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ORIGINAL PAPER

Approximation for the complete elliptic integral of the first kind

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Abstract
In the article, we present several sharp upper and lower bounds for the complete elliptic integral of the first kind in terms of inverse trigonometric and inverse hyperbolic functions. As consequences, some sharp bounds for the Gaussian arithmetic-geometric mean in terms of other bivariate means are also given.

Keywords Complete elliptic integrals · Inverse trigonometric function · Inverse hyperbolic function · Arithmetic-geometric mean

Mathematics Subject Classification 33E05 · 26E60

1 Introduction

For two distinct positive real numbers u and v, the Gaussian arithmetic–geometric mean $AG(u, v)$ [1,2], first Seiffert mean $P(u, v)$, arithmetic mean $A(u, v)$, Neuman–Sándor mean $M(u, v)$ and second Seiffert mean $T(u, v)$ are given by

$$AG(u,v) = \frac{\pi}{2\int_0^{\pi} \frac{d\theta}{\sqrt{u^2\cos^2\theta + v^2\sin^2\theta}}},$$
$$P(u,v) = \frac{u-v}{2\arcsin\left(\frac{u-v}{u+v}\right)}, \quad A(u,v) = \frac{u+v}{2}, \tag{1.1}$$

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$$M(u, v) = \frac{u-v}{2\sinh^{-1}\left(\frac{u-v}{u+v}\right)}, \tag{1.2}$$

$$T(u, v) = \frac{u-v}{2\arctan\left(\frac{u-v}{u+v}\right)}, \tag{1.3}$$

respectively, where $\sinh^{-1}(\omega) = \log(\omega + \sqrt{1+\omega^2})$ is the inverse hyperbolic sine function.

The Gaussian identity [3, Theorem 4.4] shows that

$$AG(1, \tau) = \frac{\pi}{2\mathcal{K}\left(\sqrt{1-\tau^2}\right)} \tag{1.4}$$

for all $0 < \tau < 1$, where

$$\mathcal{K}(\tau) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\tau^2\sin^2\theta}}$$

is the complete elliptic integral of the first kind [4–8] and it is the special case of the Gaussian hypergeometric function [9–15]

$$F(\alpha, \beta; \gamma; \tau) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{r^n}{n!},$$

where $(\alpha)_0 = 1$ for $\alpha \neq 0$, $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$ is the shifted factorial function and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ $(x > 0)$ is the gamma function [16–18]. Indeed,

$$\mathcal{K}(\tau) = \frac{\pi}{2}F\left(\frac{1}{2}, \frac{1}{2}; 1; \tau^2\right).$$

Let $r \in (0, 1)$. Then it is well known that the conformal modulus $\mu(r)$ of the plane Grötzsch ring [19,20] $\{z \in \mathbb{C} | |z| < 1\} \backslash [0, r]$ can be expressed by

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}\left(\sqrt{1-r^2}\right)}{\mathcal{K}(r)},$$

which paly an important role in the geometric functions theory [21–23].

Recently, the bounds for the the complete elliptic integral $\mathcal{K}(\tau)$ have attracted the attention of many mathematicians [24–30], many applications for $\mathcal{K}(\tau)$ and its related special functions in mathematics, physics and engineering can be found in the literature [31–51].

Anderson, Vamanamurthy and Vuorinen [52] provided the upper bound for $\mathcal{K}(\tau)$ in terms of the ratio of the inverse hyperbolic tangent function $\tanh^{-1}(\tau) = \log[(1+\tau)/(1-\tau)]/2$ with τ as follows:

$$\frac{2}{\pi}\mathcal{K}(\tau) < \frac{\tanh^{-1}(\tau)}{\tau} \tag{1.5}$$

for all $0 < \tau < 1$.

In [53], Yang, Qian, Chu and Zhang proved that the inequality

$$\frac{2}{\pi}\mathcal{K}(\tau) > \frac{1+(6p-7)\sqrt{1-\tau^2}}{p+(5p-6)\sqrt{1-\tau^2}}\frac{\tanh^{-1}(\tau)}{\tau} \tag{1.6}$$

holds for all $0 < \tau < 1$ if and only if $p \geq \pi/2$.

Let $u = 1$ and $v = (1-\tau)/(1+\tau) \in (0, 1)$. Then (1.1)–(1.4) lead to the identities

$$\frac{A(u,v)}{AG(u,v)} = \frac{1/(1+\tau)}{(\pi/2)\mathcal{K}[2\sqrt{\tau}/(1+\tau)]} = \frac{2}{\pi}\mathcal{K}(\tau), \tag{1.7}$$

$$\frac{A(u,v)}{P(u,v)} = \frac{\arcsin(\tau)}{\tau}, \tag{1.8}$$

$$\frac{A(u,v)}{M(u,v)} = \frac{\sinh^{-1}(\tau)}{\tau}, \quad \frac{A(u,v)}{T(u,v)} = \frac{\arctan(\tau)}{\tau}. \tag{1.9}$$

From (1.7)–(1.9) we clearly see that the well known inequalities

$$AG(u,v) < P(u,v) < M(u,v) < T(u,v)$$

for all $u, v > 0$ with $u \neq v$ is equivalent to

$$\frac{\arctan(\tau)}{\tau} < \frac{\sinh^{-1}(\tau)}{\tau} < \frac{\arcsin(\tau)}{\tau} < \frac{2}{\pi}\mathcal{K}(\tau) \tag{1.10}$$

for all $0 < \tau < 1$.

Inequalities (1.5), (1.6) and (1.10) give us the motivation to find the sharp bounds for $\mathcal{K}(\tau)$ in terms of the convex combinations of $\tanh^{-1}(\tau)/\tau$ and $\arcsin(\tau)/\tau$, $\tanh^{-1}(\tau)/\tau$ and $\sinh^{-1}(\tau)/\tau$, and $\tanh^{-1}(\tau)/\tau$ and $\arctan(\tau)/\tau$.

Our first result is Theorem 1.1 which states as follows.

Theorem 1.1 *The double inequalities*

$$\frac{\pi}{2}\left[\alpha_1\frac{\tanh^{-1}(\tau)}{\tau} + (1-\alpha_1)\frac{\arcsin(\tau)}{\tau}\right] < \mathcal{K}(\tau) < \frac{\pi}{2}\left[\beta_1\frac{\tanh^{-1}(\tau)}{\tau} + (1-\beta_1)\frac{\arcsin(\tau)}{\tau}\right], \tag{1.11}$$

$$\frac{\pi}{2}\left[\alpha_2\frac{\tanh^{-1}(\tau)}{\tau} + (1-\alpha_2)\frac{\sinh^{-1}(\tau)}{\tau}\right] < \mathcal{K}(\tau) < \frac{\pi}{2}\left[\beta_2\frac{\tanh^{-1}(\tau)}{\tau} + (1-\beta_2)\frac{\sinh^{-1}(\tau)}{\tau}\right], \tag{1.12}$$

$$\frac{\pi}{2}\left[\alpha_3\frac{\tanh^{-1}(\tau)}{\tau} + (1-\alpha_3)\frac{\arctan(\tau)}{\tau}\right] < \mathcal{K}(\tau) < \frac{\pi}{2}\left[\beta_3\frac{\tanh^{-1}(\tau)}{\tau} + (1-\beta_3)\frac{\arctan(\tau)}{\tau}\right] \tag{1.13}$$

hold for all $0 < \tau < 1$ *if and only if* $\alpha_1 \leq 1/2$, $\alpha_2 \leq 2/\pi$, $\alpha_3 \leq 2/\pi$, $\beta_1 \geq 2/\pi$, $\beta_2 \geq 5/6$ *and* $\beta_3 \geq 7/8$.

To further improve and refine the lower bound in (1.11) and the upper bounds in (1.12) and (1.13) we establish the following Theorem 1.2.

Theorem 1.2 *The double inequalities*

$$\frac{\pi\alpha_4}{2}\left[\frac{5\tanh^{-1}(\tau)}{6\tau} + \frac{\sinh^{-1}(\tau)}{6\tau}\right] + \frac{\pi(1-\alpha_4)}{2}\left[\frac{\tanh^{-1}(\tau)}{2\tau} + \frac{\arcsin(\tau)}{2\tau}\right] < \mathcal{K}(\tau) < \frac{\pi\beta_4}{2}\left[\frac{5\tanh^{-1}(\tau)}{6\tau} + \frac{\sinh^{-1}(\tau)}{6\tau}\right] + \frac{\pi(1-\beta_4)}{2}\left[\frac{\tanh^{-1}(\tau)}{2\tau} + \frac{\arcsin(\tau)}{2\tau}\right], \tag{1.14}$$

$$\frac{\pi\alpha_5}{2}\left[\frac{7\tanh^{-1}(\tau)}{8\tau}+\frac{\arctan(\tau)}{8\tau}\right]+\frac{\pi(1-\alpha_5)}{2}\left[\frac{\tanh^{-1}(\tau)}{2\tau}+\frac{\arcsin(\tau)}{2\tau}\right]<\mathcal{K}(\tau)$$
$$<\frac{\pi\beta_5}{2}\left[\frac{7\tanh^{-1}(\tau)}{8\tau}+\frac{\arctan(\tau)}{8\tau}\right]+\frac{\pi(1-\beta_5)}{2}\left[\frac{\tanh^{-1}(\tau)}{2\tau}+\frac{\arcsin(\tau)}{2\tau}\right] \quad (1.15)$$

hold for all $0<\tau<1$ *if and only if* $\alpha_4\le 3/40$, $\alpha_5\le 1/20$, $\beta_4\ge 3(4-\pi)/(2\pi)=0.4098\ldots$ *and* $\beta_5\ge 4(4-\pi)/(3\pi)=0.3643\ldots$.

2 Lemmas

In order to simplify the proofs of our Theorems 1.1 and 1.2, we need several lemmas which we present in this section.

First of all, we introduce the complete elliptic integral of the second kind [54–60]

$$\mathcal{E}(\tau)=\int_0^{\pi/2}\sqrt{1-\tau^2\sin^2\theta}d\theta=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;\tau^2\right)$$

and the elementary formulas [3, Appendix E] for $\mathcal{K}(\tau)$ and $\mathcal{E}(\tau)$ as follows

$$\frac{d\mathcal{K}(\tau)}{d\tau}=\frac{\mathcal{E}(\tau)-\tau'^2\mathcal{K}(\tau)}{\tau\tau'^2},\quad \frac{d\mathcal{E}(\tau)}{d\tau}=\frac{\mathcal{E}(\tau)-\mathcal{K}(\tau)}{\tau},\quad \frac{d\left[\mathcal{E}(\tau)-\tau'^2\mathcal{K}(\tau)\right]}{d\tau}=r\mathcal{K}(\tau),$$
$$\mathcal{E}\left(\frac{2\sqrt{\tau}}{1+\tau}\right)=\frac{2\mathcal{E}(\tau)-\tau'^2\mathcal{K}(\tau)}{1+\tau},\quad \mathcal{K}\left(\frac{2\sqrt{\tau}}{1+\tau}\right)=(1+\tau)\mathcal{K}(\tau),$$

where and in what follows $\tau'=\sqrt{1-\tau^2}$.

Lemma 2.1 (See [3, Theorem 1.25]) *Let* $-\infty<\lambda<\mu<\infty$, $f,g:[\lambda,\mu]\mapsto(-\infty,\infty)$ *be continuous on* $[\lambda,\mu]$ *and differentiable on* (λ,μ) *with* $g'(t)\ne 0$ *on* (λ,μ). *If* $f'(t)/g'(t)$ *is increasing (decreasing) on* (λ,μ), *then so are the functions*

$$\frac{f(t)-f(\lambda)}{g(t)-g(\lambda)},\qquad \frac{f(t)-f(\mu)}{g(t)-g(\mu)}.$$

If $f'(t)/g'(t)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.2 *The following statements are true:*

(1) *The function* $\tau\mapsto\left[\mathcal{E}(\tau)-\tau'^2\mathcal{K}(\tau)\right]/\tau^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4,1)$;
(2) *The function* $\tau\mapsto\left[\mathcal{E}(\tau)-\tau'^2\mathcal{K}(\tau)-\tau'^2(\mathcal{K}(\tau)-\mathcal{E}(\tau))\right]/\tau^4$ *is strictly increasing from* $(0,1)$ *onto* $(3\pi/16,1)$;
(3) *The function* $\tau\mapsto[\mathcal{K}(\tau)-\mathcal{E}(\tau)]/\tau^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4,\infty)$;
(4) *The function* $\tau\mapsto\left[(1+\tau'^2)\mathcal{K}(\tau)-2\mathcal{E}(\tau)\right]/\tau^4$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/16,\infty)$;
(5) *The function* $\tau\mapsto\phi(\tau)=\mathcal{E}(\tau)/(3+2\tau^2)$ *is strictly decreasing from* $(0,1)$ *onto* $(1/5,\pi/6)$;
(6) *The function* $\tau\mapsto\varphi(\tau)=[2\mathcal{E}(\tau)-\tau'^2\mathcal{K}(\tau)]/(3+2\tau^2)$ *is strictly decreasing from* $(0,1)$ *onto* $(2/5,\pi/6)$;
(7) *The function* $\tau\mapsto\psi(\tau)=(1+\tau^2)[\mathcal{E}(\tau)-\mathcal{K}(\tau)]/\tau^2$ *is strictly decreasing from* $(0,1)$ *onto* $(-\infty,-\pi/4)$.

Proof Parts (1)–(4) can be found in [3, Theorem 3.21(1) and Exercise 3.43(10), (11) and (29)]. For part (5), it is not difficult to verify that

$$\phi(0^+) = \frac{\pi}{6}, \qquad \phi(1^-) = \frac{1}{5}, \tag{2.1}$$

$$\phi'(\tau) = -\frac{\tau}{3+2\tau^2}\left[\frac{\mathcal{K}(\tau)-\mathcal{E}(\tau)}{\tau^2} + \frac{4\mathcal{E}(\tau)}{3+2\tau^2}\right]. \tag{2.2}$$

It follows from (2.2) and part (3) together with the monotonicity of $\mathcal{E}(\tau)$ that

$$\phi'(\tau) < 0 \tag{2.3}$$

for $0 < \tau < 1$.

Therefore, part (5) follows from (2.1) and (2.3).

For part (6), simple computations lead to

$$\varphi(0^+) = \frac{\pi}{6}, \qquad \varphi(1^-) = \frac{2}{5}, \tag{2.4}$$

$$\varphi'(\tau) = -\frac{\tau}{(3+2\tau^2)^2}\left[3\tau'^2\left(\frac{\mathcal{K}(\tau)-\mathcal{E}(\tau)}{\tau^2}\right) + 2\tau^2\left(\frac{\mathcal{E}(\tau)-\tau'^2\mathcal{K}(\tau)}{\tau^2}\right) + \mathcal{E}(\tau)\right]. \tag{2.5}$$

From parts (1) and (3), (2.5) and the monotonicity of $\mathcal{E}(\tau)$ we clearly see that

$$\varphi'(\tau) < 0 \tag{2.6}$$

for $0 < \tau < 1$.

Therefore, part (6) follows from (2.4) and (2.6).

For part (7), elaborated computations give

$$\psi(0^+) = -\frac{\pi}{4}, \qquad \psi(1^-) = -\infty, \tag{2.7}$$

$$\psi'(\tau) = -\tau\left[2\left(\frac{\mathcal{K}(\tau)-\mathcal{E}(\tau)}{\tau^2}\right) + \frac{1+\tau^2}{\tau'^2} \times \left(\frac{\mathcal{E}(\tau)-\tau'^2\mathcal{K}(\tau)-\tau'^2(\mathcal{K}(\tau)-\mathcal{E}(\tau))}{\tau^4}\right)\right]. \tag{2.8}$$

It follows from parts (2) and (3) together with (2.8) that

$$\psi'(\tau) < 0 \tag{2.9}$$

for $0 < \tau < 1$.

Therefore, part (7) follows from (2.7) and (2.9). □

Lemma 2.3 *The function*

$$F(\tau) = \frac{\frac{2}{\pi}\tau\mathcal{K}(\tau) - \arcsin(\tau)}{\tanh^{-1}(\tau) - \arcsin(\tau)}$$

is strictly increasing from $(0, 1)$ *onto* $(1/2, 2/\pi)$.

Proof Let $F_1(\tau) = 2\tau\mathcal{K}(\tau)/\pi - \arcsin(\tau)$, $F_2(\tau) = \tanh^{-1}(\tau) - \arcsin(\tau)$, $F_3(\tau) = 2\mathcal{E}(\tau)/(\pi\sqrt{1-\tau^2}) - 1$ and $F_4(\tau) = 1/\sqrt{1-\tau^2} - 1$. Then elaborated computations lead to

$$F_1(0^+) = F_2(0^+) = 0, \quad F(\tau) = \frac{F_1(\tau)}{F_2(\tau)}, \tag{2.10}$$

$$F_3(0^+) = F_4(0^+) = 0, \quad \frac{F_1'(\tau)}{F_2'(\tau)} = \frac{F_3(\tau)}{F_4(\tau)}, \tag{2.11}$$

$$\frac{F_3'(\tau)}{F_4'(\tau)} = \frac{2}{\pi}\left[\frac{\mathcal{E}(\tau) - \tau'^2\mathcal{K}(\tau)}{\tau^2}\right]. \tag{2.12}$$

It follows from Lemma 2.2(1) and (2.12) that $F_3'(\tau)/F_4'(\tau)$ is strictly increasing on (0,1). Then from (2.10) and (2.11) together with Lemma 2.1 we know that $F(\tau)$ is strictly increasing on (0,1).

Note that

$$F(0^+) = \frac{1}{2}, \quad F(1^-) = \frac{2}{\pi}. \tag{2.13}$$

Therefore, Lemma 2.3 follows from (2.13) and the monotonicity of $F(\tau)$. □

Lemma 2.4 *The function*

$$G(\tau) = \frac{\frac{2}{\pi}\tau\mathcal{K}(\tau) - \sinh^{-1}(\tau)}{\tanh^{-1}(\tau) - \sinh^{-1}(\tau)}$$

is strictly decreasing from $(0, 1)$ *onto* $(2/\pi, 5/6)$.

Proof Let $G_1(\tau) = 2\tau\mathcal{K}(\tau)/\pi - \sinh^{-1}(\tau)$, $G_2(\tau) = \tanh^{-1}(\tau) - \sinh^{-1}(\tau)$, $G_3(\tau) = 2\sqrt{1+\tau^2}\mathcal{E}(\tau)/(\pi\tau'^2) - 1$, $G_4(\tau) = \sqrt{1+\tau^2}/\tau'^2 - 1$, $G_5(\tau) = 2[(1+3\tau^2)\mathcal{E}(\tau) - \tau'^2(1+\tau^2)\mathcal{K}(\tau)]/\pi$ and $G_6(\tau) = \tau^2(3+\tau^2)$. Then elaborated computations lead to

$$G_1(0^+) = G_2(0^+) = 0, \quad G(\tau) = \frac{G_1(\tau)}{G_2(\tau)}, \tag{2.14}$$

$$G_3(0^+) = G_4(0^+) = 0, \quad \frac{G_1'(\tau)}{G_2'(\tau)} = \frac{G_3(\tau)}{G_4(\tau)}, \tag{2.15}$$

$$G_5(0^+) = G_6(0^+) = 0, \quad \frac{G_3'(\tau)}{G_4'(\tau)} = \frac{G_5(\tau)}{G_6(\tau)}, \tag{2.16}$$

$$\frac{G_5'(\tau)}{G_6'(\tau)} = \frac{1}{\pi}\left[\frac{2\mathcal{E}(\tau)}{3+2\tau^2} + \frac{3\left(2\mathcal{E}(\tau) - \tau'^2\mathcal{K}(\tau)\right)}{3+2\tau^2}\right]. \tag{2.17}$$

It follows from Lemma 2.2(5) and (6) together with (2.17) that $G_5'(\tau)/G_6'(\tau)$ is strictly decreasing on (0,1). Then from (2.14)-(2.16) and Lemma 2.1 we know that $G(\tau)$ is strictly decreasing on (0,1).

Note that

$$G(0^+) = \frac{5}{6}, \quad G(1^-) = \frac{2}{\pi}. \tag{2.18}$$

Therefore, Lemma 2.4 follows from (2.18) and the monotonicity of $G(\tau)$. □

Lemma 2.5 *The function*

$$H(\tau) = \frac{\frac{2}{\pi}\tau\mathcal{K}(\tau) - \arctan(\tau)}{\tanh^{-1}(\tau) - \arctan(\tau)}$$

is strictly decreasing from $(0, 1)$ *onto* $(2/\pi, 7/8)$.

Proof Let $H_1(\tau) = 2\tau\mathcal{K}(\tau)/\pi - \arctan(\tau)$, $H_2(\tau) = \tanh^{-1}(\tau) - \arctan(\tau)$, $H_3(\tau) = 2(1+\tau^2)\mathcal{E}(\tau)/\pi - (1-\tau^2)$ and $H_4(\tau) = 2\tau^2$. Then simple computations lead to

$$H_1(0^+) = H_2(0^+) = 0, \quad H(\tau) = \frac{H_1(\tau)}{H_2(\tau)}, \tag{2.19}$$

$$H_3(0^+) = H_4(0^+) = 0, \quad \frac{H_1'(\tau)}{H_2'(\tau)} = \frac{H_3(\tau)}{H_4(\tau)}, \tag{2.20}$$

$$\frac{H_3'(\tau)}{H_4'(\tau)} = \frac{1}{2} + \frac{1}{2\pi}\left[2\mathcal{E}(\tau) + \frac{(1+\tau^2)(\mathcal{E}(\tau) - \mathcal{K}(\tau))}{\tau^2}\right]. \tag{2.21}$$

It follows from Lemma 2.2(7) and (2.21) together with the monotonicity of $\mathcal{E}(\tau)$ that $H_3'(\tau)/H_4'(\tau)$ is strictly decreasing on (0,1). Then from (2.19), (2.20) and Lemma 2.1 we know that $H(\tau)$ is strictly decreasing on (0,1).

Note that

$$H(0^+) = \frac{7}{8}, \quad H(1^-) = \frac{2}{\pi}. \tag{2.22}$$

Therefore, Lemma 2.5 follows easily from (2.22) and the monotonicity of $H(\tau)$. □

Lemma 2.6 *The function*

$$I(\tau) = \frac{\frac{2}{\pi}\tau\mathcal{K}(\tau) - \frac{\tanh^{-1}(\tau) - \arcsin(\tau)}{2}}{\frac{\tanh^{-1}(\tau)}{3} + \frac{\sinh^{-1}(\tau)}{6} - \frac{\arcsin(\tau)}{2}}$$

is strictly increasing from $(0, 1)$ *onto* $(3/40, 3(4-\pi)/(2\pi))$.

Proof Let $I_1(\tau) = 2\tau\mathcal{K}(\tau)/\pi - [\tanh^{-1}(\tau) + \arcsin(\tau)]/2$, $I_2(\tau) = \tanh^{-1}(\tau)/3 + \sinh^{-1}(\tau)/6 - \arcsin(\tau)/2$, $I_3(\tau) = 2\mathcal{E}(\tau)/\pi - (1+\tau')/2$, $I_4(\tau) = 1/3 + \tau'^2/(6\sqrt{1+\tau^2}) - \tau'/2$, $I_5(\tau) = 1/2 - 2\tau'[\mathcal{K}(\tau) - \mathcal{E}(\tau)]/(\pi\tau^2)$ and $I_6(\tau) = 1/2 - \tau'(3+\tau^2)/[6(3+\tau^2)^{3/2}]$. Then elaborated computations lead to

$$I_1(0^+) = I_2(0^+) = 0, \quad I(\tau) = \frac{I_1(\tau)}{I_2(\tau)}, \tag{2.23}$$

$$I_3(0^+) = I_4(0^+) = 0, \quad \frac{I_1'(\tau)}{I_2'(\tau)} = \frac{I_3(\tau)}{I_4(\tau)}, \tag{2.24}$$

$$I_5(0^+) = I_6(0^+) = 0, \quad \frac{I_3'(\tau)}{I_4'(\tau)} = \frac{I_5(\tau)}{I_6(\tau)}, \tag{2.25}$$

$$\frac{I_5'(\tau)}{I_6'(\tau)} = \frac{6}{\pi}\left[\frac{(1+\tau^2)^{5/2}}{5-\tau^2}\right]\left[\frac{(1+\tau'^2)\mathcal{K}(\tau) - 2\mathcal{E}(\tau)}{\tau^4}\right]. \tag{2.26}$$

It is easy to verify that the function $\tau \mapsto (1+\tau^2)^{5/2}/(5-\tau^2)$ is positive and strictly increasing on (0, 1). Then (2.26) and Lemma 2.2(4) lead to the conclusion that $I_5'(\tau)/I_6'(\tau)$ is strictly increasing on (0,1). Thus, from (2.23), (2.24) and (2.25) together with Lemma 2.1 we know that $I(\tau)$ is strictly increasing on (0,1).

Note that

$$I(0^+) = \frac{3}{40}, \quad I(1^-) = \frac{3(4-\pi)}{2\pi}. \tag{2.27}$$

Therefore, Lemma 2.6 follows from (2.27) and the monotonicity of $I(\tau)$. □

Lemma 2.7 *The function*

$$J(\tau) = \frac{\frac{2}{\pi}\tau\mathcal{K}(\tau) - \frac{\tanh^{-1}(\tau)-\arcsin(\tau)}{2}}{\frac{3\tanh^{-1}(\tau)}{8} + \frac{\arctan(\tau)}{8} - \frac{\arcsin(\tau)}{2}}$$

is strictly increasing from $(0, 1)$ *onto* $(1/20, 4(4-\pi)/(3\pi))$.

Proof Let $J_1(\tau) = 2\tau\mathcal{K}(\tau)/\pi - [\tanh^{-1}(\tau) + \arcsin(\tau)]/2$, $J_2(\tau) = 3\tanh^{-1}(\tau)/8 + \sinh^{-1}(\tau)/8 - \arcsin(\tau)/2$, $J_3(\tau) = 2\mathcal{E}(\tau)/\pi - (1+\tau')/2$, $J_4(\tau) = 3/8 + \tau'^2/[8(1+\tau^2)] - \tau'/2$, $J_5(\tau) = 1 - 4\tau'[\mathcal{K}(\tau) - \mathcal{E}(\tau)]/(\pi\tau^2)$ and $J_6(\tau) = 1 - \tau'/(1+\tau^2)^2$. Then simple computations lead to

$$J_1(0^+) = J_2(0^+) = 0, \quad J(\tau) = \frac{J_1(\tau)}{J_2(\tau)}, \tag{2.28}$$

$$J_3(0^+) = J_4(0^+) = 0, \quad \frac{J_1'(\tau)}{J_2'(\tau)} = \frac{J_3(\tau)}{J_4(\tau)}, \tag{2.29}$$

$$J_5(0^+) = J_6(0^+) = 0, \quad \frac{J_3'(\tau)}{J_4'(\tau)} = \frac{J_5(\tau)}{J_6(\tau)}, \tag{2.30}$$

$$\frac{J_5'(\tau)}{J_6'(\tau)} = \frac{4}{\pi}\left[\frac{(1+\tau^2)^3}{5-3\tau^2}\right]\left[\frac{(1+\tau'^2)\mathcal{K}(\tau) - 2\mathcal{E}(\tau)}{\tau^4}\right]. \tag{2.31}$$

It is not difficult to verify that the function $\tau \mapsto (1+\tau^2)^3/(5-3\tau^2)$ is positive and strictly increasing on $(0, 1)$. Then (2.31) and Lemma 2.2 (4) lead to the conclusion that $J_5'(\tau)/J_6'(\tau)$ is strictly increasing on (0,1). Hence, from (2.28), (2.29) and (2.30) together with Lemma 2.1 we know that $J(\tau)$ is strictly increasing on (0,1).

Note that

$$J(0^+) = \frac{1}{20}, \quad J(1^-) = \frac{4(4-\pi)}{3\pi}. \tag{2.32}$$

Therefore, Lemma 2.7 follows from (2.32) and the monotonicity of $J(\tau)$. □

3 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1 We clearly see that inequalities (1.11)–(1.13) can be rewritten as

$$\alpha_1 < \frac{\frac{2}{\pi}\tau\mathcal{K}(\tau) - \arcsin(\tau)}{\tanh^{-1}(\tau) - \arcsin(\tau)} < \beta_1, \tag{3.1}$$

$$\alpha_2 < \frac{\frac{2}{\pi}\tau\mathcal{K}(\tau) - \sinh^{-1}(\tau)}{\tanh^{-1}(\tau) - \sinh^{-1}(\tau)} < \beta_2, \tag{3.2}$$

$$\alpha_3 < \frac{\frac{2}{\pi}\tau\mathcal{K}(\tau) - \arctan(\tau)}{\tanh^{-1}(\tau) - \arctan(\tau)} < \beta_3, \tag{3.3}$$

respectively. Therefore, Theorem 1.1 follows easily from (3.1)–(3.3) and Lemmas 2.3– 2.5. □

Proof of Theorem 1.2 We clearly see that inequalities (1.14) and (1.15) are equivalent to

$$\alpha_4 < \frac{\frac{2}{\pi}\tau\mathcal{K}(\tau) - \frac{\tanh^{-1}(\tau)-\arcsin(\tau)}{2}}{\frac{\tanh^{-1}(\tau)}{3} + \frac{\sinh^{-1}(\tau)}{6} - \frac{\arcsin(\tau)}{2}} < \beta_4, \tag{3.4}$$

and

$$\alpha_5 < \frac{\frac{2}{\pi}\tau\mathcal{K}(\tau) - \frac{\tanh^{-1}(\tau)-\arcsin(\tau)}{2}}{\frac{3\tanh^{-1}(\tau)}{8} + \frac{\arctan(\tau)}{8} - \frac{\arcsin(\tau)}{2}} < \beta_5, \tag{3.5}$$

respectively. Therefore, Theorem 1.2 follows easily from (3.4) and (3.5) together with Lemmas 2.6 and 2.7. □

Let $0 < \tau < 1$, $u = 1$, $v = (1-\tau)/(1+\tau)$ and

$$L(u, v) = \frac{u - v}{\log u - \log v}$$

be the logarithmic mean of u and v. Then we clearly see that

$$\frac{A(u, v)}{L(u, v)} = \frac{\tanh^{-1}(\tau)}{\tau}. \tag{3.6}$$

From (1.7)–(1.9), Theorems 1.1 and 1.2, and (3.6) we get Corollaries 3.1 and 3.2 immediately.

Corollary 3.1 *The double inequalities*

$$\begin{aligned}
&\frac{\pi}{4\tau}\left[\tanh^{-1}(\tau) + \arcsin(\tau)\right] < \mathcal{K}(\tau) < \frac{1}{2\tau}\left[2\tanh^{-1}(\tau) + (\pi - 2)\arcsin(\tau)\right],\\
&\frac{1}{2\tau}\left[2\tanh^{-1}(\tau) + (\pi - 2)\sinh^{-1}(\tau)\right] < \mathcal{K}(\tau) < \frac{\pi}{12\tau}\left[5\tanh^{-1}(\tau) + \sinh^{-1}(\tau)\right],\\
&\frac{1}{2\tau}\left[2\tanh^{-1}(\tau) + (\pi - 2)\arctan(\tau)\right] < \mathcal{K}(\tau) < \frac{\pi}{16\tau}\left[7\tanh^{-1}(\tau) + \arctan(\tau)\right],\\
&\frac{\pi}{160\tau}\left[42\tanh^{-1}(\tau) + \sinh^{-1}(\tau) + 37\arcsin(\tau)\right] < \mathcal{K}(\tau)\\
&\quad < \frac{1}{8\tau}\left[8\tanh^{-1}(\tau) + (4 - \pi)\sinh^{-1}(\tau) + (5\pi - 12)\arcsin(\tau)\right],\\
&\frac{\pi}{320\tau}\left[83\tanh^{-1}(\tau) + 76\arcsin(\tau) + \arctan(\tau)\right] < \mathcal{K}(\tau)\\
&\quad < \frac{1}{12\tau}\left[12\tanh^{-1}(\tau) + (7\pi - 16)\arcsin(\tau) + (4 - \pi)\arctan(\tau)\right]
\end{aligned}$$

hold for all $0 < \tau < 1$.

Corollary 3.2 *The double inequalities*

$$\begin{aligned}
&\frac{\alpha_1}{L(u,v)} + \frac{1-\alpha_1}{P(u,v)} < \frac{1}{AG(u,v)} < \frac{\beta_1}{L(u,v)} + \frac{1-\beta_1}{P(u,v)},\\
&\frac{\alpha_2}{L(u,v)} + \frac{1-\alpha_2}{M(u,v)} < \frac{1}{AG(u,v)} < \frac{\beta_2}{L(u,v)} + \frac{1-\beta_2}{M(u,v)},\\
&\frac{\alpha_3}{L(u,v)} + \frac{1-\alpha_3}{T(u,v)} < \frac{1}{AG(u,v)} < \frac{\beta_3}{L(u,v)} + \frac{1-\beta_3}{T(u,v)},\\
&\alpha_4\left[\frac{5}{6L(u,v)} + \frac{1}{6M(u,v)}\right] + (1-\alpha_4)\left[\frac{1}{2L(u,v)} + \frac{1}{2P(u,v)}\right] < \frac{1}{AG(u,v)}
\end{aligned}$$

$$< \beta_4 \left[\frac{5}{6L(u,v)} + \frac{1}{6M(u,v)}\right] + (1-\beta_4)\left[\frac{1}{2L(u,v)} + \frac{1}{2P(u,v)}\right],$$

$$\alpha_5 \left[\frac{7}{8L(u,v)} + \frac{1}{8T(u,v)}\right] + (1-\alpha_5)\left[\frac{1}{2L(u,v)} + \frac{1}{2P(u,v)}\right] < \frac{1}{AG(u,v)}$$

$$< \beta_5 \left[\frac{7}{8L(u,v)} + \frac{1}{8T(u,v)}\right] + (1-\beta_5)\left[\frac{1}{2L(u,v)} + \frac{1}{2P(u,v)}\right]$$

hold for all $u, v > 0$ *with* $u \neq v$ *if and only if* $\alpha_1 \leq 1/2$, $\alpha_2 \leq 2/\pi$, $\alpha_3 \leq 2/\pi$, $\alpha_4 \leq 3/40$, $\alpha_5 \leq 1/20$, $\beta_1 \geq 2/\pi$, $\beta_2 \geq 5/6$, $\beta_3 \geq 7/8$, $\beta_4 \geq 3(4-\pi)/(2\pi) = 0.4098\ldots$ *and* $\beta_5 \geq 4(4-\pi)/(3\pi) = 0.3643\ldots$.

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