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ORIGINAL PAPER

Inequalities related to certain inverse trigonometric and inverse hyperbolic functions

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Abstract
In this paper, we obtain a sharp double inequality between the inverse tangent and inverse hyperbolic sine functions. At the same time, we give a sharp double inequality between the inverse hyperbolic tangent and inverse sine functions.

Keywords Inequalities · Inverse tangent function · Inverse hyperbolic sine function · Inverse hyperbolic tangent function · Inverse sine function

Mathematics Subject Classification 26D05

1 Introduction

Masjed-Jamei [6] obtained the following inequality:

$$(\arctan x)^2 \leq \frac{x \ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} \tag{1.1}$$

for $|x| < 1$. By using Maple software, Masjed-Jamei [6] pointed out that the inequality (1.1) holds for $x \in \mathbb{R}$. Zhu and Malešević [13, Theorem 1.1] proved that the inequality (1.1) holds for all $x \in \mathbb{R}$, and the power number 2 is the best in (1.1). Inequality (1.1) gives the upper bound for the square of the inverse tangent function $\arctan x$ by the inverse hyperbolic sine function $\operatorname{arcsinh} x = \ln(x + \sqrt{1+x^2})$.

Zhu and Malešević [14] obtained a general result on the natural approximation of the function $(\arctan x)^2 - (x \operatorname{arcsinh} x)/\sqrt{1+x^2}$, and proved a conjecture raised by Zhu and Malešević [13].

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Zhu and Malešević [13, Theorem 1.4] showed the analogue for inverse hyperbolic tangent function $\operatorname{arctanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$ and inverse sine function $\arcsin x$. More precisely, these authors proved that the inequality

$$(\operatorname{arctanh} x)^2 \le \frac{x \arcsin x}{\sqrt{1-x^2}} \tag{1.2}$$

holds for all $x \in (-1, 1)$, and the power number 2 is the best in (1.2).

The first aim of the present paper is to develop (1.1) to produce a sharp double inequality (Theorem 2.1). The second aim of the present paper is to provide a lower bound of $(\operatorname{arctanh} x)^2$ (Theorem 2.2).

The numerical values given have been calculated using the computer program MAPLE 11.

2 Results

Theorem 2.1 develops (1.1) to produce a sharp double inequality.

Theorem 2.1 *For $x > 0$, we have*

$$\frac{x \operatorname{arcsinh} x}{\sqrt{1+x^2+\alpha x^4}} < (\arctan x)^2 < \frac{x \operatorname{arcsinh} x}{\sqrt{1+x^2+\beta x^4}}, \tag{2.1}$$

with the best possible constants

$$\alpha = \frac{2}{45} \quad \text{and} \quad \beta = 0. \tag{2.2}$$

Proof Zhu and Malešević [13, Theorem 1.1] have proved that the right-hand side of (2.1) with $\beta = 0$ is valid for $x > 0$.

We now prove that the left-hand side of (2.1) with $\alpha = \frac{2}{45}$ is valid for $x > 0$, namely,

$$\frac{x \operatorname{arcsinh} x}{\sqrt{1+x^2+\frac{2}{45}x^4}} < (\arctan x)^2, \quad x > 0. \tag{2.3}$$

The inequality (2.3) is proved by considering the function $F(x)$ defined, for $x > 0$, by

$$F(x) = (\arctan x)^2 \frac{\sqrt{1+x^2+\frac{2}{45}x^4}}{x} - \operatorname{arcsinh} x.$$

We consider two cases to prove $F(x) > 0$ for $x > 0$.

Case 1. $0 < x < 1$.

From the continued fraction [7, p.122, Eq.(4.25.4)]

$$\arctan x = \cfrac{x}{1+\cfrac{x^2}{3+\cfrac{4x^2}{5+\cfrac{9x^2}{7+\cfrac{16x^2}{9+\cfrac{25x^2}{11+\ddots}}}}}},$$

we find, for $x > 0$,

$$\frac{x}{1+\frac{x^2}{3+\frac{4x^2}{5+\frac{9x^2}{7+\frac{16x^2}{9+\frac{25x^2}{11}}}}}} < \arctan x < \frac{x}{1+\frac{x^2}{3+\frac{4x^2}{5+\frac{9x^2}{7+\frac{16x^2}{9}}}}},$$

which can be written for $x > 0$ as

$$\frac{7x(165+170x^2+33x^4)}{5(231+315x^2+105x^4+5x^6)} < \arctan x < \frac{x(945+735x^2+64x^4)}{15(63+70x^2+15x^4)}. \tag{2.4}$$

From the continued fraction [7, p.129, Eq.(4.39.2)]

$$\frac{\operatorname{arcsinh} x}{\sqrt{1+x^2}} = \frac{x}{1+\frac{1\cdot 2x^2}{3+\frac{1\cdot 2x^2}{5+\frac{3\cdot 4x^2}{7+\frac{3\cdot 4x^2}{9+\ddots}}}}},$$

we find, for $x > 0$,

$$\frac{x}{1+\frac{1\cdot 2x^2}{3+\frac{1\cdot 2x^2}{5+\frac{3\cdot 4x^2}{7}}}} < \frac{\operatorname{arcsinh} x}{\sqrt{1+x^2}} < \frac{x}{1+\frac{1\cdot 2x^2}{3+\frac{1\cdot 2x^2}{5+\frac{3\cdot 4x^2}{7+\frac{3\cdot 4x^2}{9}}}}},$$

which can be written for $x > 0$ as

$$\sqrt{1+x^2}\,\frac{5x(21+10x^2)}{3(35+40x^2+8x^4)} < \operatorname{arcsinh} x < \sqrt{1+x^2}\,\frac{15x(315+210x^2+8x^4)}{(21+28x^2+8x^4)}. \tag{2.5}$$

Using the left-hand side of (2.4) and the right-hand side of (2.5), we have

$$F(x) > F_1(x) - F_2(x),$$

where

$$F_1(x) = \left(\frac{7x(165+170x^2+33x^4)}{5(231+315x^2+105x^4+5x^6)}\right)^2 \frac{\sqrt{1+x^2+\frac{2}{45}x^4}}{x}$$

and

$$F_2(x) = \sqrt{1+x^2}\,\frac{15x(315+210x^2+8x^4)}{(21+28x^2+8x^4)}.$$

For $0 < x < 1$, we have

$$(F_1(x))^2 - (F_2(x))^2 = \frac{x^8 F_3(x)}{28125(231 + 315x^2 + 105x^4 + 5x^6)^4(21 + 28x^2 + 8x^4)^2},$$

where

$$\begin{aligned} F_3(x) = {} & 74744153426250 - 687500000x^{26} - 5000000x^{28} \\ & + x^2(579606935205375 - 344451926144788x^{12}) \\ & + x^4(1824891791790375 - 226558849074978x^{12}) \\ & + x^6(3069233024942700 - 70320861146025x^{12}) \\ & + x^8(2949115869590100 - 11855324999049x^{12}) \\ & + x^{10}(1480818426676610 - 1034978520912x^{12}) \\ & + x^{12}(103550407764650 - 41061562500x^{12}). \end{aligned}$$

Noting that $F_3(x) > 0$ for $0 < x < 1$, we obtain, for $0 < x < 1$,

$$F(x) > F_1(x) - F_2(x) > 0.$$

Case 2. $x \geq 1$.
Shafer [9] proved that for $x > 0$,

$$\frac{8x}{3 + \sqrt{25 + \frac{80}{3}x^2}} < \arctan x. \tag{2.6}$$

The inequality (2.6) can also be found in [8,10–12]. We have, by (2.6),

$$F(x) > \left(\frac{8x}{3 + \sqrt{25 + \frac{80}{3}x^2}}\right)^2 \frac{\sqrt{1 + x^2 + \frac{2}{45}x^4}}{x} - \operatorname{arcsinh} x =: F_4(x).$$

Differentiation yields

$$F_4'(x) = F_5(x) - \frac{1}{\sqrt{1 + x^2}},$$

where

$$F_5(x) = \frac{576\Big((135 + 270x^2 + 18x^4)\sqrt{225 + 240x^2} + 3375 + 3150x^2 + 450x^4 + 160x^6\Big)}{(9 + \sqrt{225 + 240x^2})^3\sqrt{225 + 240x^2}\sqrt{225 + 225x^2 + 10x^4}}.$$

For $x \geq 1$, we have

$$\big(F_5(x)\big)^2 - \left(\frac{1}{\sqrt{1 + x^2}}\right)^2 = \frac{864 F_6(x)}{5(9 + \sqrt{225 + 240x^2})^6(15 + 16x^2)(45 + 45x^2 + 2x^4)(1 + x^2)},$$

where

$$F_6(x) = F_7(x) + F_8(x) - F_9(x), \tag{2.7}$$

and

$$\begin{aligned} F_7(x) = {} & 23236875 - 46018125x^2 - 80566650x^4 + 80740800x^6 \\ & + 107019360x^8 + 18091776x^{10} + 3772416x^{12} + 655360x^{14}, \end{aligned}$$

$$F_8(x) = (147456x^{12} + 2198016x^{10} + 3073950x^4 + 3894075x^2)\sqrt{225 + 240x^2},$$

$$F_9(x) = (2371680x^8 + 6793200x^6 + 1549125)\sqrt{225 + 240x^2}.$$

Write (2.7) as

$$F_6(x) = F_7(x) - 0.5F_8(x) + 1.5F_8(x) - F_9(x),$$

We find, for $x \geq 1$,

$$\big(F_7(x)\big)^2 - \big(0.5F_8(x)\big)^2 = P_{28}(x-1) \quad \text{and} \quad \big(1.5F_8(x)\big)^2 - \big(F_9(x)\big)^2 = P_{26}(x-1),$$

where

$$P_{28}(x) = \frac{5402979408824191}{4} + 143110860826405993x + \cdots + 429496729600x^{28}$$

and

$$P_{26}(x) = \frac{149502812513856165}{4} + 297654306451453215x + \cdots + 11741366845440x^{26}.$$

The polynomials $P_{28}(x)$ and $P_{26}(x)$ have all coefficients positive, so $F_6(x) > 0$ for $x \geq 1$. We then obtain, for $x \geq 1$,

$$F_5(x) > \frac{1}{\sqrt{1+x^2}} \Longrightarrow F_4'(x) > 0.$$

Hence, $F_4(x)$ is strictly increasing for $x \geq 1$, and we have, for $x \geq 1$,

$$\begin{aligned} F(x) > F_4(x) \geq F_4(1) &= \frac{64\sqrt{115} - (455 + 15\sqrt{465})\ln(1+\sqrt{2})}{(455 + 15\sqrt{465})} \\ &= 0.0002714659399\ldots > 0. \end{aligned}$$

If we write (2.1) as

$$\beta < \frac{1}{x^4}\left[\left(\frac{x \operatorname{arcsinh} x}{(\arctan x)^2}\right)^2 - 1 - x^2\right] < \alpha, \quad x > 0,$$

we find

$$\lim_{x\to 0} \frac{1}{x^4}\left[\left(\frac{x \operatorname{arcsinh} x}{(\arctan x)^2}\right)^2 - 1 - x^2\right] = \frac{2}{45}$$

and

$$\lim_{x\to \infty} \frac{1}{x^4}\left[\left(\frac{x \operatorname{arcsinh} x}{(\arctan x)^2}\right)^2 - 1 - x^2\right] = 0.$$

Hence, the double inequality (2.1) holds for $x > 0$, with the best possible constants $\alpha = \frac{2}{45}$ and $\beta = 0$. The proof of Theorem 2.1 is complete.

We provide another proof of (2.3) in the Appendix.

Theorem 2.2 provides a lower bound of $(\operatorname{arctanh} x)^2$. □

Theorem 2.2 *For* $0 < x < 1$, *we have*

$$\frac{x \arcsin x}{1 - \frac{1}{2}x^2} < (\operatorname{arctanh} x)^2, \tag{2.8}$$

and the constant $\frac{1}{2}$ *in the lower bound is the best possible.*

Proof Let $\arcsin x = t$, $x \in (0, 1)$. Then $x = \sin t$, $t \in (0, \pi/2)$. We see that

$$\operatorname{arctanh}(\sin t) = \frac{1}{2} \ln \frac{1 + \sin t}{1 - \sin t} = \frac{1}{2} \ln \frac{(1 + \sin t)^2}{(1 - \sin t)(1 + \sin t)} = \ln \frac{1 + \sin t}{\cos t},$$

and (2.8) is equivalent to

$$\frac{t \sin t}{1 - \frac{1}{2}\sin^2 t} < \left(\ln \frac{1 + \sin t}{\cos t}\right)^2, \qquad 0 < t < \frac{\pi}{2}. \tag{2.9}$$

In order to prove (2.9), it suffices to show that for $0 < t < \pi/2$,

$$\frac{t \sin t}{1 - \frac{1}{2}\sin^2 t} < t^2 + \frac{1}{3}t^4 + \frac{1}{9}t^6 < \left(\ln \frac{1 + \sin t}{\cos t}\right)^2. \tag{2.10}$$

The left-hand side of (2.10) can be written for $0 < t < \pi/2$ as

$$\frac{1}{t + \frac{1}{3}t^3 + \frac{1}{9}t^5} < \csc t - \frac{1}{2}\sin t.$$

Using the power series expansions for $\csc t$ and $\sin t$, we have

$$\begin{aligned}
&\csc t - \frac{1}{2}\sin t - \frac{1}{t + \frac{1}{3}t^3 + \frac{1}{9}t^5} \\
&= \frac{1}{t} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2(2^{2n-1} - 1) B_{2n}}{(2n)!} t^{2n-1} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1} - \frac{1}{t + \frac{1}{3}t^3 + \frac{1}{9}t^5} \\
&= \frac{1}{t} - \frac{1}{3}t + \frac{37}{360}t^3 - \frac{2}{945}t^5 + \sum_{n=4}^{\infty} \left\{ \frac{2(2^{2n-1} - 1)|B_{2n}|}{(2n)!} - \frac{(-1)^{n-1}}{2 \cdot (2n-1)!} \right\} t^{2n-1} \\
&\quad - \frac{1}{t + \frac{1}{3}t^3 + \frac{1}{9}t^5},
\end{aligned} \tag{2.11}$$

where B_n ($n \in \mathbb{N}_0$) are the Bernoulli numbers defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \qquad |z| < 2\pi.$$

By the inequality (see [1, p. 805])

$$|B_{2n}| > \frac{2 \cdot (2n)!}{(2\pi)^{2n}}, \qquad n = 1, 2, \ldots,$$

we find, for $n \geq 4$,

$$\frac{2(2^{2n-1} - 1)|B_{2n}|}{(2n)!} > \frac{4(2^{2n-1} - 1)}{(2\pi)^{2n}} > \frac{1}{2 \cdot (2n-1)!}. \tag{2.12}$$

By induction on n, the second inequality in (2.12) can be proved (we here omit the proof). We then obtain from (2.11) that, for $0 < t < \pi/2$,

$$\csc t - \frac{1}{2}\sin t - \frac{1}{t + \frac{1}{3}t^3 + \frac{1}{9}t^5} > \frac{1}{t} - \frac{1}{3}t + \frac{37}{360}t^3 - \frac{2}{945}t^5 - \frac{1}{t + \frac{1}{3}t^3 + \frac{1}{9}t^5}$$
$$= \frac{t^3\Big((6993 - 333t^2) + t^4(729 - 16t^2)\Big)}{7560(9 + 3t^2 + t^4)} > 0.$$

Hence, the left-hand side of (2.10) holds for $0 < t < \pi/2$.

We now prove the right-hand side of (2.10). For $0 < t < \pi/2$, let

$$G(t) = \left(\ln\frac{1 + \sin t}{\cos t}\right)^2 - \left(t^2 + \frac{1}{3}t^4 + \frac{1}{9}t^6\right).$$

Differentiation yields

$$\frac{\cos t}{2}G'(t) = \ln\frac{1 + \sin t}{\cos t} - \left(t + \frac{2}{3}t^3 + \frac{1}{3}t^5\right)\cos t =: H(t),$$

and

$$H'(t) = \frac{(t^5 + 2t^3 + 3t)\sin t\cos t + 3\sin^2 t - (5t^4 + 6t^2)\cos^2 t}{3\cos t},$$

which can be written as

$$\frac{3}{\cos t}H'(t) = (t^5 + 2t^3 + 3t)\tan t + 3\tan^2 t - 5t^4 - 6t^2.$$

Using the power series expansion of $\tan t$

$$\tan t = \sum_{j=1}^{\infty}\frac{2^{2j}(2^{2j} - 1)|B_{2j}|}{(2j)!}t^{2j-1} = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \ldots,$$

we obtain, for $0 < t < \pi/2$,

$$\frac{3}{\cos t}H'(t) = (t^5 + 2t^3 + 3t)\left(t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \ldots\right) + 3\left(t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \ldots\right)^2$$
$$- 5t^4 - 6t^2$$
$$= \frac{16}{5}t^6 + \frac{142}{105}t^8 + \ldots > 0.$$

Hence, $H(t)$ is strictly increasing for $0 < t < \pi/2$, and we have

$$H(t) > H(0) = 0 \quad \text{and} \quad G'(t) > 0 \quad \text{for} \quad 0 < t < \frac{\pi}{2}.$$

Therefore, $G(t)$ is strictly increasing for $0 < t < \pi/2$, and we have

$$G(t) > G(0) = 0 \quad \text{for} \quad 0 < t < \frac{\pi}{2}.$$

This means that the right-hand side of (2.10) holds for $0 < t < \pi/2$.

If we write (2.8) as

$$\frac{1}{x^2}\left[1 - \frac{x\arcsin x}{(\operatorname{arctanh} x)^2}\right] > \frac{1}{2},$$

we find

$$\lim_{x\to 0} \frac{1}{x^2}\left[1 - \frac{x \arcsin x}{(\operatorname{arctanh} x)^2}\right] = \frac{1}{2}.$$

Hence, the inequality (2.8) holds for $0 < x < 1$, and the constant $\frac{1}{2}$ in the lower bound is the best possible. The proof of Theorem 2.2 is complete. □

Conjecture 2.1 *For* $0 < x < 1$, *we have*

$$\frac{x \arcsin x}{\left(1 - \frac{41}{45}x^2\right)^{\frac{45}{82}}} < (\operatorname{arctanh} x)^2. \tag{2.13}$$

Remark 2.1 The lower bound in (2.13) is better than the lower bound in (2.8). By using Maple software, we find the following approximation formulas near the origin:

$$(\operatorname{arctanh} x)^2 - \frac{x \arcsin x}{\left(1 - \frac{41}{45}x^2\right)^{\frac{45}{82}}} = O\left(x^8\right), \tag{2.14}$$

$$(\operatorname{arctanh} x)^2 - \frac{x \arcsin x}{1 - \frac{1}{2}x^2} = O\left(x^6\right), \tag{2.15}$$

$$(\operatorname{arctanh} x)^2 - \frac{x \arcsin x}{\sqrt{1 - x^2}} = O\left(x^6\right). \tag{2.16}$$

This shows that, among approximation formulas (2.14)–(2.16), the formula (2.14) would be the best one.

Appendix: Another proof of (2.3)

By an elementary change of variable $x = \tan t$ $(0 < t < \pi/2)$, the inequality (2.3) is equivalent to

$$\ln(\tan t + \sec t) < \frac{t^2}{\tan t}\sqrt{\sec^2 t + \frac{2}{45}\tan^4 t}, \quad 0 < t < \frac{\pi}{2}.$$

For $0 \le t < \pi/2$, let

$$f(t) = \left(\frac{t^2}{\tan t}\sqrt{\sec^2 t + \frac{2}{45}\tan^4 t}\right)^2 - \Big(\ln(\tan t + \sec t)\Big)^2, \quad t \neq 0$$

and

$$f(0) = \lim_{t\to 0} f(t) = 0.$$

Differentiation yields

$$\frac{\cos t}{2} f'(t) = \frac{t^3}{45\cos^2 t \sin^3 t}\Big(4 \sin t \cos t + 82 \sin t \cos^3 t + 4 \sin t \cos^5 t - 4t\cos^2 t - 43t\cos^4 t + 2t\Big) - \ln\frac{1 + \sin t}{\cos t} =: g(t)$$

and

$$g'(t) = \frac{h(t)}{45(1 - \cos t)^2(1 + \cos t)^2 \cos^3 t},$$

where

$$\begin{aligned}h(t) &= -12t^2\cos^8 t + 4t^3\cos^7 t\sin t - 45\cos^6 t - 234t^2\cos^6 t + 43t^4\cos^6 t\\ &\quad - 266t^3\cos^5 t\sin t + 90\cos^4 t + 234t^2\cos^4 t + 98t^4\cos^4 t - 110t^3\cos^3 t\sin t\\ &\quad + 12t^2\cos^2 t - 10t^4\cos^2 t - 45\cos^2 t + 12t^3\cos t\sin t + 4t^4.\end{aligned}$$

We now use the method from paper [4] to prove $h(t) > 0$ for $t \in (0, \pi/2)$. Let us start from the function $h(t)$ in the form of multiple angles

$$\begin{aligned}h(t) = &\underbrace{\left(\frac{2053}{32}t^4 + \frac{129}{16}t^2 + \frac{45}{32}\right)}_{(>0)}\cos 2t + \underbrace{\left(\frac{325}{16}t^4 + \frac{45}{16}\right)}_{(>0)}\cos 4t - \frac{69}{4}t^2\cos 4t\\ &+ \underbrace{\left(\frac{43}{32}t^4 - \frac{129}{16}t^2 - \frac{45}{32}\right)}_{(<0)}\cos 6t - \frac{3}{32}t^2\cos 8t - \frac{501}{8}t^3\sin 2t\\ &- \frac{745}{16}t^3\sin 4t - \frac{65}{8}t^3\sin 6t + \frac{1}{32}t^3\sin 8t + \frac{787}{16}t^4 + \frac{555}{32}t^2 - \frac{45}{16},\end{aligned}$$

for $t \in (0, \pi/2)$. Let us denote with $T_m^{\varphi,a}(x)$ Taylor development of function $\varphi(x)$ in the point $x = a$ of degree m [4]. The following inequality is true

$$\begin{aligned}h(t) &\ge P_{n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8,n_9}(t)\\ &= \left(\frac{2053}{32}t^4 + \frac{129}{16}t^2 + \frac{45}{32}\right)T^{\cos,0}_{4n_1+2}(2t) + \left(\frac{325}{16}t^4 + \frac{45}{16}\right)T^{\cos,0}_{4n_2+2}(4t) - \frac{69}{4}t^2T^{\cos,0}_{4n_3+0}(4t)\\ &\quad + \left(\frac{43}{32}t^4 - \frac{129}{16}t^2 - \frac{45}{32}\right)T^{\cos,0}_{4n_4+0}(6t) - \frac{3}{32}t^2T^{\cos,0}_{4n_5+0}(8t) - \frac{501}{8}t^3T^{\sin,0}_{4n_6+1}(2t)\\ &\quad - \frac{745}{16}t^3T^{\sin,0}_{4n_7+1}(4t) - \frac{65}{8}t^3T^{\sin,0}_{4n_8+1}(6t) + \frac{1}{32}t^3T^{\sin,0}_{4n_9+3}(8t) + \frac{787}{16}t^4 + \frac{555}{32}t^2 - \frac{45}{16},\end{aligned}$$

for each $n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9 \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$ ($\mathbb{N} = \{1, 2, \ldots\}$) and $t \in (0, \pi/2)$. For the following choice

$$n_1=n_2=n_3=n_4=n_5=n_6=n_7=n_8=n_9=6$$

we obtain the polynomial with rational coefficients

$$\begin{aligned}Q(t) &= P_{6,6,6,6,6,6,6,6,6}(t)\\ &= -\frac{36029974779582599}{519221756478231056250\,0}t^{30} + \frac{2291243636599537}{32050725708532781250}t^{28}\\ &\quad - \frac{31610419202861}{2261074124058750\,0}t^{26} + \frac{27106191652}{2406129350625}t^{24} - \frac{16899377227}{227949096375}t^{22}\\ &\quad + \frac{81737813}{212837625}t^{20} - \frac{63051259}{42567525}t^{18} + \frac{604216}{155925}t^{16} - \frac{359}{63}t^{14} + \frac{149}{63}t^{12} + \frac{8}{3}t^{10}.\end{aligned}$$

We now prove

$$Q(t) > 0 \quad \text{for } t \in (0, \pi/2).$$

One proof previously polynomial inequality, which we give here, is based on the Sturm's algorithm [3, Section 4 in Chapter 6]. Namely, using Sturm's algorithm it is possible verify that polynomial $Q(t)$ not have zeros in interval $(0, b)$, using for the right bound the rational number $b = (22/7)/2 > \pi/2$. Let us remark for example $Q(1) > 0$ is true and therefore we obtain the following conclusion

$$h(t) > Q(t) > 0 \quad \text{for } t \in (0, \pi/2).$$

On this way we present one proof of inequality $h(t) > 0$, for $t \in (0, \pi/2)$. Let us emphasize that previous proof in all steps is available and via system SIMTHEP—SIMple THE Prover for automatic proving of inequalities of mixed trigonometric polynomial functions class [2].

We then obtain $g'(t) > 0$ for $0 < t < \pi/2$. Hence, $g(t)$ is strictly increasing for $0 < t < \pi/2$, and we have

$$g(t) > \lim_{u \to 0} g(u) = 0 \quad \text{and} \quad f'(t) > 0 \quad \text{for } 0 < t < \frac{\pi}{2}.$$

Therefore, $f(t)$ is strictly increasing for $0 < t < \pi/2$, and we have

$$f(t) > f(0) = 0 \quad \text{for } 0 < t < \frac{\pi}{2}.$$

This means that (2.3) holds for $x > 0$.

Remark 2.2 The inequality (1.1) can be traced back by the generalization of the famous Cauchy-Schwarz inequality, which can be found in [5] and the references cited therein.

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