RACSAM (2020) 114:14
https://doi.org/10.1007/s13398-019-00775-5

ORIGINAL PAPER

Ground state solutions for general Choquard equations with a variable potential and a local nonlinearity

Sitong Chen[1] · Xianhua Tang[1]

Received: 16 January 2019 / Accepted: 18 November 2019 / Published online: 6 December 2019

Abstract
This paper deals with the following Choquard equation with a local nonlinear perturbation:

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * F(u))f(u) + g(u), & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential, $N \geq 3, \alpha \in (0, N)$, $F(t) = \int_0^t f(s)\mathrm{d}s \geq 0\ (\not\equiv 0)$, $V \in \mathcal{C}^1(\mathbb{R}^N, [0, \infty))$ and $f, g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ satisfying the subcritical growth. Under some suitable conditions on V, we prove that the above problem admits ground state solutions without super-linear conditions near infinity or monotonicity properties on f and g. In particular, some new tricks are used to overcome the combined effects and the interaction of the nonlocal nonlinear term and the local nonlinear term. Our results improve and extends the previous related ones in the literature.

Keywords Choquard equation · Local nonlinear perturbation · Ground state solution · Pohožaev mainfold

Mathematics Subject Classification 35J20 · 35J62 · 35Q55

1 Introduction

In this paper, we consider the existence of ground state solution for the following equation:

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * F(u))f(u) + g(u), & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

This work is partially supported by the National Natural Science Foundation of China (no. 11571370).

✉ Xianhua Tang
tangxh@mail.csu.edu.cn

Sitong Chen
mathsitongchen@163.com

1 School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, People's Republic of China

where $N \geq 3$, $\alpha \in (0, N)$ and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) 2^\alpha \pi^{N/2} |x|^{N-\alpha}}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

$F(t) = \int_0^t f(s)\mathrm{d}s$, $V : \mathbb{R}^N \to \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

(V1) $V \in \mathcal{C}(\mathbb{R}^N, [0, \infty))$ and $V(x) \leq V_\infty := \lim_{|y|\to\infty} V(y)$ for all $x \in \mathbb{R}^N$;
(V2) $V \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$ and there exists $\theta \in [0, 1)$ such that either of the following cases holds:

(i) $\nabla V(x) \cdot x \leq \frac{\theta(N-2)^2}{2|x|^2}$ for all $x \in \mathbb{R}^N \setminus \{0\}$,
(ii) $\| \max\{\nabla V(x) \cdot x, 0\}\|_{N/2} \leq 2\theta S$, where $S = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}$;

(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $f(t) = o(t^{\alpha/N})$ as $t \to 0$ and $f(t) = o\left(t^{(\alpha+2)/(N-2)}\right)$ as $|t| \to \infty$;
(F2) $F(t) \geq 0$ for all $t \in \mathbb{R}$ and $\mathrm{meas}\{t \in \mathbb{R} : F(t) = 0\} = 0$;
(G1) $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $g(t) = o(|t|)$ as $t \to 0$ and $g(t) = o(|t|^{(N-2)/2N})$ as $|t| \to \infty$.

By (V1), (F1), (G1), the Hardy–Littlewood–Sobolev and the Sobolev embedding theorem, the energy functional $\mathcal{I} : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated with (1.1) is continuously differentiable defined by

$$\mathcal{I}(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla u|^2 + V(x)u^2\right] \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\mathrm{d}x - \int_{\mathbb{R}^N} G(u)\mathrm{d}x, \tag{1.2}$$

and its critical points correspond to the weak solutions of (1.1). A solution is called a ground state solution if its energy is minimal among all nontrivial solutions.

Equation (1.1) can be viewed as a local nonlinear perturbation of the following Choquard equation

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * F(u))f(u), & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.3}$$

which has a strong physical meaning, and appears in several physical contexts, for example, for $N = 3$, $\alpha = 2$, $V(x) = 1$ and $f(u) = u$, it was used to study the quantum theory of a polaron at rest by Pekar [24]; to describe an electron trapped in its own hole by Choquard [17]; to model a self-gravitating matter by Penrose [20]. In a few decade, Eq. (1.3) has been studied by variational methods. For the special form of (1.3):

$$\begin{cases} -\Delta u + u = (I_\alpha * |u|^q)|u|^{q-2}u, & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.4}$$

Moroz and Van Schaftingen [21] obtained the existence of ground state solutions and qualitative properties of solutions within an optimal range exponents q satisfying by the intercriticality condition: $1 + \alpha/N < q < (N + \alpha)/(N - 2)$, and showed that it has no nontrivial solution when either $q \leq 1 + \alpha/N$ or $q \geq (N + \alpha)/(N - 2)$, where endpoints of the above interval are lower and upper critical exponents for the Choquard equation. Later, in another paper [22], they proved the existence of a ground state solution for (1.3) with $V = 1$ under Berestycki–Lions assumptions on f, by using a scaling technique introduced by Jeanjean [11] whose key is to construct a Palais–Smale sequence ((PS) sequence in short) that satisfies asymptotically the Pohožaev identity (a Pohožaev–Palais–Smale sequence in short). For more existence results on (1.3) or (1.4), we refer to [1–3,8,10,18,19,23,26,33].

Recently, many researchers began to focus on the existence of ground state solutions for the Choquard equation with a local nonlinear perturbation like (1.1). It seems that the first result is due to Van Schaftingen and Xia [31]. By the mountain pass lemma and a concentration compactness argument, they proved the existence and symmetry of ground state solutions for (1.1) where $V = 1$, $f(u) = |u|^{\frac{\alpha}{N}-1}u$ and g satisfies (G1) and the following super-linear conditions:

(G2) there exists $\mu > 2$ such that $0 < \mu G(t) \le g(t)t$ for all $t \ne 0$;

(G3) there exists $\Lambda_0 > 0$ such that $\liminf_{|t|\to 0} \frac{G(t)}{|t|^{N/4+2}} \ge \Lambda_0$.

Note that (G2) is a well-known assumption of Ambrosetti–Rabinowitz type, which can help verify the mountain pass geometry and the boundedness of (PS) sequence for the corresponding functional. Inspired by [31], Li et al. [16] considered the following equation:

$$\begin{cases} -\Delta u + u = (I_\alpha * |u|^q)|u|^{q-2}u + |u|^{p-2}u, & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.5}$$

with $1+\alpha/N < q < (N+\alpha)/(N-2)$ and $2 < p < 2^*$, and obtained a ground state solution of mountain pass type under some additional assumptions on p and q. Regarding existence results for (1.1) with $V = 1$ and $f(u) = |u|^{\frac{(N+\alpha)}{(N-2)}-2}u$, we quote Ao [4], Li and Tang [14] and Li and Ma [15]. It is worth pointing out that the nonlinearity f is always a power function and the local nonlinear perturbation g is super-linear at infinity in the above-mentioned papers. In fact, the approaches used in these papers rely heavily on the homogeneous of degree s ($s = 1 + \alpha/N, q, (N + \alpha)/(N - 2)$), the constant potential V and the super-linear growth of g. It is difficult to generalize the results on existence of ground state solutions for (1.5) to (1.1) with a variable potential V and general interaction functions f and g.

Motivated by the above works, especially [16], in this paper, we shall establish the existence of ground state solutions for (1.1), and improve and generalize the results on (1.5) obtained in [16] to (1.1). In particular, different from the existing literature, in our argument, f and g only need to satisfy (F1), (F2) and (G1). Compared with the related results, we must overcome the difficulties due to the following unpleasant facts.

(a) Since f and g satisfy neither the Ambrosetti–Rabinowitz growth condition nor monotonicity properties, the usual Nehari manifold to obtain existence of nontrivial solutions does not work anymore.
(b) No condition is imposed on the nonlinear terms f and g near infinity, except $f(t) = o(t^{(\alpha+2)/(N-2)})$ and $g(t) = o(|t|^{(N-2)/2N})$ as $|t| \to \infty$. So we have to take care of the combined effects and the interaction of the nonlocal nonlinear term and the local nonlinear term to verify the mountain pass geometry.
(c) The fact that $V \not\equiv$ constant in (1.1) prevents us from constructing a Pohožaev–Palais–Smale sequence as in [22]. Moreover, we have to introduce other skills to recover the compactness since our work space is $H^1(\mathbb{R}^N)$ not $H^1_r(\mathbb{R}^N)$.

These difficulties enforce the implementation of new ideas and techniques. To the best of our knowledge, there seem to be no results for (1.1) on this topic until now.

Now, we are in a position to state our first result.

Theorem 1.1 *Assume that* V, f *and* g *satisfy* (V1), (V2), (F1), (F2) *and* (G1). *Then* (1.1) *has a ground state solution* $\bar{u} \in H^1(\mathbb{R}^N)$ *such that* $\mathcal{I}(\bar{u}) = \inf_{\mathcal{K}} \mathcal{I}$, *where*

$$\mathcal{K} := \{u \in H^1(\mathbb{R}^N)\backslash\{0\} : \mathcal{I}'(u) = 0\}.$$

Applying Theorem 1.1 to the following perturbed problem:

$$\begin{cases} -\Delta u + [V_\infty - \varepsilon h(x)]u = (I_\alpha * F(u))f(u) + G(u), & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.6}$$

where V_∞ is a positive constant and the function $h \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$ verifies:

(H1) $h(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $\lim_{|x|\to\infty} h(x) = 0$;
(H2) $\sup_{x\in\mathbb{R}^N}[-|x|^2\nabla h(x) \cdot x] < \infty$,

we have the following corollary.

Corollary 1.2 *Assume that h, f and g satisfy* (H1), (H2), (F1), (F2) *and* (G1). *Then there exists a constant $\hat{\varepsilon} > 0$ such that* (1.6) *has a ground state solution $\bar{u}_\varepsilon \in H^1(\mathbb{R}^N)\setminus\{0\}$ for all $0 < \varepsilon \leq \hat{\varepsilon}$.*

Next, we further provide a minimax characterization of the ground state energy. Inspired by [6,7,9], we introduce a monotonicity condition on V as follows:

(V3) $V \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$ and $t \mapsto NV(tx) + \nabla V(tx) \cdot (tx) + \frac{(N-2)^3}{4t^2|x|^2}$ is nonincreasing on $(0, \infty)$ for all $x \in \mathbb{R}^N\setminus\{0\}$.

And we define the following Pohožaev functional on $H^1(\mathbb{R}^N)$:

$$\begin{aligned} \mathcal{P}(u) := & \frac{N-2}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}[NV(x) + \nabla V(x) \cdot x]u^2 \mathrm{d}x \\ & - \frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x - N\int_{\mathbb{R}^N} G(u)\mathrm{d}x. \end{aligned} \tag{1.7}$$

In view of [16, Proposition 3.1], if $\bar{u}$ is a solution of (1.1), then it satisfies Pohožaev identity $\mathcal{P}(\bar{u}) = 0$. Let

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^N)\setminus\{0\} : \mathcal{P}(u) = 0\}. \tag{1.8}$$

Then every solution of (1.1) is contained in $\mathcal{M}$, we call $\mathcal{M}$ the Pohožaev manifold of $\mathcal{I}$. Our second main result is as follows.

Theorem 1.3 *Assume that V, f and g satisfy* (V1)–(V3), (F1), (F2) *and* (G1). *Then* (1.1) *has a solution $\bar{u} \in H^1(\mathbb{R}^N)$ such that $\mathcal{I}(\bar{u}) = \inf_{\mathcal{M}} \mathcal{I} = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} \mathcal{I}(u_t) > 0$, where $u_t(x) := u(x/t)$.*

Applying Theorem 1.3 to the limiting problem:

$$\begin{cases} -\Delta u + V_\infty u = (I_\alpha * F(u))f(u) + g(u), & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.9}$$

we have the following corollary.

Corollary 1.4 *Assume that f and g satisfy* (F1), (F2) *and* (G1). *Then* (1.9) *has a solution $\bar{u} \in H^1(\mathbb{R}^N)$ such that $\mathcal{I}^\infty(\bar{u}) = \inf_{\mathcal{M}^\infty} \mathcal{I}^\infty = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} \mathcal{I}^\infty(u_t) > 0$, where*

$$\mathcal{I}^{\infty}(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla u|^2 + V_\infty u^2\right]\mathrm{d}x \\ -\frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x - \int_{\mathbb{R}^N} G(u)\mathrm{d}x, \tag{1.10}$$

$$\mathcal{P}^{\infty}(u) = \frac{N-2}{2}\|\nabla u\|_2^2 + \frac{NV_\infty}{2}\|u\|_2^2 \\ -\frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x - N\int_{\mathbb{R}^N} G(u)\mathrm{d}x = 0 \tag{1.11}$$

and

$$\mathcal{M}^{\infty} := \{u \in H^1(\mathbb{R}^N)\backslash\{0\} : \mathcal{P}^{\infty}(u) = 0\}. \tag{1.12}$$

Remark 1.5 Applying Corollary 1.4 to Eq. (1.5) considered in [16], (1.5) admits a ground state solution provided $1 + \alpha/N < q < (N+\alpha)/(N-2)$ and $2 < p < 2^*$, while the extra conditions on p and q used in [16] are removed. Our results generalize and improve the main results in [16], and also extend the previous related ones in the literature.

To prove Theorem 1.1, following an approximation procedure developed by Jeanjean and Toland [13], we construct a sequence $\{u_n\}$ of exact critical points of nearby functionals which satisfies $\lambda_n \uparrow 1$, $\mathcal{I}'_{\lambda_n}(u_n) = 0$ and $\mathcal{I}_{\lambda_n}(u_n) \to c_* > 0$, where

$$\mathcal{I}_\lambda(u) = \mathcal{I}(u) + (1-\lambda)\int_{\mathbb{R}^N}\left[\frac{1}{2}(I_\alpha * F(u))F(u) + G(u)\right]\mathrm{d}x, \\ \forall\, u \in H^1(\mathbb{R}^N),\ \lambda \in [1/2, 1].$$

Note that the variable potential $V(x)$ in (1.1) breaks down the invariance under translations in $\mathbb{R}^N$. To circumvent this obstacle, we borrow the idea used in [25] which rely on a comparison of the mountain pass level with the ground state energy for the corresponding limit problem (1.9). But, in our assumptions the function $\frac{1}{2}(I_\alpha * F(u))F(u) + G(u)$ may be sign-changing and the ground state solutions of the limit problem (1.9) are not positive definite. These facts, together with the appearance of the nonlocal nonlinear term would require our extra efforts. More precisely, as in [29], we give a new minimax characterization of the ground state energy for the limit functional $\mathcal{I}_\lambda^\infty$ (see (3.3) below), and establish the key inequality:

$$c_\lambda < m_\lambda^\infty := \inf_{u \in \mathcal{M}_\lambda^\infty} \mathcal{I}_\lambda^\infty(u) = \inf_{u \in H^1(\mathbb{R}^N)\backslash\{0\}} \max_{t>0} \mathcal{I}_\lambda^\infty(u_t)$$

for $\lambda \in (\bar{\lambda}, 1]$ by using some new analytical skills and finer calculations (see Lemma 3.5), and then prove the strong convergence of critical points $\{u_n\}$ based on the above inequality and the global compactness lemma.

To prove Theorem 1.3, inspired by the works in [5,29], we look for a minimizer for the minimization problem $m := \inf_{\mathcal{M}} \mathcal{I}$ and then prove that the minimizer is a ground state solution of (1.1). More precisely, we first choose a minimizing sequence $\{u_n\}$ of $\mathcal{I}$ on $\mathcal{M}$ satisfying

$$\mathcal{I}(u_n) \to m = \inf_{\mathcal{M}} \mathcal{I}, \qquad \mathcal{P}(u_n) = 0. \tag{1.13}$$

Then we show, with a concentration-compactness argument and "the least energy squeeze approach", that there exist $\hat{u} \in H^1(\mathbb{R}^N)\backslash\{0\}$ and $\hat{t} > 0$ such that, after a translation and extraction of a subsequence, $u_n \rightharpoonup \hat{u}$ in $H^1(\mathbb{R}^N)$, and $(\hat{u})_{\hat{t}} \in \mathcal{M}$ is a minimizer of m (see Lemma 2.11), since the lack of information on $\mathcal{I}'(u_n)$ prevents us from using the global compactness lemma in $H^1(\mathbb{R}^N)$. In the final, we prove that $\bar{u}$ is a critical point of $\mathcal{I}$ by combining the deformation lemma and intermediary theorem for continuous functions (see Lemma 2.12).

Throughout the paper we make use of the following notations:

- $H^1(\mathbb{R}^N)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv)\mathrm{d}x, \quad \|u\| = (u, u)^{1/2}, \quad \forall\, u, v \in H^1(\mathbb{R}^N).$$

- $L^s(\mathbb{R}^N)(1 \le s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_{\mathbb{R}^N} |u|^s \mathrm{d}x\right)^{1/s}$.
- For any $u \in H^1(\mathbb{R}^N)\setminus\{0\}$, $u_t(x) := u(t^{-1}x)$ for $t > 0$.
- For any $x \in \mathbb{R}^N$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^N : |y - x| < r\}$.
- $C_1, C_2, \cdots$ denote positive constants possibly different in different places.

Under (V1), there exists a constant $\gamma_0 > 0$ such that

$$\gamma_0\|u\|^2 \le \int_{\mathbb{R}^N} \left[|\nabla u|^2 + V(x)u^2\right] \mathrm{d}x \le \max\{1, V_\infty\}\|u\|^2, \qquad \forall\, u \in H^1(\mathbb{R}^N). \tag{1.14}$$

By (F1) and Hardy–Littlewood–Sobolev inequality, for some $\kappa \in (2, 2^*)$ and any $\epsilon > 0$, one has

$$\begin{aligned} &\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\mathrm{d}x \\ &= \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) 2^\alpha \pi^{N/2}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(x))F(u(y))}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \le C_1\|F(u)\|^2_{2N/(N+\alpha)} \\ &\le \epsilon\left(\|u\|_2^{2(N+\alpha)/N} + \|u\|_{2^*}^{2(N+\alpha)/(N-2)}\right) + C_\epsilon\|u\|_\kappa^{(N+\alpha)\kappa/N}, \quad \forall\, u \in H^1(\mathbb{R}^N). \end{aligned} \tag{1.15}$$

The rest of the paper is organized as follows. As mentioned above, since the proof of Theorem 1.1 require the helps of a ground state solution for the limiting problem (1.9) and a minimax characterization of its energy, for the sake of convenience, the proof of Theorem 1.3 is provided in Sect. 2. Section 3 is devoted to the proof of Theorem 1.1.

2 Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. Since $V(x) \equiv V_\infty$ satisfies (V1)–(V3), thus all conclusions on $\mathcal{I}$ are also true for $\mathcal{I}^\infty$. For (1.9), we always assume that $V_\infty > 0$.

First, by a simple calculation, we can verify the following lemma.

Lemma 2.1 *The following two inequalities hold:*

$$2 - Nt^{N-2} + (N-2)t^N > 0, \quad \forall\, t \in [0, 1) \cup (1, +\infty), \tag{2.1}$$

$$\beta(t) := \alpha - (N+\alpha)t^N + Nt^{N+\alpha} > \beta(1) = 0, \quad \forall\, t \in [0, 1) \cup (1, +\infty). \tag{2.2}$$

Moreover, (V3) *implies the following inequality holds:*

$$\begin{aligned} &Nt^N [V(x) - V(tx)] + \left(t^N - 1\right)\nabla V(x)\cdot x \\ &\ge -\frac{(N-2)^2\left[2 - Nt^{N-2} + (N-2)t^N\right]}{4|x|^2}, \quad \forall\, t \ge 0,\ x \in \mathbb{R}^N\setminus\{0\}. \end{aligned} \tag{2.3}$$

Inspired by [30], we establish a key functional inequality as follows.

Lemma 2.2 *Assume that* (V1), (V3), (F1), (F2) *and* (G1) *hold. Then*

$$\mathcal{I}(u) \geq \mathcal{I}(u_t) + \frac{1-t^N}{N}\mathcal{P}(u) + \frac{\beta(t)}{2N}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x, \quad \forall\, u \in H^1(\mathbb{R}^N),\ \ t>0. \tag{2.4}$$

Proof According to Hardy inequality, we have

$$\|\nabla u\|_2^2 \geq \frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}\mathrm{d}x, \quad \forall\, u \in H^1(\mathbb{R}^N). \tag{2.5}$$

Note that

$$\begin{aligned} \mathcal{I}(u_t) = & \frac{t^{N-2}}{2}\|\nabla u\|_2^2 + \frac{t^N}{2}\int_{\mathbb{R}^N}V(tx)u^2\mathrm{d}x \\ & -\frac{t^{N+\alpha}}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x - t^N\int_{\mathbb{R}^N}G(u)\mathrm{d}x. \end{aligned} \tag{2.6}$$

Thus, by (1.2), (1.7), (2.1), (2.3), (2.5) and (2.6), one has

$$\begin{aligned} & \mathcal{I}(u) - \mathcal{I}(u_t) \\ & = \frac{1-t^{N-2}}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}\left[V(x) - t^N V(tx)\right]u^2\mathrm{d}x \\ & \quad -\frac{1-t^{N+\alpha}}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x - (1-t^N)\int_{\mathbb{R}^N}G(u)\mathrm{d}x \\ & = \frac{1-t^N}{N}\left\{\frac{N-2}{2}\|\nabla u\|_2^2\right. \\ & \quad +\frac{1}{2}\int_{\mathbb{R}^N}[NV(x) + \nabla V(x)\cdot x]u^2\mathrm{d}x - \frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x \\ & \quad \left.-N\int_{\mathbb{R}^N}G(u)\mathrm{d}x\right\} + \frac{2 - Nt^{N-2} + (N-2)t^N}{2N}\|\nabla u\|_2^2 \\ & \quad +\frac{1}{2}\int_{\mathbb{R}^N}\left\{t^N[V(x) - V(tx)] - \frac{1-t^N}{N}\nabla V(x)\cdot x\right\}u^2\mathrm{d}x \\ & \quad +\frac{\alpha - (N+\alpha)t^N + Nt^{N+\alpha}}{2N}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x \\ & \geq \frac{1-t^N}{N}\mathcal{P}(u) + \frac{1}{2N}\int_{\mathbb{R}^N}\left\{\frac{(N-2)^2\left[2 - Nt^{N-2} + (N-2)t^N\right]}{4|x|^2}\right. \\ & \quad \left.+Nt^N[V(x) - V(tx)] - (1-t^N)\nabla V(x)\cdot x\right\}u^2\mathrm{d}x + \frac{\beta(t)}{2N}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x \\ & \geq \frac{1-t^N}{N}\mathcal{P}(u) + \frac{\beta(t)}{2N}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x. \end{aligned}$$

This shows that (2.4) holds. □

From Lemma 2.2, we have the following two corollaries.

Corollary 2.3 *Assume that* (F1), (F2) *and* (G1) *hold. Then*

$$\mathcal{I}^{\infty}(u) \geq \mathcal{I}^{\infty}(u_t) + \frac{1-t^N}{N}\mathcal{P}^{\infty}(u) + \frac{2-Nt^{N-2}+(N-2)t^N}{2N}\|\nabla u\|_2^2,$$
$$\forall\, u \in H^1(\mathbb{R}^N), \quad t > 0. \tag{2.7}$$

Corollary 2.4 *Assume that* (V1), (V3), (F1), (F2) *and* (G1) *hold. Then for* $u \in \mathcal{M}$

$$\mathcal{I}(u) = \max_{t>0} \mathcal{I}(u_t). \tag{2.8}$$

Next, we shall construct a saddle point structure with respect to the fibre $\{u_t : t > 0\} \subset H^1(\mathbb{R}^N)$ for $u \in H^1(\mathbb{R}^3)\setminus\{0\}$. For this purpose, we need the following inequality.

Lemma 2.5 *Assume that* (V1) *and* (V3) *hold. Then*

(i) $|\nabla V(x)\cdot x| \to 0$ *as* $|x| \to \infty$*;*
(ii) *there exist two constants* $\gamma_1, \gamma_2 > 0$ *such that for all*

$$\gamma_1\|u\|^2 \leq (N-2)\|\nabla u\|_2^2 + \int_{\mathbb{R}^N}[NV(x)+\nabla V(x)\cdot x]u^2\mathrm{d}x$$
$$\leq \gamma_2\|u\|^2, \quad \forall\, u \in H^1(\mathbb{R}^N). \tag{2.9}$$

Proof (i) Arguing by contradiction, we assume that there exist $\{x_n\} \subset \mathbb{R}^N$ and $\delta > 0$ such that

$$|x_n| \to \infty, \quad \text{and} \quad \nabla V(x_n)\cdot x_n \geq \delta \text{ or } \nabla V(x_n)\cdot x_n \leq -\delta, \quad \forall\, n \in \mathbb{N}. \tag{2.10}$$

Now, we only distinguish two cases: (1) $\nabla V(x_n)\cdot x_n \geq \delta, \forall\, n \in \mathbb{N}$ and (2) $\nabla V(x_n)\cdot x_n \leq -\delta, \forall\, n \in \mathbb{N}$.

Case (1) $\nabla V(x_n)\cdot x_n \geq \delta, \forall\, n \in \mathbb{N}$. Note that (2.3) with $t = 0$ gives

$$\nabla V(x)\cdot x \leq \frac{(N-2)^2}{2|x|^2}, \quad \forall\, x \in \mathbb{R}^N\setminus\{0\}. \tag{2.11}$$

Then (2.11) implies that

$$\delta \leq \nabla V(x_n)\cdot x_n \leq \frac{(N-2)^2}{2|x_n|^2} = o(1), \tag{2.12}$$

which is a obvious contradiction.

Case (2) $\nabla V(x_n)\cdot x_n \leq -\delta$ for all $n \in \mathbb{N}$. Clearly, (2.1) yields

$$2 - 2^{N-2}N + 2^N(N-2) > 0. \tag{2.13}$$

From (2.3), with $t = 2$, and (2.13), we derive

$$\begin{aligned} -\delta &\geq \nabla V(x_n)\cdot x_n \\ &\geq \frac{N2^N[V(2x_n)-V(x_n)]}{2^N-1} - \frac{(N-2)^2[2-2^{N-2}N+2^N(N-2)]}{4(2^N-1)|x_n|^2} = o(1). \end{aligned}$$

Again this contradiction proves that (i) holds.

(ii) Note the item (i) implies that $\nabla V(x)\cdot x$ is bounded for all $x \in \mathbb{R}^N$. From (V1), (2.3), with $t \to \infty$, and (2.11), we deduce

$$-\frac{(N-2)^3}{4|x|^2} + NV_\infty \leq NV(x) + \nabla V(x)\cdot x \leq NV_\infty + \frac{(N-2)^2}{2|x|^2}, \quad \forall\, x \in \mathbb{R}^N\setminus\{0\}. \tag{2.14}$$

Thus it follows from (V1) and (2.14) that

$$
\begin{aligned}
&(N-2)\|\nabla u\|_2^2+\int_{\mathbb{R}^N}[NV(x)+\nabla V(x)\cdot x]\,u^2\mathrm{d}x\\
&\le (N-2+2)\|\nabla u\|_2^2+NV_\infty\|u\|_2^2\\
&\le [N-2+2+NV_\infty]\|u\|^2:=\gamma_2\|u\|^2,\quad \forall\, u\in H^1(\mathbb{R}^N).
\end{aligned}\tag{2.15}
$$

Next, we prove the first inequality in (2.9). Arguing by contradiction, suppose that there exists a sequence $\{u_n\}\subset H^1(\mathbb{R}^N)$ such that

$$
\|u_n\|=1,\quad (N-2)\|\nabla u_n\|_2^2+\int_{\mathbb{R}^N}[NV(x)+\nabla V(x)\cdot x]\,u_n^2\mathrm{d}x=o(1).\tag{2.16}
$$

Thus there exists $\bar u\in H^1(\mathbb{R}^N)$ such that $u_n\rightharpoonup\bar u$ in $H^1(\mathbb{R}^N)$. Then $u_n\to\bar u$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2\le s<2^*$ and $u_n\to\bar u$ a.e. in $\mathbb{R}^N$. By (V1) and (2.14), one has

$$
V(x)\to V_\infty,\ \ |\nabla V(x)\cdot x|\to 0 \text{ as } |x|\to\infty.\tag{2.17}
$$

This implies that there exists a constant $R_0>0$ such that

$$
NV(x)+\nabla V(x)\cdot x\ge\frac{N}{2}V_\infty,\quad \forall\,|x|\ge R_0.\tag{2.18}
$$

Since $u_n\to\bar u$ in $L^2(B_{R_0}(0))$, it follows from (2.5), (2.14), (2.15), (2.16), (2.18), the weak semicontinuity of norm and Fatou's Lemma that

$$
\begin{aligned}
0&=\lim_{n\to\infty}\left\{(N-2)\|\nabla u_n\|_2^2+\int_{|x|<R_0}[NV(x)+\nabla V(x)\cdot x]\,u_n^2\mathrm{d}x\right.\\
&\qquad\left.+\int_{|x|\ge R_0}[NV(x)+\nabla V(x)\cdot x]\,u_n^2\mathrm{d}x\right\}\\
&\ge (N-2)\|\nabla\bar u\|_2^2+\int_{|x|<R_0}[NV(x)+\nabla V(x)\cdot x]\,\bar u^2\mathrm{d}x+\frac{NV_\infty}{2}\liminf_{n\to\infty}\int_{|x|\ge R_0}u_n^2\mathrm{d}x\\
&\ge\int_{|x|<R_0}\left[\frac{(N-2)^3}{4|x|^2}+NV(x)+\nabla V(x)\cdot x\right]\bar u^2\mathrm{d}x+\frac{NV_\infty}{2}\int_{|x|\ge R_0}\bar u^2\mathrm{d}x\\
&\ge\frac{NV_\infty}{2}\|\bar u\|_2^2,
\end{aligned}
$$

which implies $\bar u=0$. Thus, from (V1) and (2.14), one has

$$
\int_{\mathbb{R}^N}[N(V(x)-V_\infty)+\nabla V(x)\cdot x]\,u_n^2\mathrm{d}x=o(1).\tag{2.19}
$$

Both (2.16) and (2.19) imply

$$
\begin{aligned}
o(1)&=(N-2)\|\nabla u_n\|_2^2+\int_{\mathbb{R}^N}[NV(x)+\nabla V(x)\cdot x]\,u_n^2\mathrm{d}x\\
&=(N-2)\|\nabla u_n\|_2^2+NV_\infty\|u_n\|_2^2+o(1)\\
&\ge\min\{N-2,NV_\infty\}\|u_n\|^2+o(1)\\
&=\min\{N-2,NV_\infty\}+o(1).
\end{aligned}
$$

This contradiction shows that there exists γ_1 such that the first inequality in (2.9) holds. □

Based on the above lemmas, we establish the following important property for $\mathcal{M}$.

Lemma 2.6 *Assume that* (V1), (V3), (F1), (F2) *and* (G1) *hold. Then for any* $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, *there exists a unique* $t_u > 0$ *such that* $u_{t_u} \in \mathcal{M}$.

Proof Let $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ be fixed and define a function $\zeta(t) := \mathcal{I}(u_t)$ on $(0, \infty)$. Clearly, by (1.7) and (2.6), we have

$$\begin{aligned} \zeta'(t) = 0 \ \Leftrightarrow \ & \frac{N-2}{2} t^{N-2} \|\nabla u\|_2^2 + \frac{t^N}{2} \int_{\mathbb{R}^N} [NV(tx) + \nabla V(tx) \cdot (tx)] u^2 \mathrm{d}x \\ & - \frac{(N+\alpha) t^{N+\alpha}}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \mathrm{d}x - N t^N \int_{\mathbb{R}^N} G(u) \mathrm{d}x = 0 \\ \Leftrightarrow \ & \mathcal{P}(u_t) = 0 \ \Leftrightarrow \ u_t \in \mathcal{M}. \end{aligned} \tag{2.20}$$

By (V1), (F1) and (G1), one has $\zeta(t) = 0$ and $\zeta(t) > 0$ for $t > 0$ small. Noting that (F2) implies $\int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \mathrm{d}x > 0$, we get easily $\zeta(t) < 0$ for t large. Therefore $\max_{t \in (0,\infty)} \zeta(t)$ is achieved at $t_u > 0$ so that $\zeta'(t_u) = 0$ and $u_{t_u} \in \mathcal{M}$.

Next we claim that t_u is unique for any $u \in H^1(\mathbb{R}^N)\backslash\{0\}$. Otherwise, for some $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, there exists two positive constants $t_1 \neq t_2$ such that $u_{t_1}, u_{t_2} \in \mathcal{M}$, and so $\mathcal{P}\left(u_{t_1}\right) = \mathcal{P}\left(u_{t_2}\right) = 0$. From (2.2) and (2.4), we have

$$\mathcal{I}\left(u_{t_1}\right) > \mathcal{I}\left(u_{t_2}\right) + \frac{t_1^N - t_2^N}{N t_1^N} \mathcal{P}\left(u_{t_1}\right) = \mathcal{I}\left(u_{t_2}\right)$$

and

$$\mathcal{I}\left(u_{t_2}\right) > \mathcal{I}\left(u_{t_1}\right) + \frac{t_2^N - t_1^N}{N t_2^N} \mathcal{P}\left(u_{t_2}\right) = \mathcal{I}\left(u_{t_1}\right).$$

This contradiction shows that $t_u > 0$ is unique for any $u \in H^1(\mathbb{R}^N)\backslash\{0\}$. □

Corollary 2.7 *Assume that* (F1), (F2) *and* (G1) *hold. Then for any* $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, *there exists a unique* $t_u > 0$ *such that* $u_{t_u} \in \mathcal{M}^\infty$.

From Corollary 2.4, Lemma 2.6 and Corollary 2.7, we have $\mathcal{M} \neq \emptyset$, $\mathcal{M}^\infty \neq \emptyset$ and the following minimax characterization.

Lemma 2.8 *Assume that* (V1), (V3), (F1), (F2) *and* (G1) *hold. Then*

$$\inf_{u \in \mathcal{M}} \mathcal{I}(u) = m = \inf_{u \in H^1(\mathbb{R}^N)\backslash\{0\}} \max_{t>0} \mathcal{I}(u_t).$$

Lemma 2.9 *Assume that* (V1)-(V3), (F1), (F2) *and* (G1) *hold. Then*

(i) *there exists* $\rho_0 > 0$ *such that* $\|u\| \geq \rho_0$, $\forall\, u \in \mathcal{M}$;
(ii) $m = \inf_{u \in \mathcal{M}} \mathcal{I}(u) > 0$.

Proof (i). Since $\mathcal{P}(u) = 0$ for all $u \in \mathcal{M}$, by (1.7), (1.15), (2.9) and Sobolev embedding theorem, one has

$$\begin{aligned} \frac{\gamma_1}{2} \|u\|^2 & \leq \frac{N-2}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} [NV(x) + \nabla V(x) \cdot x] u^2 \mathrm{d}x \\ & = \frac{N+\alpha}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \mathrm{d}x + N \int_{\mathbb{R}^N} G(u) \mathrm{d}x \\ & \leq \|u\|^{2(N+\alpha)/N} + C_1 \|u\|^{2(N+\alpha)/(N-2)} + \frac{\gamma_1}{4} \|u\|^2 + C_2 \|u\|^{2N/(N-2)}, \end{aligned} \tag{2.21}$$

which implies

$$\|u\| \geq \rho_0 := \min\left\{1, \left[\frac{\gamma_1}{4(1+C_1+C_2)}\right]^{\max\{N/2\alpha,(N-2)/4\}}\right\}, \quad \forall\, u \in \mathcal{M}. \tag{2.22}$$

(ii). Let $\{u_n\} \subset \mathcal{M}$ be such that $\mathcal{I}(u_n) \to m$. There are two possible cases: (1) $\inf_{n\in\mathbb{N}} \|\nabla u_n\|_2 > 0$ and (2) $\inf_{n\in\mathbb{N}} \|\nabla u_n\|_2 = 0$.

Case (1) $\inf_{n\in\mathbb{N}} \|\nabla u_n\|_2 := \varrho_0 > 0$. Note that by (1.2) and (1.7), one has

$$\begin{aligned}\mathcal{I}(u) - \frac{1}{N}\mathcal{P}(u) &= \frac{1}{N}\|\nabla u\|_2^2 - \frac{1}{2N}\int_{\mathbb{R}^N} \nabla V(x)\cdot x u^2 \mathrm{d}x \\ &\quad + \frac{\alpha}{2N}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x, \quad \forall\, u \in H^1(\mathbb{R}^N).\end{aligned} \tag{2.23}$$

If (i) of (V2) holds, then it follows from Hardy inequality (2.5) that

$$\int_{\mathbb{R}^N} \nabla V(x)\cdot x u^2 \mathrm{d}x \leq \frac{\theta(N-2)^2}{2}\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}\mathrm{d}x \leq 2\theta\|\nabla u\|_2^2, \quad \forall\, u \in H^1(\mathbb{R}^N). \tag{2.24}$$

If (ii) of (V2) holds, then it follows from the Sobolev embedding inequality that

$$\begin{aligned}&\int_{\mathbb{R}^N} \nabla V(x)\cdot x u^2 \mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^N} |\max\{\nabla V(x)\cdot x, 0\}|^{N/2}\mathrm{d}x\right)^{2/N}\left(\int_{\mathbb{R}^N}|u|^{2N/(N-2)}\mathrm{d}x\right)^{(N-2)/N} \\ &\leq \frac{\|\max\{\nabla V(x)\cdot x, 0\}\|_{N/2}}{S}\|\nabla u\|_2^2 \leq 2\theta\|\nabla u\|_2^2, \quad \forall\, u \in H^1(\mathbb{R}^N).\end{aligned} \tag{2.25}$$

By (2.23) and (2.24) or (2.25), we have

$$m + o(1) = \mathcal{I}(u_n) = \mathcal{I}(u_n) - \frac{1}{N}\mathcal{P}(u_n) \geq \frac{1-\theta}{N}\|\nabla u_n\|_2^2 \geq \frac{1-\theta}{N}\varrho_0^2.$$

Case (2) $\inf_{n\in\mathbb{N}} \|\nabla u_n\|_2 = 0$. In this case, by (2.22), passing to a subsequence, one has

$$\|\nabla u_n\|_2 \to 0, \quad \|u_n\|_2 \geq \frac{1}{2}\rho_0. \tag{2.26}$$

By (1.15) and the Sobolev inequality, one has for all $u \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}&\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x \\ &\leq C_3\left(\|u\|_2^{2(N+\alpha)/N} + \|u\|_{2^*}^{2(N+\alpha)/(N-2)}\right) \\ &\leq C_3\left(\|u\|_2^{2(N+\alpha)/N} + S^{-(N+\alpha)/(N-2)}\|\nabla u\|_2^{2(N+\alpha)/(N-2)}\right).\end{aligned} \tag{2.27}$$

By (V1), there exists $R > 0$ such that $V(x) \geq \frac{V_\infty}{2}$ for $|x| \geq R$. This implies

$$\int_{|tx|\geq R} V(tx)u^2\mathrm{d}x \geq \frac{V_\infty}{2}\int_{|tx|\geq R} u^2\mathrm{d}x, \quad \forall\, t > 0,\ u \in H^1(\mathbb{R}^N). \tag{2.28}$$

Making use of the Hölder inequality and the Sobolev inequality, we get

$$\begin{aligned}\int_{|tx|<R} u^2\mathrm{d}x &\leq \left(\frac{\omega_N R^N}{t^N}\right)^{(2^*-2)/2^*}\left(\int_{|tx|<R} u^{2^*}\mathrm{d}x\right)^{2/2^*} \\ &\leq \omega_N^{2/N} R^2 t^{-2} S^{-1}\|\nabla u\|_2^2, \quad \forall\, t > 0,\ u \in H^1(\mathbb{R}^N),\end{aligned} \tag{2.29}$$

where ω_N denotes the volume of the unit ball of $\mathbb{R}^N$. Let

$$\delta_0 = \min\left\{V_\infty, SR^{-2}\omega_N^{-2/N}\right\} \tag{2.30}$$

and

$$t_n = \left(\frac{\delta_0}{8C_3}\right)^{1/\alpha} \|u_n\|_2^{-2/N}. \tag{2.31}$$

By (G1) and the Sobolev inequality, one has

$$\int_{\mathbb{R}^N} G(u)\mathrm{d}x \le \frac{\delta_0}{8}\|u\|_2^2 + C_4\|u\|_{2^*}^{2^*} \le \frac{\delta_0}{8}\|u\|_2^2 + C_4 S^{-N/(N-2)}\|\nabla u\|_2^{2N/(N-2)}. \tag{2.32}$$

Since (2.26) implies $\{t_n\}$ is bounded, then it follows from (2.6), (2.8), (2.26), (2.27), (2.28), (2.29), (2.30), (2.31) and (2.32) that

$$\begin{aligned}
m + o(1) &= \mathcal{I}(u_n) \ge \mathcal{I}\left((u_n)_{t_n}\right)\\
&= \frac{t_n^{N-2}}{2}\|\nabla u_n\|_2^2 + \frac{t_n^N}{2}\int_{\mathbb{R}^N} V(t_n x)u_n^2\mathrm{d}x - \frac{t_n^{N+\alpha}}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(u_n)\mathrm{d}x\\
&\quad - t_n^N\int_{\mathbb{R}^N} G(u_n)\mathrm{d}x\\
&\ge \frac{S}{2R^2\omega_N^{2/N}} t_n^N \int_{|t_n x|<R} u_n^2\mathrm{d}x + \frac{1}{4}V_\infty t_n^N \int_{|t_n x|\ge R} u_n^2\mathrm{d}x\\
&\quad -\frac{1}{2}C_3 t_n^{N+\alpha}\|u_n\|_2^{2(N+\alpha)/N} - \frac{C_3}{2S^{(N+\alpha)/(N-2)}} t_n^{N+\alpha}\|\nabla u_n\|_2^{2(N+\alpha)/(N-2)}\\
&\quad -\frac{\delta_0 t_n^N}{8}\|u_n\|_2^2 - C_4 S^{-N/(N-2)} t_n^N \|\nabla u_n\|_2^{2N/(N-2)}\\
&\ge \frac{1}{8}\delta_0 t_n^N\|u_n\|_2^2 - \frac{1}{2}C_3 t_n^{N+\alpha}\|u_n\|_2^{2(N+\alpha)/N} + o(1)\\
&= \frac{1}{8}t_n^N\|u_n\|_2^2\left(\delta_0 - 4C_3 t_n^\alpha\|u_n\|_2^{2\alpha/N}\right) + o(1)\\
&= \frac{\delta_0}{16}\left(\frac{\delta_0}{8C_3}\right)^{N/\alpha} + o(1).
\end{aligned} \tag{2.33}$$

Cases (1) and (2) show that $m = \inf_{u\in\mathcal{M}}\mathcal{I}(u) > 0$. □

Lemma 2.10 *Assume that* (V1)–(V3), (F1), (F2) *and* (G1) *hold. Then* $m \le m^\infty$.

Proof Arguing by contradiction, we assume that $m > m^\infty$. Let $\varepsilon := m - m^\infty$. Then there exists u_ε^∞ such that

$$u_\varepsilon^\infty \in \mathcal{M}^\infty \quad \text{and} \quad m^\infty + \frac{\varepsilon}{2} > \mathcal{I}^\infty(u_\varepsilon^\infty). \tag{2.34}$$

In view of Lemma 2.6, there exists $t_\varepsilon > 0$ such that $(u_\varepsilon^\infty)_{t_\varepsilon} \in \mathcal{M}$. Thus, it follows from (V1), (1.2), (1.10), (2.7) and (2.34) that

$$m^\infty + \frac{\varepsilon}{2} > \mathcal{I}^\infty(u_\varepsilon^\infty) \ge \mathcal{I}^\infty\left((u_\varepsilon^\infty)_{t_\varepsilon}\right) \ge \mathcal{I}\left((u_\varepsilon^\infty)_{t_\varepsilon}\right) \ge m.$$

This contradiction shows the conclusion of Lemma 2.10 is true. □

Lemma 2.11 *Assume that* (V1)–(V3), (F1), (F2) *and* (G1) *hold. Then m is achieved.*

Proof In view of Lemma 2.9, we have $m > 0$. Let $\{u_n\} \subset \mathcal{M}$ be such that $\mathcal{I}(u_n) \to m$. Since $\mathcal{P}(u_n) = 0$, it follows from (2.23) and (2.24) or (2.25) that

$$m + o(1) = \mathcal{I}(u_n) \geq \frac{1-\theta}{N} \|\nabla u_n\|_2^2. \tag{2.35}$$

Moreover, from (V2), (G1), (1.2), (1.7), (1.14), (2.5), the Sobolev embedding inequality and (2.24) or (2.25), we derive

$$\begin{aligned}
\mathcal{I}(u_n) &= \mathcal{I}(u_n) - \frac{1}{N+\alpha}\mathcal{P}(u_n) \\
&= \frac{2+\alpha}{2(N+\alpha)}\|\nabla u_n\|_2^2 + \frac{1}{2(N+\alpha)}\int_{\mathbb{R}^N} [\alpha V(x) - \nabla V(x)\cdot x]\, u_n^2 \mathrm{d}x \\
&\quad - \frac{\alpha}{N+\alpha}\int_{\mathbb{R}^N} G(u_n)\mathrm{d}x \\
&\geq \frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}\left[|\nabla u_n|^2 + V(x)u_n^2\right]\mathrm{d}x - \frac{\alpha\gamma_0}{4(N+\alpha)}\|u_n\|_2^2 - C_5\|u_n\|_{2^*}^{2^*} \\
&\geq \frac{\alpha\gamma_0}{4(N+\alpha)}\|u_n\|^2 - C_5 S^{-2^*/2}\|\nabla u_n\|_2^{2^*},
\end{aligned} \tag{2.36}$$

which, together with (2.35), implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Passing to a subsequence, we have $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^N)$. Then $u_n \to \bar{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq s < 2^*$ and $u_n \to \bar{u}$ a.e. in $\mathbb{R}^N$. Inspired by [28, Lemma 3.2], we now distinguish the following two cases: (i) $\bar{u} = 0$ and (ii). $\bar{u} \neq 0$.

Case (i). $\bar{u} = 0$, i.e. $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$. Then $u_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq s < 2^*$ and $u_n \to 0$ a.e. in $\mathbb{R}^N$. By (2.17), it is easy to show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}[V_\infty - V(x)]u_n^2\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^N}\nabla V(x)\cdot x u_n^2 \mathrm{d}x = 0. \tag{2.37}$$

From (1.2), (1.7), (1.10), (1.11) and (2.37), one can get

$$\mathcal{I}^\infty(u_n) \to m, \quad \mathcal{P}^\infty(u_n) \to 0. \tag{2.38}$$

From Lemma 2.11 (i), (1.11) and (2.38), one has

$$\begin{aligned}
&\min\{N-2, NV_\infty\}\rho_0^2 \leq \min\{N-2, NV_\infty\}\|u_n\|^2 \\
&\quad \leq (N-2)\|\nabla u_n\|_2^2 + NV_\infty\|u_n\|_2^2 \\
&\quad = (N+\alpha)\int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(u_n)\mathrm{d}x + 2N\int_{\mathbb{R}^N} G(u_n)\mathrm{d}x + o(1).
\end{aligned} \tag{2.39}$$

Using (1.15), (2.39) and Lions' concentration compactness principle [32, Lemma 1.21], we can prove that there exist $\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n)} |u_n|^2\mathrm{d}x > \delta$. Let $\hat{u}_n(x) = u_n(x+y_n)$. Then we have $\|\hat{u}_n\| = \|u_n\|$ and

$$\mathcal{I}^\infty(\hat{u}_n) \to m, \quad \mathcal{P}^\infty(\hat{u}_n) = o(1), \quad \int_{B_1(0)} |\hat{u}_n|^2\mathrm{d}x > \delta. \tag{2.40}$$

Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that, passing to a subsequence,

$$\begin{cases} \hat{u}_n \rightharpoonup \hat{u}, & \text{in } H^1(\mathbb{R}^N); \\ \hat{u}_n \to \hat{u}, & \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^N), \ \forall\, s \in [1, 2^*); \\ \hat{u}_n \to \hat{u}, & \text{a.e. on } \mathbb{R}^N. \end{cases} \tag{2.41}$$

Let $w_n = \hat{u}_n - \hat{u}$. Then (2.41) and the Brezis–Lieb type Lemma (see [18, Lemmas 5.1–5.3], [27, Lemmas 2.7 and 2.8] and [32]) lead to

$$\mathcal{I}^{\infty}(\hat{u}_n) = \mathcal{I}^{\infty}(\hat{u}) + \mathcal{I}^{\infty}(w_n) + o(1) \tag{2.42}$$

and

$$\mathcal{P}^{\infty}(\hat{u}_n) = \mathcal{P}^{\infty}(\hat{u}) + \mathcal{P}^{\infty}(w_n) + o(1). \tag{2.43}$$

Set

$$\begin{aligned}\Psi_0(u) &= \mathcal{I}^{\infty}(u) - \frac{1}{N}\mathcal{P}^{\infty}(u) \\ &= \frac{1}{N}\|\nabla u\|_2^2 + \frac{\alpha}{2N}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x, \quad \forall\, u \in H^1(\mathbb{R}^N).\end{aligned} \tag{2.44}$$

From (2.40), (2.42), (2.43) and (2.44), one has

$$\Psi_0(w_n) = m - \Psi_0(\hat{u}) + o(1), \quad \mathcal{P}^{\infty}(w_n) = -\mathcal{P}^{\infty}(\hat{u}) + o(1). \tag{2.45}$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then going to this subsequence, we have

$$\mathcal{I}^{\infty}(\hat{u}) = m, \quad \mathcal{P}^{\infty}(\hat{u}) = 0. \tag{2.46}$$

Next, we assume that $w_n \neq 0$. We claim that $\mathcal{P}^{\infty}(\hat{u}) \leq 0$. Otherwise, if $\mathcal{P}^{\infty}(\hat{u}) > 0$, then (2.45) implies $\mathcal{P}^{\infty}(w_n) < 0$ for large n. In view of Corollary 2.7, there exists $t_n > 0$ such that $(w_n)_{t_n} \in \mathcal{M}^{\infty}$ for large n. From (2.7), (2.44) and (2.45), we obtain

$$\begin{aligned}m - \Psi_0(\hat{u}) + o(1) = \Psi_0(w_n) &= \mathcal{I}^{\infty}(w_n) - \frac{1}{N}\mathcal{P}^{\infty}(w_n) \\ &\geq \mathcal{I}^{\infty}\left((w_n)_{t_n}\right) - \frac{t_n^N}{N}\mathcal{P}^{\infty}(w_n) \\ &\geq m^{\infty} - \frac{t_n^N}{N}\mathcal{P}^{\infty}(w_n) \geq m^{\infty},\end{aligned}$$

which implies $\mathcal{P}^{\infty}(\hat{u}) \leq 0$ due to $m \leq m^{\infty}$ and $\Psi_0(\hat{u}) > 0$. Since $\hat{u} \neq 0$ and $\mathcal{P}^{\infty}(\hat{u}) \leq 0$, in view of Corollary 2.7, there exists $t_\infty > 0$ such that $\hat{u}_{t_\infty} \in \mathcal{M}^{\infty}$. From (2.1), (2.7), (2.40), (2.44), the weak semicontinuity of norm and Fatou's lemma, one has

$$\begin{aligned}m &= \lim_{n\to\infty}\left[\mathcal{I}^{\infty}(\hat{u}_n) - \frac{1}{N}\mathcal{P}^{\infty}(\hat{u}_n)\right] \\ &= \lim_{n\to\infty}\Psi_0(\hat{u}_n) \geq \Psi_0(\hat{u}) \\ &= \mathcal{I}^{\infty}(\hat{u}) - \frac{1}{N}\mathcal{P}^{\infty}(\hat{u}) \geq \mathcal{I}^{\infty}\left(\hat{u}_{t_\infty}\right) - \frac{(t_\infty)^N}{N}\mathcal{P}^{\infty}(\hat{u}) \\ &\geq m^{\infty} - \frac{(t_\infty)^N}{N}\mathcal{P}^{\infty}(\hat{u}) \geq m - \frac{(t_\infty)^N}{N}\mathcal{P}^{\infty}(\hat{u}) \geq m,\end{aligned}$$

which implies (2.46) holds also. In view of Lemma 2.6, there exists $\hat{t} > 0$ such that $\hat{u}_{\hat{t}} \in \mathcal{M}$, moreover, it follows from (V1), (1.2), (1.10), (2.46) and Corollary 2.3 that

$$m \leq \mathcal{I}(\hat{u}_{\hat{t}}) \leq \mathcal{I}^{\infty}(\hat{u}_{\hat{t}}) \leq \mathcal{I}^{\infty}(\hat{u}) = m.$$

This shows that m is achieved at $\hat{u}_{\hat{t}} \in \mathcal{M}$.

Case (ii). $\bar{u} \neq 0$. Let $v_n = u_n - \bar{u}$. Then the Brezis–Lieb type Lemma (see [18, Lemmas 5.1-5.3], [27, Lemmas 2.7 and 2.8] and [32]) leads to

$$\mathcal{I}(u_n) = \mathcal{I}(\bar{u}) + \mathcal{I}(v_n) + o(1) \tag{2.47}$$

and

$$\mathcal{P}(u_n) = \mathcal{P}(\bar{u}) + \mathcal{P}(v_n) + o(1). \tag{2.48}$$

Set

$$\Psi(u) = \mathcal{I}(u) - \frac{1}{N}\mathcal{P}(u), \quad \forall\, u \in H^1(\mathbb{R}^N). \tag{2.49}$$

Since $\mathcal{I}(u_n) \to m$ and $\mathcal{P}(u_n) = 0$, then it follows from (2.47), (2.48) and (2.49) that

$$\Psi(v_n) = m - \Psi(\bar{u}) + o(1), \quad \mathcal{P}(v_n) = -\mathcal{P}(\bar{u}) + o(1). \tag{2.50}$$

If there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $v_{n_i} = 0$, then going to this subsequence, we have

$$\mathcal{I}(\bar{u}) = m, \quad \mathcal{P}(\bar{u}) = 0, \tag{2.51}$$

which implies the conclusion of Lemma 2.11 holds. Next, we assume that $v_n \neq 0$. We claim that $\mathcal{P}(\bar{u}) \leq 0$. Otherwise $\mathcal{P}(\bar{u}) > 0$, then (2.50) implies $\mathcal{P}(v_n) < 0$ for large n. In view of Lemma 2.6, there exists $t_n > 0$ such that $(v_n)_{t_n} \in \mathcal{M}$ for large n. From (2.4), (2.49) and (2.50), we obtain

$$\begin{aligned} m - \Psi(\bar{u}) + o(1) = \Psi(v_n) &= \mathcal{I}(v_n) - \frac{1}{N}\mathcal{P}(v_n) \\ &\geq \mathcal{I}\left((v_n)_{t_n}\right) - \frac{t_n^N}{N}\mathcal{P}(v_n) \\ &\geq m - \frac{t_n^N}{N}\mathcal{P}(v_n) \geq m, \end{aligned}$$

which implies $\mathcal{P}(\bar{u}) \leq 0$ due to $\Psi(\bar{u}) > 0$. Since $\bar{u} \neq 0$ and $\mathcal{P}(\bar{u}) \leq 0$, in view of Lemma 2.6, there exists $\bar{t} > 0$ such that $\bar{u}_{\bar{t}} \in \mathcal{M}$. From (2.4), (2.23), (2.49), the weak semicontinuity of norm, Fatou's lemma and (2.24) or (2.25), one has

$$\begin{aligned} m &= \lim_{n\to\infty}\left[\mathcal{I}(u_n) - \frac{1}{N}\mathcal{P}(u_n)\right] = \lim_{n\to\infty}\Psi(u_n) \geq \Psi(\bar{u}) \\ &= \mathcal{I}(\bar{u}) - \frac{1}{N}\mathcal{P}(\bar{u}) \geq \mathcal{I}(\bar{u}_{\bar{t}}) - \frac{\bar{t}^N}{N}\mathcal{P}(\bar{u}) \\ &\geq m - \frac{\bar{t}^N}{N}\mathcal{P}(\bar{u}) \geq m, \end{aligned}$$

which implies (2.51) also holds. □

Lemma 2.12 *Assume that* (V1)–(V3), (F1), (F2) *and* (G1) *hold. If* $\bar{u} \in \mathcal{M}$ *and* $\mathcal{I}(\bar{u}) = m$, *then* $\bar{u}$ *is a critical point of* $\mathcal{I}$.

Proof Following the idea of [5, Lemma 2.14], we can prove the above conclusion. For the sake of completeness, we give some details. Assume that $\mathcal{I}'(\bar{u}) \neq 0$. Then there exist $\delta > 0$ and $\varrho > 0$ such that

$$\|u - \bar{u}\| \leq 3\delta \Rightarrow \|\mathcal{I}'(u)\| \geq \varrho. \tag{2.52}$$

As in the proof of [5, (2.40)], one has

$$\lim_{t\to 1} \|\bar{u}_t - \bar{u}\| = 0. \tag{2.53}$$

Thus, there exists $\delta_1 \in (0, 1/4)$ such that

$$|t-1| < \delta_1 \Rightarrow \|\bar{u}_t - \bar{u}\| < \delta. \tag{2.54}$$

In view of (2.2) and (2.4), one has

$$\mathcal{I}(\bar{u}_t) \le \mathcal{I}(\bar{u}) - \frac{\beta(t)}{2N}\int_{\mathbb{R}^N}(I_\alpha * F(\bar{u}))F(\bar{u})\mathrm{d}x < m, \quad \forall\, t \in (0,1)\cup(1,\infty). \tag{2.55}$$

From (F1), (F2), (G1) and (1.7), there exist $T_1 \in (0, 1)$ and $T_2 \in (1, \infty)$ such that

$$\mathcal{P}\left(\bar{u}_{T_1}\right) > 0, \quad \mathcal{P}\left(\bar{u}_{T_2}\right) < 0. \tag{2.56}$$

Moreover, (2.55) implies

$$\chi := \max\left\{\mathcal{I}\left(\bar{u}_{T_1}\right), \mathcal{I}\left(\bar{u}_{T_2}\right)\right\} < m. \tag{2.57}$$

Let $\varepsilon := \min\{(m-\chi)/3, 1, \varrho\delta/8\}$ and $S := B(\bar{u}, \delta)$. Then [32, Lemma 2.3] yields a deformation $\eta \in \mathcal{C}([0,1]\times H^1(\mathbb{R}^N), H^1(\mathbb{R}^N))$ such that

(i) $\eta(1,u) = u$ if $\mathcal{I}(u) < m - 2\varepsilon$ or $\mathcal{I}(u) > m + 2\varepsilon$;
(ii) $\eta\left(1, \mathcal{I}^{m+\varepsilon}\cap B(\bar{u},\delta)\right) \subset \mathcal{I}^{m-\varepsilon}$;
(iii) $\mathcal{I}(\eta(1,u)) \le \mathcal{I}(u),\ \forall\, u \in H^1(\mathbb{R}^N)$;
(iv) $\eta(1,u)$ is a homeomorphism of $H^1(\mathbb{R}^N)$.

By Corollary 2.4, $\mathcal{I}(\bar{u}_t) \le \mathcal{I}(\bar{u}) = m$ for $t > 0$, then it follows from (2.54) and (ii) that

$$\mathcal{I}(\eta(1, \bar{u}_t)) \le m - \varepsilon, \quad \forall\, t > 0,\ |t-1| < \delta_1. \tag{2.58}$$

On the other hand, by (iii) and (2.55), one has

$$\mathcal{I}(\eta(1, \bar{u}_t)) \le \mathcal{I}(\bar{u}_t) < m, \quad \forall\, t > 0,\ |t-1| \ge \delta_1. \tag{2.59}$$

Combining (2.58) with (2.59), we have

$$\max_{t\in[T_1,T_2]} \mathcal{I}(\eta(1, \bar{u}_t)) < m. \tag{2.60}$$

Define $\Phi_0(t) := \mathcal{P}(\eta(1, \bar{u}_t))$ for $t > 0$. It follows from (2.55) and i) that $\eta(1, \bar{u}_t) = \bar{u}_t$ for $t = T_1$ and $t = T_2$, which, together with (2.56), implies

$$\Phi_0(T_1) = \mathcal{P}\left(\bar{u}_{T_1}\right) > 0, \quad \Phi_0(T_2) = \mathcal{P}\left(\bar{u}_{T_2}\right) < 0.$$

Since $\Phi_0(t)$ is continuous on $(0,\infty)$, then we have that $\eta(1, \bar{u}_t) \cap \mathcal{M} \neq \emptyset$ for some $t_0 \in [T_1, T_2]$, which contradicts to the definition of m. □

Proof of Theorem 1.3 In view of Lemmas 2.9, 2.11 and 2.12 , there exists $\bar{u} \in \mathcal{M}$ such that

$$\mathcal{I}(\bar{u}) = m = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} \mathcal{I}(u_t) > 0, \quad \mathcal{I}'(\bar{u}) = 0.$$

This shows that $\bar{u}$ is a ground state solution of (1.1).

3 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. To find critical points of $\mathcal{I}$, we apply the following proposition due to Jeanjean and Toland [13].

Proposition 3.1 *Let X be a Banach space and let $J \subset \mathbb{R}^+$ be an interval, and*

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \forall\, \lambda \in J,$$

be a family of $\mathcal{C}^1$-functional on X such that

(i) *either $A(u) \to +\infty$ or $B(u) \to +\infty$, as $\|u\| \to \infty$;*
(ii) *B maps every bounded set of X into a set of $\mathbb{R}$ bounded below;*
(iii) *there are two points v_1, v_2 in X such that*

$$\tilde{c}_\lambda := \inf_{\gamma \in \tilde{\Gamma}} \max_{t \in [0,1]} \Phi_\lambda(\gamma(t)) > \max\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\}, \tag{3.1}$$

where

$$\tilde{\Gamma} = \{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\lambda \in J$, there exists a sequence $\{u_n(\lambda)\}$ such that

(i) *$\{u_n(\lambda)\}$ is bounded in X;*
(ii) *$\Phi_\lambda(u_n(\lambda)) \to c_\lambda$;*
(iii) *$\Phi'_\lambda(u_n(\lambda)) \to 0$ in X^*, where X^* is the dual of X.*

To apply Proposition 3.1, for $\lambda \in [1/2, 1]$, we consider two families of functionals $\mathcal{I}_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{I}_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x - \lambda \int_{\mathbb{R}^N} \left[\frac{1}{2}(I_\alpha * F(u))F(u) + G(u)\right] \mathrm{d}x \tag{3.2}$$

and

$$\mathcal{I}^\infty_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_\infty u^2\right) \mathrm{d}x - \lambda \int_{\mathbb{R}^N} \left[\frac{1}{2}(I_\alpha * F(u))F(u) + G(u)\right] \mathrm{d}x. \tag{3.3}$$

Similar to the proof of [16, Proposition 3.1], we can obtain the following lemma.

Lemma 3.2 *Assume that* (V1), (V2), (F1), (F2) and (G1) *hold. Let u be a critical point of $\mathcal{I}_\lambda$ in $H^1(\mathbb{R}^N)$, then we have the following Pohožaev type identity*

$$\begin{aligned} \mathcal{P}_\lambda(u) := \frac{N-2}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} [NV(x) + \nabla V(x) \cdot x]\, u^2 \mathrm{d}x \\ -\frac{(N+\alpha)\lambda}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\mathrm{d}x - N\lambda \int_{\mathbb{R}^N} G(u)\mathrm{d}x = 0. \end{aligned} \tag{3.4}$$

For $\lambda \in [1/2, 1]$, we define the following functional on $H^1(\mathbb{R}^N)$:

$$\mathcal{P}^\infty_\lambda(u) = \frac{N-2}{2}\|\nabla u\|_2^2 + NV_\infty\|u\|_2^2 - \frac{(N+\alpha)\lambda}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\mathrm{d}x. \tag{3.5}$$

By Corollary 2.3, we have the following lemma.

Lemma 3.3 *Assume that* (F1), (F2) *and* (G1) *hold. Then*

$$\mathcal{I}_\lambda^\infty(u) \geq \mathcal{I}_\lambda^\infty(u_t) + \frac{1-t^N}{N}\mathcal{P}_\lambda^\infty(u) + \frac{2-Nt^{N-2}+(N-2)t^N}{2N}\|\nabla u\|_2^2, \quad \forall\, u \in H^1(\mathbb{R}^N), \quad t > 0. \tag{3.6}$$

In view of Corollary 1.4, $\mathcal{I}_1^\infty = \mathcal{I}^\infty$ has a minimizer $u_1^\infty \neq 0$ on $\mathcal{M}_1^\infty = \mathcal{M}^\infty$, i.e.

$$u_1^\infty \in \mathcal{M}_1^\infty, \quad (\mathcal{I}_1^\infty)'(u_1^\infty) = 0 \quad \text{and} \quad m_1^\infty = \mathcal{I}_1^\infty(u_1^\infty), \tag{3.7}$$

where

$$\mathcal{M}_\lambda^\infty = \{u \in H^1(\mathbb{R}^N)\backslash\{0\} : \mathcal{P}_\lambda^\infty(u) = 0\} \tag{3.8}$$

and

$$m_\lambda^\infty = \inf_{u\in\mathcal{M}_\lambda^\infty} \mathcal{I}_\lambda^\infty(u) = \inf_{u\in H^1(\mathbb{R}^N)\backslash\{0\}} \max_{t>0} \mathcal{I}_\lambda^\infty(u_t). \tag{3.9}$$

Since (1.9) is autonomous, $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and $V(x) \leq V_\infty$ but $V(x) \not\equiv V_\infty$, then there exist $\bar{x} \in \mathbb{R}^N$ and $\bar{r} > 0$ such that

$$V_\infty - V(x) > 0, \quad |u_1^\infty(x)| > 0 \quad a.e.\ |x - \bar{x}| \leq \bar{r}. \tag{3.10}$$

Lemma 3.4 *Assume that* (V1), (V2), (F1), (F2) *and* (G1) *hold. Then*

(i) *there exists a constant* $T > 0$ *independent of* λ *such that* $\mathcal{I}_\lambda((u_1^\infty)_T) < 0$ *for all* $\lambda \in [1/2, 1]$*;*
(ii) *there exists a positive constant* κ_0 *independent of* λ *such that for all* $\lambda \in [1/2, 1]$*,*

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)) \geq \kappa_0 > \max\left\{\mathcal{I}_\lambda(0), \mathcal{I}_\lambda\left((u_1^\infty)_T\right)\right\},$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = (u_1^\infty)_T\}; \tag{3.11}$$

(iii) c_λ *is bounded for* $\lambda \in [1/2, 1]$*;*
(iv) m_λ^∞ *is non-increasing on* $\lambda \in [1/2, 1]$*;*
(v) $\limsup_{\lambda\to\lambda_0} c_\lambda \leq c_{\lambda_0}$ *for* $\lambda_0 \in (1/2, 1]$.

Proof Note that

$$\begin{aligned}\mathcal{I}_\lambda(u_t) = {} & \frac{t^{N-2}}{2}\|\nabla u\|_2^2 + \frac{t^N}{2}\int_{\mathbb{R}^N} V(tx)u^2\mathrm{d}x - \frac{\lambda t^{N+\alpha}}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\mathrm{d}x \\ & - \lambda t^N \int_{\mathbb{R}^N} G(u)\mathrm{d}x, \quad \forall\, u \in H^1(\mathbb{R}^N).\end{aligned} \tag{3.12}$$

Since $\mathcal{P}_1^\infty(u_1^\infty) = 0$, we have

$$\begin{aligned}& \lambda \int_{\mathbb{R}^N} \left[NV_\infty|u_1^\infty|^2 - (N+\alpha)(I_\alpha * F(u_1^\infty))F(u_1^\infty) - 2NG(u_1^\infty)\right]\mathrm{d}x \\ & \leq 2\mathcal{P}^\infty(u_1^\infty) - (N-2)\|\nabla u_1^\infty\|_2^2 - (1-\lambda)\int_{\mathbb{R}^N} NV_\infty|u_1^\infty|^2\mathrm{d}x \\ & \leq -(N-2)\|\nabla u_1^\infty\|_2^2 < 0.\end{aligned} \tag{3.13}$$

By (3.12) and (3.13), we have

$$
\begin{aligned}
\mathcal{I}_\lambda((u_1^\infty)_t) &= \frac{t^{N-2}}{2}\|\nabla u_1^\infty\|_2^2 + \frac{t^N}{2}\int_{\mathbb{R}^N} V(tx)|u_1^\infty|^2 \mathrm{d}x - \frac{\lambda t^N}{2}\int_{\mathbb{R}^N} V_\infty |u_1^\infty|^2 \mathrm{d}x \\
&\quad + \lambda t^N \int_{\mathbb{R}^N}\left[\frac{1}{2}V_\infty|u_1^\infty|^2 - \frac{N+\alpha}{2N}(I_\alpha * F(u_1^\infty))F(u_1^\infty) - G(u)\right]\mathrm{d}x \\
&\quad + \frac{\lambda t^N(N+\alpha-Nt^\alpha)}{2N}\int_{\mathbb{R}^N}(I_\alpha * F(u_1^\infty))F(u_1^\infty)\mathrm{d}x \\
&\le \frac{t^{N-2}}{2}\|\nabla u_1^\infty\|_2^2 + \frac{t^N}{4}\int_{\mathbb{R}^N} V_\infty|u_1^\infty|^2\mathrm{d}x \\
&\quad + \frac{t^N}{2}\int_{\mathbb{R}^N}\left[\frac{1}{2}V_\infty|u_1^\infty|^2 - \frac{N+\alpha}{2N}(I_\alpha * F(u_1^\infty))F(u_1^\infty) - G(u)\right]\mathrm{d}x \\
&\quad + \frac{t^N(N+\alpha-Nt^\alpha)}{2N}\int_{\mathbb{R}^N}(I_\alpha * F(u_1^\infty))F(u_1^\infty)\mathrm{d}x, \\
&\quad \forall\, t > 2^{1/\alpha},\ \lambda\in[1/2,1],
\end{aligned}
$$

which implies that (i) holds. The proof of (ii)–(iv) in Lemma 3.4 is standard, (v) can be proved in the same way as [12, Lemma 2.3], so we omit it. □

Lemma 3.5 *Assume that* (V1), (V2), (F1), (F2) *and* (G1) *hold. Then there exists* $\bar{\lambda}\in[1/2,1)$ *such that* $c_\lambda < m_\lambda^\infty$ *for* $\lambda\in(\bar{\lambda},1]$.

Proof It is easy to see that $\mathcal{I}_\lambda((u_1^\infty)_t)$ is continuous on $t\in(0,\infty)$. Hence for any $\lambda\in[1/2,1]$, we can choose $t_\lambda\in(0,T)$ such that $\mathcal{I}_\lambda((u_1^\infty)_{t_\lambda}) = \max_{t\in(0,T]}\mathcal{I}_\lambda((u_1^\infty)_t)$. Set

$$
\gamma_0(t) = \begin{cases} (u_1^\infty)_{(tT)}, & \text{for } t>0, \\ 0, & \text{for } t=0. \end{cases} \tag{3.14}
$$

Then $\gamma_0\in\Gamma$ defined by Lemma 3.4 (ii). Moreover

$$
\mathcal{I}_\lambda\left((u_1^\infty)_{t_\lambda}\right) = \max_{t\in[0,1]}\mathcal{I}_\lambda(\gamma_0(t)) \ge c_\lambda. \tag{3.15}
$$

Let

$$
\zeta_0 := \min\{3\bar{r}/8(1+|\bar{x}|),\, 1/4\}. \tag{3.16}
$$

Then it follows from (3.10) and (3.16) that

$$
|x-\bar{x}| \le \frac{\bar{r}}{2} \ \text{ and } \ s\in[1-\zeta_0, 1+\zeta_0] \Rightarrow |sx-\bar{x}|\le\bar{r}. \tag{3.17}
$$

Let

$$
\begin{aligned}
\bar{\lambda} := \max\Bigg\{ &\frac{1}{2},\, 1 - \frac{(1-\zeta_0)^N \min_{s\in[1-\zeta_0,1+\zeta_0]}\int_{\mathbb{R}^N}[V_\infty - V(sx)]\,|u_1^\infty|^2\mathrm{d}x}{T^N\int_{\mathbb{R}^N}\left[T^\alpha(I_\alpha * F(u_1^\infty))F(u_1^\infty) + 2|G(u_1^\infty)|\right]\mathrm{d}x}, \\
&1 - \frac{\min_{t\in\{1-\zeta_0,1+\zeta_0\}}\left\{2 - Nt^{N-2} + (N-2)t^N\right\}\|\nabla u_1^\infty\|_2^2}{NT^N\int_{\mathbb{R}^N}\left[T^\alpha(I_\alpha * F(u_1^\infty))F(u_1^\infty) + 2|G(u_1^\infty)|\right]\mathrm{d}x}\Bigg\}.
\end{aligned} \tag{3.18}
$$

Then it follows from (2.1), (2.2), (3.10) and (3.17) that $1/2\le\bar{\lambda}<1$. We have two cases to distinguish:

Case (i) $t_\lambda \in [1-\zeta_0, 1+\zeta_0]$. From (3.2), (3.3), (3.6)–(3.15), (3.17), (3.18) and Lemma 3.4 (iv), we have

$$\begin{aligned}
m_\lambda^\infty &\geq m_1^\infty = \mathcal{I}_1^\infty(u_1^\infty) \geq \mathcal{I}_1^\infty\left((u_1^\infty)_{t_\lambda}\right) \\
&= \mathcal{I}_\lambda\left((u_1^\infty)_{t_\lambda}\right) - \frac{(1-\lambda)t_\lambda^{N+\alpha}}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u_1^\infty))F(u_1^\infty)\mathrm{d}x \\
&\quad -(1-\lambda)t_\lambda^N\int_{\mathbb{R}^N}G(u_1^\infty)\mathrm{d}x + \frac{t_\lambda^N}{2}\int_{\mathbb{R}^N}[V_\infty - V(t_\lambda x)]|u_1^\infty|^2\mathrm{d}x \\
&\geq c_\lambda - \frac{(1-\lambda)T^N}{2}\int_{\mathbb{R}^N}\left[T^\alpha(I_\alpha * F(u_1^\infty))F(u_1^\infty) + 2|G(u_1^\infty)|\right]\mathrm{d}x \\
&\quad + \frac{(1-\zeta_0)^N}{2}\min_{s\in[1-\zeta_0,1+\zeta_0]}\int_{\mathbb{R}^N}[V_\infty - V(sx)]\,|u_1^\infty|^2\mathrm{d}x \\
&> c_\lambda, \quad \forall\, \lambda \in (\bar{\lambda}, 1].
\end{aligned}$$

Case (ii) $t_\lambda \in (0, 1-\zeta_0) \cup (1+\zeta_0, T]$. From (2.1), (2.2), (3.2), (3.3), (3.6), (3.7), (3.15), (3.18) and Lemma 3.4 (iv), we have

$$\begin{aligned}
m_\lambda^\infty &\geq m_1^\infty = \mathcal{I}_1^\infty(u_1^\infty) = \mathcal{I}_1^\infty\left((u_1^\infty)_{t_\lambda}\right) + \frac{2 - Nt_\lambda^{N-2} + (N-2)t_\lambda^N}{2N}\|\nabla u_1^\infty\|_2^2 \\
&= \mathcal{I}_\lambda\left((u_1^\infty)_{t_\lambda}\right) - \frac{(1-\lambda)t_\lambda^{N+\alpha}}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u_1^\infty))F(u_1^\infty)\mathrm{d}x \\
&\quad -(1-\lambda)t_\lambda^N\int_{\mathbb{R}^N}G(u_1^\infty)\mathrm{d}x + \frac{t_\lambda^N}{2}\int_{\mathbb{R}^N}[V_\infty - V(t_\lambda x)]|u_1^\infty|^2\mathrm{d}x \\
&\quad + \frac{2 - Nt_\lambda^{N-2} + (N-2)t_\lambda^N}{2N}\|\nabla u_1^\infty\|_2^2 \\
&\geq c_\lambda - \frac{(1-\lambda)T^N}{2}\int_{\mathbb{R}^N}\left[T^\alpha(I_\alpha * F(u_1^\infty))F(u_1^\infty) + 2|G(u_1^\infty)|\right]\mathrm{d}x \\
&\quad + \frac{\min_{t\in\{1-\zeta_0,1+\zeta_0\}}\left\{2 - Nt^{N-2} + (N-2)t^N\right\}\|\nabla u_1^\infty\|_2^2}{2N} \\
&> c_\lambda, \quad \forall\, \lambda \in (\bar{\lambda}, 1].
\end{aligned}$$

In both cases, we obtain that $c_\lambda < m_\lambda^\infty$ for $\lambda \in (\bar{\lambda}, 1]$. □

Similar to the proof of [18, Proposition 3.1], we can obtain the following global compactness lemma.

Lemma 3.6 *Assume that* (V1), (V2), (F1), (F2) *and* (G1) *hold. Let* $\{u_n\}$ *be a bounded* (PS)-*sequence for* $\mathcal{I}_\lambda$, *for* $\lambda \in [1/2, 1]$. *Then there exists a subsequence of* $\{u_n\}$, *still denoted by* $\{u_n\}$, *an integer* $l \in \mathbb{N} \cup \{0\}$, *a sequence* $\{y_n^k\}$ *and* $w^k \in H^1(\mathbb{R}^N)$ *for* $1 \leq k \leq l$, *such that*

(i) $u_n \rightharpoonup u_0$ *with* $\mathcal{I}_\lambda'(u_0) = 0$;
(ii) $w^k \neq 0$ *and* $(\mathcal{I}_\lambda^\infty)'(w^k) = 0$ *for* $1 \leq k \leq l$;
(iii) $\left\|u_n - u_0 - \sum_{k=1}^l w^k(\cdot + y_n^k)\right\| \to 0$;
(iv) $\mathcal{I}_\lambda(u_n) \to \mathcal{I}_\lambda(u_0) + \sum_{i=1}^l \mathcal{I}_\lambda^\infty(w^i)$;

where we agree that in the case $l = 0$ *the above holds without* w^k.

Lemma 3.7 *Assume that* (V1), (V2), (F1), (F2) *and* (G1) *hold. Then for almost every* $\lambda \in (\bar{\lambda}, 1]$, *there exists* $u_\lambda \in H^1(\mathbb{R}^N)\backslash\{0\}$ *such that*

$$\mathcal{I}'_\lambda(u_\lambda) = 0, \quad \mathcal{I}_\lambda(u_\lambda) = c_\lambda. \tag{3.19}$$

Proof Lemma 3.4 implies that $\mathcal{I}_\lambda(u)$ satisfies the assumptions of Proposition 3.1 with $X = H^1(\mathbb{R}^N)$, $\Phi_\lambda = \mathcal{I}_\lambda$ and $J = (\bar{\lambda}, 1]$. So for almost every $\lambda \in (\bar{\lambda}, 1]$, there exists a bounded sequence $\{u_n(\lambda)\} \subset H^1(\mathbb{R}^N)$ (for simplicity, we denote it by $\{u_n\}$) such that

$$\mathcal{I}_\lambda(u_n) \to c_\lambda > 0, \quad \mathcal{I}'_\lambda(u_n) \to 0. \tag{3.20}$$

By Lemma 3.6, there exist a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, $u_\lambda \in H^1(\mathbb{R}^N)$, an integer $l \in \mathbb{N} \cup \{0\}$, and $w^1, \ldots, w^l \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that

$$u_n \rightharpoonup u_\lambda \ \text{ in } H^1(\mathbb{R}^N), \quad \mathcal{I}'_\lambda(u_\lambda) = 0, \tag{3.21}$$

$$(\mathcal{I}^\infty_\lambda)'(w^k) = 0, \quad \mathcal{I}^\infty_\lambda(w^k) \geq m^\infty_\lambda, \quad 1 \leq k \leq l \tag{3.22}$$

and

$$c_\lambda = \mathcal{I}_\lambda(u_\lambda) + \sum_{k=1}^{l} \mathcal{I}^\infty_\lambda(w^k). \tag{3.23}$$

Since $\mathcal{I}'_\lambda(u_\lambda) = 0$, then it follows from Lemma 3.2 that

$$\begin{aligned} \mathcal{P}_\lambda(u_\lambda) = &\frac{N-2}{2}\|\nabla u_\lambda\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} [NV(x) + \nabla V(x) \cdot x]\, u_\lambda^2 \mathrm{d}x \\ &-\frac{(N+\alpha)\lambda}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u_\lambda))F(u_\lambda)\mathrm{d}x - N\lambda\int_{\mathbb{R}^N} G(u_\lambda)\mathrm{d}x = 0. \end{aligned} \tag{3.24}$$

Since $\|u_n\| \nrightarrow 0$, we deduce from (3.22) and (3.23) that if $u_\lambda = 0$ then $l \geq 1$ and

$$c_\lambda = \mathcal{I}_\lambda(u_\lambda) + \sum_{k=1}^{l} \mathcal{I}^\infty_\lambda(w^k) \geq m^\infty_\lambda,$$

which contradicts Lemma 3.5. Thus $u_\lambda \neq 0$. It follows from (3.2), (3.24) and (2.24) or (2.25) that

$$\begin{aligned} \mathcal{I}_\lambda(u_\lambda) &= \mathcal{I}_\lambda(u_\lambda) - \frac{1}{N}\mathcal{P}_\lambda(u_\lambda) \\ &= \frac{1}{N}\|\nabla u_\lambda\|_2^2 - \frac{1}{2N}\int_{\mathbb{R}^N} \nabla V(x) \cdot x u_\lambda^2 \mathrm{d}x + \frac{\alpha\lambda}{2N}\int_{\mathbb{R}^N} (I_\alpha * F(u_\lambda))F(u_\lambda)\mathrm{d}x \\ &\geq \frac{1-\theta}{N}\|\nabla u_\lambda\|_2^2 > 0. \end{aligned} \tag{3.25}$$

From (3.23) and (3.25), one has

$$c_\lambda = \mathcal{I}_\lambda(u_\lambda) + \sum_{k=1}^{l} \mathcal{I}^\infty_\lambda(w^k) > l m^\infty_\lambda. \tag{3.26}$$

By Lemma 3.5, we have $c_\lambda < m^\infty_\lambda$ for $\lambda \in (\bar{\lambda}, 1]$, which, together with (3.26), implies that $l = 0$ and $\mathcal{I}_\lambda(u_\lambda) = c_\lambda$. □

Lemma 3.8 *Assume that* (V1), (V2), (F1), (F2) *and* (G1) *hold. Then there exists* $\bar{u} \in H^1(\mathbb{R}^N)\backslash\{0\}$ *such that*

$$\mathcal{I}'(\bar{u}) = 0, \quad 0 < \mathcal{I}(\bar{u}) \leq c_1. \tag{3.27}$$

Proof In view of Lemma 3.4 (iii) and Lemma 3.7, there exist two sequences $\{\lambda_n\} \subset (\bar{\lambda}, 1]$ and $\{u_{\lambda_n}\} \subset H^1(\mathbb{R}^N)\backslash\{0\}$, denoted by $\{u_n\}$, such that

$$\lambda_n \to 1, \quad c_{\lambda_n} \to c_*, \quad \mathcal{I}'_{\lambda_n}(u_n) = 0, \quad \mathcal{I}_{\lambda_n}(u_n) = c_{\lambda_n}. \tag{3.28}$$

Then it follows from (3.28) and Lemma 3.2 that $\mathcal{P}_{\lambda_n}(u_n) = 0$. By (3.2), (3.4), (3.25), (3.28) and Lemma 3.4 (iii), one has

$$C_4 \geq c_{\lambda_n} = \mathcal{I}_{\lambda_n}(u_n) - \frac{1}{N}\mathcal{P}_{\lambda_n}(u_n) \geq \frac{1-\theta}{N}\|\nabla u_n\|_2^2. \tag{3.29}$$

As in the proof of (2.36), we deduce that $\{\|u_n\|\}$ is bounded in $H^1(\mathbb{R}^N)$. In view of Lemma 3.4 (v), we have $\lim_{n\to\infty} c_{\lambda_n} = c_* \leq c_1$. Hence, it follows from (3.2) and (3.28) that

$$\mathcal{I}(u_n) \to c_*, \quad \mathcal{I}'(u_n) \to 0. \tag{3.30}$$

This shows that $\{u_n\}$ satisfies (3.20) with $c_\lambda = c_*$. In view of the proof of Lemma 3.7, we can show that there exists $\bar{u} \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that (3.27) holds. □

Proof of Theorem 1.1 Let $\hat{m} := \inf_{u\in\mathcal{K}} \mathcal{I}(u)$. Then Lemma 3.8 shows that $\mathcal{K} \neq \emptyset$ and $\hat{m} \leq c_1$. For any $u \in \mathcal{K}$, Lemma 3.2 implies $\mathcal{P}(u) = \mathcal{P}_1(u) = 0$. Hence it follows from (3.25) that $\mathcal{I}(u) = \mathcal{I}_1(u) > 0$ for all $u \in \mathcal{K}$, and so $\hat{m} \geq 0$. Let $\{u_n\} \subset \mathcal{K}$ such that

$$\mathcal{I}'(u_n) = 0, \quad \mathcal{I}(u_n) \to \hat{m}. \tag{3.31}$$

In view of Lemma 3.5, $\hat{m} \leq c_1 < m_1^\infty$. Arguing as in the proof of Lemma 3.7, we can deduce that there exists $\bar{u} \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that

$$\mathcal{I}'(\bar{u}) = 0, \quad \mathcal{I}(\bar{u}) = \hat{m}. \tag{3.32}$$

This shows that $\bar{u}$ is a ground state solution of (1.1).

Acknowledgements The authors also expresses their sincere gratitude to the anonymous referee for all insightful comments and valuable suggestions.

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