D}$, then the inequality (1.4) holds for $|z| = r \leq \rho_s = 0.161353\cdots$, where ρ_s is the positive real root of the equation*

$$\frac{2r^2}{(1-r)^2} - \ln(1-r) = \frac{1}{4}$$

(compare with (1.3) with $k = 1$).

One of the aims of this article is to prove sharp versions of Theorem A(3) and (4) which in turn imply that Conjecture B is true. In fact, we prove these in a general setting along with the sharp version of Theorem A(1).

Inspired by Theorem A and the notion of quasi-subordination in the setting of Bohr's inequality, as discussed in [7], we obtain the following results.

Theorem 1 *Let $f = h + \overline{g} \in H_{K, h_0 \prec_q \varphi_0}(\mathbb{D})$, $k = \frac{K-1}{K+1}$, $h_0(z) = h(z) - h(0)$ and $\varphi_0(z) = \varphi(z) - \varphi(0)$. We have the following:*

(1) *If φ is analytic and univalent in $\mathbb{D}$, then*

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leq \text{dist}(\varphi(0), \partial\varphi(\mathbb{D})) \tag{1.5}$$

for $|z| = r \leq r_u$, where $r_u = r_u(k) = \frac{1}{2k+3+\sqrt{(2k+3)^2-1}} \in (0, 1/3)$. The result is sharp.

(2) *If φ is univalent and convex in $\mathbb{D}$, then (1.5) holds for $|z| = r \leq \frac{K+1}{5K+1}$. The result is sharp.*

Remark 1 The conclusion of Theorem 1(1) continues to hold under the assumption that $f = h + \overline{g} \in H_{K, h \prec \varphi}(\mathbb{D})$ and thus, Theorem 1 contains a sharp version of Theorem A(1) or [28, Theorem 3], namely, with the subordination $h_0 \prec \varphi_0$ (which is equivalent to $h \prec \varphi$) in place of $h_0 \prec_q \varphi_0$. Note that in the case of quasi-subordination, $h_0 \prec_q \varphi_0$ is not equivalent to $h \prec_q \varphi$ unless $h(0) = \varphi(0) = 0$. In particular, if we set $k = 0$ (i.e. $K = 1$) in the case of subordination, then we get the result of Abu-Muhanna [1, Theorem 1] as a special case of Theorem 1.

Remark 2 Recall that the notion of quasi-subordination generalizes both the concept of sub-ordination and the principle of majorization and thus, Theorem 1(2) is an extension of

Theorem A(2) or [28, Theorem 1]. In particular, the conclusion of Theorem 1(2) holds if we replace the condition $h_0 \prec_q \varphi_0$ by the majorization condition $|h_0(z)| \leq |\varphi_0(z)|$ for $z \in \mathbb{D}$.

Theorem 2 *Let* $f = h + \overline{g} \in H_{K,\, h \prec \varphi}(\mathbb{D})$, $b_1 = g'(0) = 0$, *and* $k = \frac{K-1}{K+1}$. *We have the following:*

(1) *If* φ *is univalent and convex in* $\mathbb{D}$, *then*

$$\sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=2}^{\infty} |b_n| r^n \leq \text{dist}\, (\varphi(0), \partial\varphi(\mathbb{D})) \tag{1.6}$$

for $|z| = r \leq r_c$, *where* $r_c = r_c(k)$ *is the unique positive root in* $(0, 1/3)$ *of the equation*

$$\frac{2(1+k)r}{1-r} + 2k \ln(1-r) = 1. \tag{1.7}$$

The result is sharp.

(2) *If* φ *is analytic and univalent in* $\mathbb{D}$, *then* (1.6) *holds for* $|z| = r \leq r_s$, *where* $r_s = r_s(k)$ *is the unique positive root in* $(0, 1/3)$ *of the equation*

$$\frac{r(1-k+2kr)}{(1-r)^2} - k \ln(1-r) = \frac{1}{4}. \tag{1.8}$$

The result is sharp.

Remark 3 Theorem 2(1) is the sharp version of Theorem A(3) or [28, Theorem 2]. Also, Theorem 2(2) is the sharp version of Theorem A(4) or [28, Theorem 4]. Setting $k = 0$ in Theorem 2(1), we also get the classical version of the Bohr theorem. Finally, we remark that the whole proof of Theorem 2 can be imitated to establish Conjecture B only by replacing k with 1.

The paper is organized as follows. In Sect. 3, we present the proof of Theorems 1 and 2. In Sect. 4, we state and prove two theorems which extend two recent results of Ponnusamy et al. [34,35] from the case of analytic functions to the case of sense-preserving harmonic mappings. Finally, in Sect. 5, we investigate the Bohr radius for the derivatives of analytic functions in the setting of quasi-subordination.

2 Preliminaries

In order to establish Theorems 1 and 2, we need the following lemmas. It is easy to obtain the following two well-known lemmas from the latest monograph of Avkhadiev and Wirths [8]. See also [18, pp. 195–196] and [1,20].

Lemma C *Let* φ *be an analytic univalent map from* $\mathbb{D}$ *onto a simply connected domain* $\Omega = \varphi(\mathbb{D})$. *We have the following:*

(1) $\frac{1}{4}|\varphi'(0)| \leq \text{dist}\, (\varphi(0), \partial\Omega) \leq |\varphi'(0)|.$

(2) *If* $g(z) = \sum_{n=0}^{\infty} b_n z^n \prec \varphi(z)$, *then* $|b_n| \leq n|\varphi'(0)| \leq 4n\, \text{dist}\, (\varphi(0), \partial\Omega).$

Lemma D *Let* φ *be an analytic univalent map from* $\mathbb{D}$ *onto a convex domain* $\Omega = \varphi(\mathbb{D})$. *We have the following:*

(1) $\dfrac{1}{2}|\varphi'(0)| \le \mathrm{dist}\,(\varphi(0), \partial\Omega) \le |\varphi'(0)|.$

(2) If $g(z) = \sum_{n=0}^{\infty} b_n z^n \prec \varphi(z)$, then $|b_n| \le |\varphi'(0)| \le 2\mathrm{dist}\,(\varphi(0), \partial\Omega).$

The following two lemmas will play a key role in the proofs of our main results in Sect. 3.

Lemma E (Alkhaleefah et al. [7]) *Let $f(z)$ and $g(z)$ be two analytic functions in $\mathbb{D}$ with the Taylor series expansions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ for $z \in \mathbb{D}$. We have the following:*

(1) *If $f(z) \prec_q g(z)$ in $\mathbb{D}$, then*

$$\sum_{n=0}^{\infty} |a_n| r^n \le \sum_{n=0}^{\infty} |b_n| r^n \quad \text{for all } r \le \frac{1}{3}.$$

(2) *If $|g'(z)| \le k|h'(z)|$ in $\mathbb{D}$ for some $k \in (0, 1]$, then*

$$\sum_{n=1}^{\infty} |b_n| r^n \le k \sum_{n=1}^{\infty} |a_n| r^n \quad \text{for all } r \le \frac{1}{3}.$$

Proof The proof of the first part of Lemma E is available in [7] while the second part follows easily from this. Indeed, by assumption, we obtain that $g'(z) \prec_q kh'(z)$ which quickly gives from Lemma E(1) that

$$\sum_{n=1}^{\infty} n|b_n| r^{n-1} \le \sum_{n=1}^{\infty} kn\,|a_n| r^{n-1} \quad \text{for all } r \le \frac{1}{3}$$

and integrating this with respect to r gives the desired inequality. $\qquad\square$

3 The proofs of Theorems 1 and 2

3.1 Proof of Theorem 1

Assume that $f = h + \overline{g} \in H_{K,\,h_0 \prec_q \varphi_0}(\mathbb{D})$. Then $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$, where $|g'(z)| \le k|h'(z)|$ in $\mathbb{D}$ for some $k \in [0, 1)$ and $h(z) - h(0) \prec_q \varphi(z) - \varphi(0)$ in $\mathbb{D}$.

(1) By assumption, $\varphi(z) = \sum_{n=0}^{\infty} c_n z^n$ is analytic and univalent in $\mathbb{D}$. Now, by Lemmas C and E(1), we obtain respectively the inequalities

$$|c_n| \le n|\varphi'(0)| \le 4n\,\mathrm{dist}\,(\varphi(0), \partial\varphi(\mathbb{D})), \quad \text{for } n = 1, 2, \ldots,$$

and

$$\sum_{n=1}^{\infty} |a_n| r^n \le \sum_{n=1}^{\infty} |c_n| r^n \quad \text{for } r \le \frac{1}{3},$$

which by the previous inequality leads to

$$\sum_{n=1}^{\infty} |a_n| r^n \le 4\mathrm{dist}(\varphi(0), \partial\varphi(\mathbb{D})) \frac{r}{(1-r)^2} \quad \text{for all } r \le \frac{1}{3}. \tag{3.1}$$

Again, as $|g'(z)| \leq k|h'(z)|$, Lemma E(2) and (3.1) show that

$$\sum_{n=1}^{\infty} |b_n| r^n \leq k \sum_{n=1}^{\infty} |a_n| r^n \leq 4k \text{dist}(\varphi(0), \partial\varphi(\mathbb{D})) \frac{r}{(1-r)^2} \quad \text{for all } r \leq \frac{1}{3}.$$

Consequently, by combining the last two inequalities, we have

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leq 4(1+k) \text{dist}(\varphi(0), \partial\varphi(\mathbb{D})) \frac{r}{(1-r)^2}$$

which is less than or equal to $\text{dist}(\varphi(0), \partial\varphi(\mathbb{D}))$ if and only if

$$4(1+k)r - (1-r)^2 = (r - r_u)(2k + 3 + \sqrt{(2k+3)^2 - 1} - r) \leq 0.$$

The last inequality holds for $r \leq r_u$, where $r_u = r_u(k) = \frac{1}{2k+3+\sqrt{(2k+3)^2-1}} \in (0, 1/3)$ and $k = \frac{K-1}{K+1}$.

To prove the sharpness, we consider $f = h + \overline{g}$ such that $g'(z) = k\lambda h'(z)$, where $\lambda \in \mathbb{D}$,

$$h(z) = a_0 + \frac{z}{(1-z)^2} \quad \text{and} \quad \varphi(z) = c_0 + \frac{z}{(1-z)^2} = c_0 + \sum_{n=1}^{\infty} nz^n.$$

Then $\text{dist}(\varphi(0), \partial\varphi(\mathbb{D})) = 1/4$ and so it is easy to see that

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n = (1+k|\lambda|) \frac{r}{(1-r)^2}$$

which is bigger than or equal to $1/4$ if and only if $4(1+k|\lambda|)r - (1-r)^2 \geq 0$. Solving the last inequality shows that the number $r_u = \frac{1}{2k+3+\sqrt{(2k+3)^2-1}}$ cannot be improved since $|\lambda|$ could be chosen so close to 1 from left. This completes the proof of the first part (1).

(2) For the proof of the second part, we just need to assume that φ is convex and then proceed with the above method of proof, but using Lemma D in place of Lemma C. This change after minor computation leads to

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leq 2(1+k) \text{dist}(\varphi(0), \partial\varphi(\mathbb{D})) \frac{r}{1-r}$$

which is less than or equal to $\text{dist}(\varphi(0), \partial\varphi(\mathbb{D}))$ for $r \leq \frac{1}{3+2k} \leq \frac{1}{3}$. Substituting $k = \frac{K-1}{K+1}$ gives the desired result.

Again, to prove the sharpness, we consider $f = h + \overline{g}$ such that

$$h(z) = a_0 + \frac{z}{1-z}, \quad \varphi(z) = c_0 + \frac{z}{1-z} = c_0 + \sum_{n=1}^{\infty} z^n,$$

and $g'(z) = k\lambda h'(z)$, where $\lambda \in \mathbb{D}$. Then a simple computation yields

$$\text{dist}(\varphi(0), \partial\varphi(\mathbb{D})) = \frac{1}{2} \quad \text{and} \quad g(z) = k\lambda \frac{z}{1-z}$$

so that for this function we have

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n = (1+k|\lambda|) \frac{r}{1-r}$$

which is bigger than or equal to $1/2$ if and only if

$$r \geq \frac{1}{3 + 2k|\lambda|} = \frac{K + 1}{3K + 3 + 2|\lambda|(K - 1)}.$$

This shows that the number $\frac{K+1}{5K+1}$ cannot be improved since $|\lambda|$ could be chosen so close to 1 from left. This completes the proof of the second part.   □

3.2 Proof of Theorem 2

Suppose that $f = h + \overline{g} \in H_{K, h \prec \varphi}(\mathbb{D})$. We consider the first part of the theorem, where $b_1 = g'(0) = 0$, $h \prec \varphi$ and φ is univalent and convex in $\mathbb{D}$. It follows from Lemma D(2) that $|a_n| \leq |\varphi'(0)|$ for $n \geq 1$, and thus

$$\sum_{n=1}^{\infty} |a_n| r^n \leq |\varphi'(0)| \frac{r}{1 - r}. \tag{3.2}$$

Because $g'(0) = 0$, by Schwarz's lemma, we obtain that $\omega = \frac{g'}{h'}$ is analytic in $\mathbb{D}$ and $|\omega(z)| \leq k|z|$ in $\mathbb{D}$. Therefore, we have $|g'(z)| \leq |kzh'(z)|$, or $g'(z) \prec_q kzh'(z)$ in $\mathbb{D}$.

By Lemma E(1), we have

$$\sum_{n=1}^{\infty} n|b_n| r^{n-1} \leq \sum_{n=1}^{\infty} kn|a_n| r^n \leq k|\varphi'(0)| \sum_{n=1}^{\infty} nr^n = k|\varphi'(0)| \frac{r}{(1-r)^2} \quad \text{for } r \leq \frac{1}{3}.$$

Integrate this inequality on $[0, r]$, where $r \leq 1/3$, we obtain

$$\sum_{n=1}^{\infty} |b_n| r^n \leq k|\varphi'(0)| \int_0^r \frac{t}{(1-t)^2} \, dt = k|\varphi'(0)| \left(\ln(1 - r) + \frac{r}{1 - r} \right).$$

Consequently, by combining (3.2) with the last inequality, we find that

$$\sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leq |\varphi'(0)| \left[\frac{r}{1 - r} + k \left(\ln(1 - r) + \frac{r}{1 - r} \right) \right]$$

$$\leq 2\text{dist}(\varphi(0), \partial\varphi(\mathbb{D})) \left[\frac{(1 + k)r}{1 - r} + k \ln(1 - r) \right]$$

$$\leq \text{dist}(\varphi(0), \partial\varphi(\mathbb{D})),$$

where the last inequality holds if and only if

$$\frac{2(1 + k)r}{1 - r} + 2k \ln(1 - r) \leq 1.$$

The above inequality holds for $r \leq r_c(k)$, where $r_c(k)$ is the unique positive root in $(0, 1/3)$ of Eq. (1.7). In order to verify the fact, we let

$$F(r) = \frac{2(1 + k)r}{1 - r} + 2k \ln(1 - r) - 1.$$

Then $F(r)$ is continuous in $[0, 1/3]$,

$$F(0) = -1 < 0, \quad F(1/3) = k \ln \frac{4e}{9} > 0, \quad F'(r) = \frac{2 + 2kr}{(1 - r)^2} > 0 \text{ for } r \in [0, 1/3],$$

and therefore, it follows from the intermediate value theorem that the equation (1.7) has a unique root $r_c = r_c(k)$ in $(0, 1/3)$.

To prove the sharpness, we consider $f = h + \overline{g}$, where h, g and φ are such that $g'(z) = kzh'(z)$ and $h(z) = \varphi(z) = 1/(1-z)$. Then for these choices we find that $\text{dist}(\varphi(0), \partial\varphi(\mathbb{D})) = 1/2$ and it is easy to compute the corresponding sum

$$\sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=1}^{\infty} |b_n|r^n = \frac{(1+k)r}{1-r} + k\ln(1-r),$$

which is less than or equal to $1/2$ only for $r \leq r_c(k)$, where $r_c(k)$ is the unique positive root in $(0, 1/3)$ of the Eq. (1.7). This shows that the number $r_c(k)$ cannot be improved. This completes the proof of the first part of the theorem.

Now we consider the second part, where φ is analytic and univalent in $\mathbb{D}$. It follows from Lemma C that $|a_n| \leq |\varphi'(0)|n \leq 4n\,\text{dist}(\varphi(0), \partial\varphi(\mathbb{D}))$ for $n \geq 1$, and thus

$$\sum_{n=1}^{\infty} |a_n|r^n \leq |\varphi'(0)|\frac{r}{(1-r)^2}. \tag{3.3}$$

As in the proof of the previous case, we have

$$\sum_{n=1}^{\infty} n|b_n|r^{n-1} \leq \sum_{n=1}^{\infty} kn|a_n|r^n \leq k|\varphi'(0)| \sum_{n=1}^{\infty} n^2 r^n = k|\varphi'(0)|\frac{r(1+r)}{(1-r)^3} \quad \text{for } r \leq \frac{1}{3}$$

and thus, by integration, we obtain easily that

$$\sum_{n=1}^{\infty} |b_n|r^n \leq k|\varphi'(0)| \left(\frac{2r^2 - r}{(1-r)^2} - \ln(1-r) \right) \quad \text{for } r \leq \frac{1}{3}.$$

Consequently, by combining (3.3) with the last inequality, we find that

$$\sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=1}^{\infty} |b_n|r^n \leq |\varphi'(0)| \left[\frac{r}{(1-r)^2} + k\left(\frac{2r^2 - r}{(1-r)^2} - \ln(1-r) \right) \right]$$

$$\leq 4\text{dist}(\varphi(0), \partial\varphi(\mathbb{D})) \left[\frac{r(1-k+2kr)}{(1-r)^2} - k\ln(1-r) \right]$$

$$\leq \text{dist}(\varphi(0), \partial\varphi(\mathbb{D})),$$

where the last inequality holds if and only if

$$\frac{r(1-k+2kr)}{(1-r)^2} - k\ln(1-r) \leq \frac{1}{4},$$

which holds for $r \leq r_s(k)$, where $r_s(k)$ is the unique positive root in $(0, 1/3)$ of Eq. (1.8)—a fact which is easy to verify as in the proof of the case (1.7) above.

To prove the sharpness, we consider the function $f = h + \overline{g}$, where

$$h(z) = \varphi(z) = \frac{z}{(1-z)^2} = a_0 + \sum_{n=1}^{\infty} nz^n,$$

and $g'(z) = kzh'(z)$. Then we find that $\text{dist}(\varphi(0), \partial\varphi(\mathbb{D})) = \frac{1}{4}$, and as before we have

$$\sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=2}^{\infty} |b_n|r^n = \sum_{n=1}^{\infty} nr^n + k\sum_{n=2}^{\infty} \left(n + \frac{1}{n} - 2 \right)r^n = \frac{r(1-k+2kr)}{(1-r)^2} - k\ln(1-r),$$

which is less than or equal to 1/4 only in the case where $r \leq r_s(k)$, where $r_s(k)$ is the unique positive root in $(0, 1/3)$ of Eq. (1.8). This shows that the number $r_s(k)$ cannot be improved. The proof of the theorem is complete.   □

4 Improved Bohr's phenomenon associated with quasi-subordination

Recently, Ponnusamy et al. [34,35] established several refined versions of Bohr's inequality in the case of bounded analytic functions. In this section, following the ideas of [34,35], we will discuss improved Bohr's phenomenon for two classes of sense-preserving harmonic mappings associated with quasi-subordination.

Theorem 3 *Suppose that* $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ *is a sense-preserving harmonic mapping in* $\mathbb{D}$ *and* $h(z) - h(0) \prec_q \varphi(z) - \varphi(0)$ *in* $\mathbb{D}$, *where* $\varphi(z)$ *is univalent and convex in* $\mathbb{D}$. *Also, let* $\lambda = \mathrm{dist}(\varphi(0), \partial\varphi(\mathbb{D})) < 1$ *and* $\|f_0\|_r = \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)r^{2n}$, *where* $f_0(z) = f(z) - f(0)$. *Then*

$$T_f(r) := \sum_{n=1}^{\infty}(|a_n| + |b_n|)r^n + \left(\frac{1}{2-\lambda} + \frac{r}{1-r}\right)\|f_0\|_r \leq \lambda \quad \text{for } |z| = r \leq r_*,$$

where $r_* \approx 0.15867508$ *is the unique root in* $(0, 1)$ *of equation*

$$5r^3 - 9r^2 - 5r + 1 = 0.$$

Moreover, for any $\lambda \in (0, 1)$, *there exists a uniquely defined* $r_0 \in (r_*, \frac{1}{5})$ *such that* $T_f(r) \leq \lambda$ *for* $r \in [0, r_0]$. *The radius* r_0 *can be calculated as the solution of the equation*

$$\Phi(\lambda, r) = 8r^3\lambda^2 - (13r^3 + 7r^2 - 5r + 1)\lambda + 10r^3 - 2r^2 - 10r + 2 = 0.$$

Proof Let $\varphi(z) = \sum_{n=0}^{\infty} c_n z^n$. Then, as before, Lemma D implies that $|c_n| \leq |\varphi'(0)| \leq 2\lambda$ for $n \geq 1$. Because $h(z) - h(0) \prec_q \varphi(z) - \varphi(0)$ and $|g'(z)| \leq |f'(z)|$ for $z \in \mathbb{D}$, we have

$$\sum_{n=1}^{\infty}|b_n|r^n \leq \sum_{n=1}^{\infty}|a_n|r^n \leq \sum_{n=1}^{\infty}|c_n|r^n \quad \text{for } r \leq \frac{1}{3}.$$

In addition we also have

$$\sum_{n=1}^{\infty}|a_n|^2 r^{2n} \leq \sum_{n=1}^{\infty}|c_n|^2 r^{2n} =: \|\varphi_0\|_r \quad \text{and} \quad \sum_{n=1}^{\infty}n^2|b_n|^2 r^{2n-2} \leq \sum_{n=1}^{\infty}n^2|a_n|^2 r^{2n-2}$$

so that

$$\sum_{n=1}^{\infty}|b_n|^2 r^{2n} \leq \sum_{n=1}^{\infty}|a_n|^2 r^{2n} \leq \sum_{n=1}^{\infty}|c_n|^2 r^{2n} = \|\varphi_0\|_r.$$

Consequently, $\|f_0\|_r \leq 2\|\varphi_0\|_r$ and thus, $T_f(r) \leq 2T_\varphi(r)$ for $r \leq 1/3$ only, where

$$T_\varphi(r) = \sum_{n=1}^{\infty}|c_n|r^n + \left(\frac{1}{2-\lambda} + \frac{r}{1-r}\right)\|\varphi_0\|_r.$$

Clearly, the desired conclusion follows if we can show that $T_\varphi(r) \le \lambda/2$. Finally, because $|c_n| \le 2\lambda$ for $n \ge 1$, we have

$$2T_\varphi(r) \le 4\lambda \sum_{n=1}^{\infty} r^n + \left(\frac{1}{2-\lambda} + \frac{r}{1-r}\right) 8\lambda^2 \sum_{n=1}^{\infty} r^{2n}$$

$$= \frac{4\lambda r}{1-r} + \frac{1+(1-\lambda)r}{(2-\lambda)(1-r)} \cdot \frac{8\lambda^2 r^2}{1-r^2}$$

$$= \lambda - \lambda \left[\frac{1-5r}{1-r} - \frac{8\lambda r^2(1+(1-\lambda)r)}{(2-\lambda)(1-r)(1-r^2)}\right]$$

$$= \lambda - \lambda \cdot \frac{\Phi(\lambda, r)}{(2-\lambda)(1-r)(1-r^2)},$$

where

$$\Phi(\lambda, r) = (1-5r)(2-\lambda)(1-r^2) - 8\lambda r^2(1+(1-\lambda)r)$$
$$= 8r^3\lambda^2 - \left[(1-5r)(1-r^2) + 8r^2(1+r)\right]\lambda + 2(1-5r)(1-r^2).$$

It is easy to see that $\Phi(\lambda, r) < 0$ for $r > 1/5$ and $0 < \lambda \le 1$, so that

$$\lambda - \lambda \cdot \frac{\Phi(\lambda, r)}{(2-\lambda)(1-r)(1-r^2)} > \lambda$$

for $r > 1/5$ and $0 < \lambda \le 1$.

We claim that $\Phi(\lambda, r) \ge 0$ for every $r \le r_*$ and for $\lambda \in (0, 1]$. In fact, we have

$$\frac{\partial^2 \Phi(\lambda, r)}{\partial \lambda^2} = 16r^3 \ge 0 \quad \text{for every } \lambda \in (0, 1],$$

and thus $\frac{\partial \Phi(\lambda, r)}{\partial \lambda}$ is an increasing function of λ. This gives

$$\frac{\partial \Phi(\lambda, r)}{\partial \lambda} \le \frac{\partial \Phi}{\partial \lambda}(1, r) = 16r^3 - (1-5r)(1-r^2) - 8r^2(1+r)$$
$$= -(1-r)^2(1-3r)$$

so that $\Phi(\lambda, r)$ is a decreasing function of λ on $(0, 1]$ for $r \le 1/3$ which implies that

$$\Phi(\lambda, r) \ge \Phi(1, r) = 5r^3 - 9r^2 - 5r + 1,$$

which is greater than or equal to 0 for all $r \le r_*$, where $r_* \approx 0.15867508$ is the unique root of equation $5r^3 - 9r^2 - 5r + 1 = 0$, which lies in $(0, \frac{1}{5})$.

Since $\Phi(0, r) = 2(1-5r)(1-r^2)$, we have $\Phi(0, r) \ge 0$ for $r \le 1/5$ and $\Phi(0, r) < 0$ for $r > 1/5$. Furthermore,

$$\Phi'(1, r) = 15r^2 - 18r - 5 = 15r(r-1) - 3r - 5 < 0$$

which implies that $\Phi(1, r) \ge 0$ for $r \le r_*$ and $\Phi(1, r) < 0$ for $r > r_*$. According to the fact that $\Phi(\lambda, r)$ is a monotonic decreasing function of λ on $(0, 1]$ for $r \le \frac{1}{3}$, we see that for any $r \in (r_*, \frac{1}{5})$, $\Phi(0, r) \ge 0$, $\Phi(1, r) < 0$, there is a uniquely defined $\lambda(r) \in (0, 1)$ such that $\Phi(\lambda(r), r) = 0$.

To prove the last assertion, we have to show that $\frac{d\lambda(r)}{dr} < 0$. Indeed, since

$$\frac{d\lambda(r)}{dr} = -\frac{\frac{\partial \Phi(\lambda, r)}{\partial r}}{\frac{\partial \Phi(\lambda, r)}{\partial \lambda}},$$

it is sufficient to prove that

$$\frac{\partial \Phi(\lambda, r)}{\partial r} = 24r^2\lambda^2 - (39r^2 + 14r - 5)\lambda + 30r^2 - 4r - 10 < 0$$

for $\lambda \in (0, 1]$ and $r \in (r_*, \frac{1}{5})$.

To that end, we use that for the intervals in question the inequalities

$$\begin{cases} 30r^2 - 4r - 10 < -8, \text{ and} \\ 24r^2\lambda^2 - (39r^2 + 14r)\lambda + 5\lambda - 8 < 24r^2\lambda^2 - 24r^2\lambda - (15r^2 + 14r)\lambda < 0 \end{cases}$$

are valid. This completes the proof of Theorem 3   □

Theorem 4 *Assume the hypotheses of Theorem 3 with a relaxed condition on φ, namely, that $\varphi(z)$ is analytic and univalent in $\mathbb{D}$. Then*

$$T_f(r) = \sum_{n=1}^{\infty}(|a_n| + |b_n|)r^n + \left(\frac{1}{2-\lambda} + \frac{r}{1-r}\right)\|f_0\|_r \leq \lambda$$

for $|z| = r \leq r_u^*$, where $r_u^* \approx 0.0808958838$ is the unique root in $(0, 1)$ of the equation

$$(1 - 10r + r^2)(1 - r)^2(1 + r)^3 - 32r^2(1 + r^2) = 0. \tag{4.1}$$

Proof Following the notation and the method of proof of Theorem 3, we easily have $|c_n| \leq 4n\lambda$ for $n \geq 1$ and $T_f(r) \leq 2T_\varphi(r)$, where

$$2T_\varphi(r) \leq 8\lambda \sum_{n=1}^{\infty} nr^n + \left(\frac{1}{2-\lambda} + \frac{r}{1-r}\right)32\lambda^2 \sum_{n=1}^{\infty} n^2 r^{2n}$$

$$= \frac{8\lambda r}{(1-r)^2} + \frac{1 + (1-\lambda)r}{(2-\lambda)(1-r)} \cdot \frac{32\lambda^2 r^2(1+r^2)}{(1-r^2)^3}$$

$$= \lambda - \lambda \left[\frac{(1-r)^2 - 8r}{(1-r)^2} - \frac{32\lambda r^2(1+r^2)(1+(1-\lambda)r)}{(2-\lambda)(1-r)(1-r^2)^3}\right]$$

$$= \lambda - \lambda \cdot \frac{\Psi(\lambda, r)}{(2-\lambda)(1-r)(1-r^2)^3}.$$

Here

$$\Psi(\lambda, r) = (1 - 10r + r^2)(2 - \lambda)(1 - r)^2(1 + r)^3 - 32\lambda r^2(1 + r^2)(1 + (1-\lambda)r).$$

To complete the proof, it suffices to show that $2T_\varphi(r) \leq \lambda$ for $r \leq r_u^*$.

We claim that $\Psi(\lambda, r) \geq 0$ for every $r \leq r_u^* \approx 0.0808958838$ and for $\lambda \in (0, 1]$. In fact, we have

$$\frac{\partial^2 \Psi(\lambda, r)}{\partial \lambda^2} = 64r^3(1 + r^2) \geq 0 \quad \text{for every } \lambda \in (0, 1],$$

and thus $\frac{\partial \Psi(\lambda, r)}{\partial \lambda}$ is an increasing function of λ. This gives

$$\frac{\partial \Psi(\lambda, r)}{\partial \lambda} \leq \frac{\partial \Psi}{\partial \lambda}(1, r) = -(1-r)\left[32r^2(1 + r^2) + (1 - 10r + r^2)(1 - r)(1 + r)^3\right]$$

$$= -(1-r)(1 - 8r + 13r^2 + 51r^4 + 8r^5 - r^6) \leq 0$$

for $r \le \frac{1}{5}$. Indeed, let $F(r) = 1 - 8r + 13r^2 + 51r^4 + 8r^5 - r^6$. Then we only need to prove $F(r) \ge 0$ for $r \le \frac{1}{5}$. Since

$$F'(r) = -8 + 26r + 204r^3 + 40r^4 - 6r^5 \le -8 + \frac{26}{5} + \frac{204}{125} + \frac{40}{625} < 0$$

for $r \le \frac{1}{5}$, this implies that $F(r)$ is a decreasing function on $[0, 1/5]$ and thus, we conclude that

$$F(r) \ge F(1/5) = 1 - \frac{8}{5} + \frac{13}{25} + \frac{51}{625} + \frac{8}{5^5} - \frac{1}{5^6}$$
$$= -\frac{2}{25} + \frac{51}{625} + \frac{8}{5^5} - \frac{1}{5^6} = \frac{1}{625} + \frac{39}{5^6} > 0$$

for $r \le \frac{1}{5}$. Hence $\frac{\partial \Psi(\lambda, r)}{\partial \lambda} \le \frac{\partial \Psi}{\partial \lambda}(1, r) \le 0$ for $r \le \frac{1}{5}$. It follows that $\Psi(\lambda, r)$ is a decreasing function of λ on $(0, 1]$ for $r \le \frac{1}{5}$, so that

$$\Psi(\lambda, r) \ge \Psi(1, r) = (1 - 10r + r^2)(1 - r)^2(1 + r)^3 - 32r^2(1 + r^2),$$

which is greater than or equal to 0 for all $r \le r_u^*$, where $r_u^* \approx 0.0808958838$ is the unique root of Eq. (4.1). This completes the proof of Theorem 4 □

5 The Bohr radius of the derivatives of analytic functions

In [10], Bhowmik and Das investigated the Bohr radius of the derivatives of analytic functions. In particular, they established the following results.

Proposition 1 [10] *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two analytic functions in $\mathbb{D}$. Then $M_{f+g}(r) \le M_f(r) + M_g(r)$ and $M_{f g}(r) \le M_f(r) M_g(r)$ for any $|z| = r \in [0, 1)$, where $M_f(r)$ denotes the majorant series of f.*

Theorem F [10] *Let $w(z)$ be an analytic self map of $\mathbb{D}$ with $w(0) = 0$. Then $M_{w'}(r) \le 1$ for $|z| = r \le r_0 = 1 - \sqrt{2/3}$. This radius r_0 is the best possible.*

Using the similar method as in the proof of Theorem F, we can easily prove the following.

Lemma 1 *Let $w(z)$ be an analytic self map of $\mathbb{D}$. Then $M_{zw'}(r) + M_w(r) \le 1$ for $|z| = r \le r_0 = 1 - \sqrt{2/3}$.*

In this section, we determine the Bohr radius for the derivatives of analytic functions associated with quasi-subordination. More precisely, we have

Theorem 5 *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two analytic functions in $\mathbb{D}$. If $f(z) - f(0) \prec_q g(z) - g(0)$ in $\mathbb{D}$, then $M_{f'}(r) \le M_{g'}(r)$ for $|z| = r \le r_0 = 1 - \sqrt{2/3}$. The radius r_0 cannot be improved.*

Proof Suppose that $f(z) - f(0) \prec_q g(z) - g(0)$ in $\mathbb{D}$. Then there exist two functions $\Phi \in \mathcal{B}$ and $\omega \in \mathcal{B}_0$ such that $f(z) - f(0) = \Phi(z)(g(\omega(z)) - g(0))$. Thus we have

$$f'(z) = \Phi'(z)(g(\omega(z)) - g(0)) + \Phi(z)g'(\omega(z))\omega'(z),$$

which implies

$$M_{f'}(r) \le M_{z\Phi'(z)}(r)M_{\frac{\omega(z)}{z}}(r)M_{\frac{g(\omega(z))-g(0)}{\omega(z)}}(r) + M_\Phi(r)M_{g' \circ \omega}(r)M_{\omega'}(r).$$

As $g' \circ \omega \prec g'$ and $\frac{g(\omega(z)) - g(0)}{\omega(z)} \prec \frac{g(z) - g(0)}{z}$, by Lemma E(1), we have

$$M_{g' \circ \omega}(r) \leq M_{g'}(r) \quad \text{and} \quad M_{\frac{g(\omega(z)) - g(0)}{\omega(z)}}(r) \leq M_{\frac{g(z) - g(0)}{z}}(r) \leq M_{g'}(r) \quad \text{for } r \leq 1/3.$$

From Theorem F, $M_{\omega'}(r) \leq 1$ for $r \leq r_0 = 1 - \sqrt{2/3} < 1/3$. Further, we observe that $M_{\frac{\omega(z)}{z}}(r) \leq M_{\omega'}(r) \leq 1$ for $r \leq r_0 = 1 - \sqrt{2/3} < 1/3$. Consequently,

$$M_{f'}(r) \leq \left(M_{z\Phi'(z)}(r) + M_{\Phi}(r) \right) M_{g'}(r).$$

Moreover Lemma 1 yields that

$$M_{z\Phi'(z)}(r) + M_{\Phi}(r) \leq 1 \quad \text{for } r \leq r_0 = 1 - \sqrt{2/3} < 1/3.$$

The desired inequality follows from the last two inequalities.

Following the method of proof of [10, Theorem 2], we can easily obtain that the radius r_0 cannot be improved. So, we omit the details. The proof is complete. $\qquad\square$

Remark 4 It is worth pointing out that [10, Theorem 2] is a special case of Theorem 5.

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