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ORIGINAL PAPER

Spaces of locally homogeneous affine surfaces

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Abstract
We examine the topology of various spaces of locally homogeneous affine surfaces which arise from the classification result of Opozda (Differ Geom Appl 21:173–198, 2004) as orbits of the action of $GL(2, \mathbb{R})$ (Type $\mathcal{A}$) and the $ax+b$ group (Type $\mathcal{B}$). We determine the topology of the spaces of Type $\mathcal{A}$ models in relation to the rank of the Ricci tensor. We determine the topology of the spaces of Type $\mathcal{B}$ models which either are flat or where the Ricci tensor is alternating.

Keywords Homogeneous affine surface · Linear equivalence · Ricci tensor

Mathematics Subject Classification 53A15 · 53C05 · 53B05

1 Introduction

1.1 Notational conventions

An *affine surface* is a pair $\mathcal{M} = (M, \nabla)$ where M is a smooth surface and where ∇ is a torsion free connection on the tangent bundle of M. Let $x = (x^1, x^2)$ be a system of local coordinates on M. Adopt the *Einstein convention* and sum over repeated indices to express $\nabla_{\partial_{x^i}} \partial_{x^j} = \Gamma_{ij}{}^k \partial_{x^k}$. The *Christoffel symbols* $\Gamma = \{\Gamma_{ij}{}^k\}$ determine the connection in the coordinate chart. Let ρ be the associated Ricci tensor. The Ricci tensor carries the geometry

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in dimension 2; an affine surface is flat if and only if $\rho = 0$. Since the Ricci tensor of an affine manifold is not necessarily symmetric, let $\rho_s(X, Y) = \frac{1}{2}\{\rho(X, Y) + \rho(Y, X)\}$ and $\rho_a(X, Y) = \frac{1}{2}\{\rho(X, Y) - \rho(Y, X)\}$ be the *symmetric* and *alternating* Ricci tensors.

1.2 Locally homogeneous affine surface geometries

Work of Opozda [11] shows that any locally homogeneous affine surface $\mathcal{M}$ is modeled on one of the following geometries.

- **Type $\mathcal{A}$.** $\mathcal{M} = (\mathbb{R}^2, \nabla)$ with constant Christoffel symbols $\Gamma_{ij}{}^k = \Gamma_{ji}{}^k$. This geometry is homogeneous; the Type $\mathcal{A}$ connections are the left invariant connections on the Lie group $\mathbb{R}^2$.
- **Type $\mathcal{B}$.** $\mathcal{M} = (\mathbb{R}^+ \times \mathbb{R}, \nabla)$ with Christoffel symbols $\Gamma_{ij}{}^k = (x^1)^{-1} A_{ij}{}^k$ where $A_{ij}{}^k = A_{ji}{}^k$ is constant. This geometry is homogeneous; the Type $\mathcal{B}$ connections are the left invariant connections on the $ax + b$ group.
- **Type $\mathcal{C}$.** $\mathcal{M} = (M, \nabla)$ where ∇ is the Levi-Civita connection of the round sphere S^2.

This result has been applied by many authors. Kowalski and Sekizawa [10] used it to examine Riemannian extensions of affine surfaces, Vanžurová [13] used it to study the metrizability of locally homogeneous affine surfaces, and Důsek [5] used it to study homogeneous geodesics. It plays a central role in the study of locally homogeneous connections with torsion of Arias-Marco and Kowalski [1] (see also [2] for a unified treatment independently of the torsion tensor). Although we will work with the local theory, the compact setting has been examined in [8,12].

The Ricci tensor ρ of an affine surface determines the full curvature tensor. In Sect. 2, we examine the spaces where the Ricci tensor has fixed rank in the Type $\mathcal{A}$ setting. In Sect. 3, we consider the spaces where either the Ricci tensor vanishes identically or where the Ricci tensor is alternating and non-trivial in the Type $\mathcal{B}$ setting.

1.3 Type $\mathcal{A}$ geometries

Let $\mathcal{M}(a, b, c, d, e, f) := (\mathbb{R}^2, \nabla)$ where the Christoffel symbols of ∇ are constant and given by

$$\begin{aligned} &\Gamma_{11}{}^1 = a, \qquad \Gamma_{11}{}^2 = b,\ \Gamma_{12}{}^1 = \Gamma_{21}{}^1 = c,\\ &\Gamma_{12}{}^2 = \Gamma_{21}{}^2 = d,\ \Gamma_{22}{}^1 = e,\ \Gamma_{22}{}^2 = f\,. \end{aligned} \tag{1.1}$$

This identifies the set of Type $\mathcal{A}$ geometries with $\mathbb{R}^6$. The linear transformations $T(x^1, x^2) = (a_1^1 x^1 + a_2^1 x^2, a_1^2 x^1 + a_2^2 x^2)$ where $(a_i^j) \in \mathrm{GL}(2, \mathbb{R})$ act on the set of Type $\mathcal{A}$ geometries. We say that two Type $\mathcal{A}$ surface models are *linearly equivalent* if there exists $T \in \mathrm{GL}(2, \mathbb{R})$ intertwining the two structures. One has that two Type $\mathcal{A}$ surfaces with non-degenerate Ricci tensor are affine equivalent if and only if they are linearly equivalent (see [3]). On the contrary, there exist Type $\mathcal{A}$ surfaces with degenerate Ricci tensor which are not linearly equivalent but which nevertheless are affine equivalent. We refer to the discussion in [6] for further details.

We consider the induced action of $GL(2, R)$ on $\mathbb{R}^6$ and identify the linear orbit of a Type $\mathcal{A}$ model $\mathcal{M}$ with $\mathcal{S}(\mathcal{M}) = GL(2, \mathbb{R})/\mathcal{I}(\mathcal{M})$ where $\mathcal{I}(\mathcal{M})$ is the isotropy group $\mathcal{I}(M) = \{T \in GL(2, \mathbb{R}); T^*\mathcal{M} = \mathcal{M}\}$.

It was shown in [6] that any flat Type $\mathcal{A}$ model is linearly equivalent to one of the following:

$$\begin{aligned}
&\mathcal{M}_0^0 := \mathcal{M}(0,0,0,0,0,0), \quad \mathcal{M}_1^0 := \mathcal{M}(1,0,0,1,0,0),\\
&\mathcal{M}_2^0 := \mathcal{M}(-1,0,0,0,0,1), \mathcal{M}_3^0 := \mathcal{M}(0,0,0,0,0,1),\\
&\mathcal{M}_4^0 := \mathcal{M}(0,0,0,0,1,0), \quad \mathcal{M}_5^0 := \mathcal{M}(1,0,0,1,-1,0).
\end{aligned} \tag{1.2}$$

The structure $\mathcal{M}_0^0$ is a singular cone point. The next result shows that the remaining orbits $\mathcal{S}(\mathcal{M}_i^0) := \mathrm{GL}(2,\mathbb{R}) \cdot \mathcal{M}_i^0$ for $1 \le i \le 5$ glue together to define a smooth 4-dimensional submanifold of $\mathbb{R}^6$. Let $\mathbb{1}$ be the trivial line bundle over the circle S^1, let $\mathbb{L}$ be the Möbius line bundle over S^1, and let $\mathcal{A}^0 \subset \mathbb{R}^6 \backslash \{0\}$ be the set of all flat Type $\mathcal{A}$ geometries other than the cone point $\mathcal{M}_0^0$.

Theorem 1.1 *$\mathcal{A}^0$ is a smooth submanifold of $\mathbb{R}^6\backslash\{0\}$ diffeomorphic to the total space of $\mathbb{L} \oplus \mathbb{1} \oplus \mathbb{1}$ minus the zero section.*

The Ricci tensor of any Type $\mathcal{A}$ model is symmetric. Let $\mathcal{A}_\pm^1 \subset \mathbb{R}^6$ be the set of all Type $\mathcal{A}$ geometries where the Ricci tensor has rank 1 and is positive semi-definite (+) or negative semi-definite (−). Any element in $\mathcal{A}_\pm^1$ is linearly equivalent to one of the following, where $c \in \mathbb{R}$ and $c_1 \in \mathbb{R}\backslash\{0, -1\}$ (see [3,6]):

$$\begin{aligned}
\mathcal{M}_1^1 &:= \mathcal{M}(-1,0,1,0,0,2),\\
\mathcal{M}_2^1(c_1) &:= \mathcal{M}(-1,0,c_1,0,0,1+2c_1),\\
\mathcal{M}_3^1(c_1) &:= \mathcal{M}(0,0,c_1,0,0,1+2c_1),\\
\mathcal{M}_4^1(c) &:= \mathcal{M}(0,0,1,0,c,2),\\
\mathcal{M}_5^1(c) &:= \mathcal{M}(1,0,0,0,1+c^2,2c).
\end{aligned} \tag{1.3}$$

We will see in Lemma 2.3 that the orbit structure of the action of $\mathrm{GL}(2,\mathbb{R})$ on $\mathcal{A}_\pm^1$ is quite complicated. It is therefore, perhaps, a bit surprising that the set of all orbits $\cup_{i,c}\mathcal{M}_i^1(c) \cdot \mathrm{GL}(2,\mathbb{R})$ is smooth as shown in the following result.

Theorem 1.2 *$\mathcal{A}_\pm^1$ is a smooth submanifold of $\mathbb{R}^6$ diffeomorphic to $S^1 \times S^1 \times \mathbb{R}^3$.*

The remaining geometries where the Ricci tensor has rank 2 form an open subset $\mathbb{R}^6\backslash\{\{0\} \cup \mathcal{A}^0 \cup \mathcal{A}_+^1 \cup \mathcal{A}_-^1\}$.

These results should be contrasted with the results in [9] where it is shown that any Type $\mathcal{A}$ affine surface is linearly equivalent to a surface determined by at most two non-zero parameters.

1.4 Type $\mathcal{B}$ geometries

Let $\mathcal{N}(a,b,c,d,e,f) := (\mathbb{R}^+ \times \mathbb{R}, \nabla)$ where the Christoffel symbols of ∇ are given by

$$\begin{aligned}
&\Gamma_{11}{}^1 = \frac{a}{x^1}, \qquad \Gamma_{11}{}^2 = \frac{b}{x^1},\ \Gamma_{12}{}^1 = \Gamma_{21}{}^1 = \frac{c}{x^1},\\
&\Gamma_{12}{}^2 = \Gamma_{21}{}^2 = \frac{d}{x^1},\ \Gamma_{22}{}^1 = \frac{e}{x^1},\ \Gamma_{22}{}^2 = \frac{f}{x^1}.
\end{aligned} \tag{1.4}$$

This identifies the space of Type $\mathcal{B}$ geometries with $\mathbb{R}^6$.

The natural structure group here is not the full general linear group, but rather the $ax+b$ group. We let $T_{a,b}(x^1,x^2) := (x^1, ax^2 + bx^1)$ define an action of the $ax+b$ group on $\mathbb{R}^+ \times \mathbb{R}$; this acts on the Type $\mathcal{B}$ geometries by reparametrization and defines the natural notion of linear equivalence in this setting. Thus, two Type $\mathcal{B}$ models $\mathcal{N}_1$ and $\mathcal{N}_2$ are said to be *linearly equivalent* if and only if there exists an affine transformation of the form $\Psi(x^1, x^2) = (x^1, a_1^2 x^1 + a_2^2 x^2)$ for $a_2^2 \neq 0$ intertwining the two structures. It follows from the work in [3,4] that two Type $\mathcal{B}$ surfaces which are neither flat nor of Type $\mathcal{A}$ are affine isomorphic if and only if they are linearly isomorphic. This is a non-trivial observation as there are non-linear affine transformations from one model to another if the dimension of the space of affine Killing vector fields is 4-dimensional or if the geometry is flat and thus the dimension of the space of affine Killing vector fields is 6-dimensional.

It was shown in [7] that a flat Type $\mathcal{B}$ model is linearly equivalent to one of the following models:

$$\begin{aligned}
&\mathcal{N}_0^0 := \mathcal{N}(0,0,0,0,0,0), && \mathcal{N}_1^0(\pm) := \mathcal{N}(1,0,0,0,\pm 1,0),\\
&\mathcal{N}_2^0(c_1) := \mathcal{N}(c_1-1,0,0,c_1,0,0),\ c_1 \neq 0, && \mathcal{N}_3^0 := \mathcal{N}(-2,1,0,-1,0,0),\\
&\mathcal{N}_4^0 := \mathcal{N}(0,1,0,0,0,0), && \mathcal{N}_5^0 := \mathcal{N}(-1,0,0,0,0,0),\\
&\mathcal{N}_6^0(c_2) := \mathcal{N}(c_2,0,0,0,0,0),\ c_2 \neq 0,-1.
\end{aligned}$$

Let $\mathcal{B}^0 \subset \mathbb{R}^6$ be the space of flat Type $\mathcal{B}$ geometries other than the cone point $\mathcal{N}_0^0$ determined by the origin in $\mathbb{R}^6$. Unlike the Type $\mathcal{A}$ setting described in Theorem 1.1, $\mathcal{B}^0$ is not a smooth manifold but consists of the union of 3 smooth submanifolds of $\mathbb{R}^6$ which intersect transversally along the union of 3 smooth curves in $\mathbb{R}^6$. Define

$$\begin{aligned}
&\mathcal{U}_1(r,s) := \mathcal{N}(1+rs^2, -s(1+rs^2), rs, -rs^2, r, -rs), && \mathcal{B}_1 := \text{Range}\{\mathcal{U}_1\},\\
&\mathcal{U}_2(u,v) := \mathcal{N}(u,v,0,0,0,0), && \mathcal{B}_2 := \text{Range}\{\mathcal{U}_2\},\\
&\mathcal{U}_3(u,v) := \mathcal{N}(u,v,0,1+u,0,0), && \mathcal{B}_3 := \text{Range}\{\mathcal{U}_3\}.
\end{aligned} \tag{1.5}$$

Theorem 1.3 $\mathcal{B}^0 = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. $\mathcal{B}_2$ *and* $\mathcal{B}_3$ *are closed smooth surfaces in* $\mathbb{R}^6$ *which are diffeomorphic to* $\mathbb{R}^2$ *and which intersect transversally along the curve* $\mathcal{N}(-1, v, 0, 0, 0, 0)$ *for* $v \in \mathbb{R}$. $\mathcal{B}_1$ *can be completed to a smooth closed surface* $\tilde{\mathcal{B}}_1$ *which intersects* $\mathcal{B}_2$ *transversally along the curve* $\mathcal{N}(1, v, 0, 0, 0, 0)$ *and which intersects* $\mathcal{B}_3$ *transversally along the curve* $\mathcal{N}(0, v, 0, 1, 0, 0)$ *for* $v \in \mathbb{R}$.

In the Type $\mathcal{B}$ setting, it is possible for the symmetric Ricci tensor ρ_s to vanish without the geometry being flat; this is not possible in the Type $\mathcal{A}$ setting. The alternating Ricci tensor, ρ_a, carries the geometry in this context.

Let $\mathcal{B}_a$ be the set of all Type $\mathcal{B}$ structures where $\rho_s = 0$ but $\rho_a \neq 0$. Set

$$\begin{aligned}
&\mathcal{V}_1(r,s,t) := \mathcal{N}(s,t,r,0,0,r),\\
&\mathcal{V}_2(u,v,w) := \mathcal{N}(1-2uw+vw^2, w(1-uw+vw^2), u-vw, -vw^2, v, u+vw)
\end{aligned} \tag{1.6}$$

and let $\mathcal{D}_1 := \text{Range}\{\mathcal{V}_1\}$ and $\mathcal{D}_2 := \text{Range}\{\mathcal{V}_2\}$.

Theorem 1.4 $\mathcal{B}_a = \mathcal{D}_1 \cup \mathcal{D}_2$. $\mathcal{V}_i$ *defines smoothly embedded 3-dimensional submanifolds of* $\mathbb{R}^6$ *for* $r \neq 0$ *and* $u \neq 0$ *which intersect transversally along a smooth 2-dimensional submanifold.*

2 The space of type $\mathcal{A}$ models

Let $\mathcal{M}(a,b,c,d,e,f) := (\mathbb{R}^2, \nabla)$ be given by Eq. (1.1) where the parameters (a,b,c,d,e,f) are real constants. The associated Ricci tensor is symmetric.

2.1 The space of flat type $\mathcal{A}$ models

Since the Ricci tensor determines the curvature in dimension two, flat surfaces are determined by a vanishing Ricci tensor. We provide the proof of the first result of the paper as follows.

The proof of Theorem 1.1 Let $\theta \in [0, 2\pi]$ be the usual periodic parameter where we identify 0 with 2π to define the circle $S^1 = (\cos\theta, \sin\theta)$. Let (x^1, x^2, x^3) be a point of $\mathbb{R}^3$. The bundle $\mathbb{L} \oplus \mathbb{1} \oplus \mathbb{1}$ is then defined by identifying (θ, x^1, x^2, x^3) with $(\theta + \pi, -x^1, x^2, x^3)$; this puts the necessary half twist in the first x-coordinate. We require that (x^1, x^2, x^3) belongs to $\mathbb{R}^3 - \{0\}$ to remove the 0-section.

The parametrization of Eq. (1.1) is not a very convenient one for studying the Ricci tensor. We make a linear change of coordinates on $\mathbb{R}^6$ and let $\mathcal{M}_1(p, q, t, s, v, w)$ be defined by

$$\begin{array}{lll} \Gamma_{11}{}^1 = 2q, & \Gamma_{11}{}^2 = p + t, & \Gamma_{12}{}^1 = \Gamma_{21}{}^1 = w, \\ \Gamma_{12}{}^2 = \Gamma_{21}{}^2 = q + s, & \Gamma_{22}{}^1 = v, & \Gamma_{22}{}^2 = p - t\,. \end{array}$$

We substitute these values in Eq. (2.4) to obtain

$$\rho = \begin{pmatrix} p^2 + q^2 - s^2 - t^2 - pw - tw & -(p+t)v + (q+s)w \\ -(p+t)v + (q+s)w & qv - sv + (p - t - w)w \end{pmatrix}.$$

We set $\rho = 0$. If $v^2 + w^2 \neq 0$, we obtain

$$\begin{aligned} p &= (v^2 + w^2)^{-1}\{2svw + t(w^2 - v^2) + w^3\}, \text{ and} \\ q &= (v^2 + w^2)^{-1}\{s(v^2 - w^2) + vw(2t + w)\}\,. \end{aligned} \tag{2.1}$$

If $v^2 + w^2 = 0$, we obtain a single equation

$$p^2 + q^2 - s^2 - t^2 = 0\,. \tag{2.2}$$

We introduce polar coordinates $v = r\cos(\theta)$ and $w = r\sin(\theta)$ to remove the singularity at $(v, w) = (0, 0)$ in Eq. (2.1). We may then combine Eqs. (2.1) and Eq. (2.2) into a single expression:

$$\begin{aligned} p &= p(\theta, r, s, t) := r\sin^3(\theta) + s\sin(2\theta) - t\cos(2\theta), \\ q &= q(\theta, r, s, t) := r\cos(\theta)\sin^2(\theta) + s\cos(2\theta) + t\sin(2\theta)\,. \end{aligned} \tag{2.3}$$

We assume $(r, s, t) \neq (0, 0, 0)$ to avoid the trivial structure $\mathcal{M}_0^0$ as the parametrization of Eq. (2.3) is singular there. We have $\theta \in [0, 2\pi]$ and $(r, s, t) \in \mathbb{R}^3 - \{0\}$; since we are permitting r to be negative in polar coordinates, we must identify (θ, r) with $(\theta + \pi, -r)$ and obtain thereby the bundle $\mathbb{L} \oplus \mathbb{1} \oplus \mathbb{1}$ minus the zero section over $[0, \pi]$. □

Remark 2.1 The isotropy subgroups of the structures $\mathcal{M}_i^0$ vary with i and the dimension of the orbit space varies correspondingly. We list below the associated isotropy subgroups.

$$\begin{aligned} \mathcal{I}(\mathcal{M}_0^0) &= \mathrm{GL}(2, \mathbb{R}), \\ \mathcal{I}(\mathcal{M}_1^0) &= \left\{T : T(x^1, x^2) = (x^1, ax^2) \text{ for } a \neq 0\right\}, \\ \mathcal{I}(\mathcal{M}_2^0) &= \{id, T\}, \text{ where } T(x^1, x^2) = (-x^2, -x^1), \\ \mathcal{I}(\mathcal{M}_3^0) &= \left\{T : T(x^1, x^2) = (ax^1, x^2) \text{ for } a \neq 0\right\}, \\ \mathcal{I}(\mathcal{M}_4^0) &= \left\{T : T(x^1, x^2) = (a^2x^1 + bx^2, ax^2) \text{ for } a \neq 0,\ b \in \mathbb{R}\right\}, \\ \mathcal{I}(\mathcal{M}_5^0) &= \left\{T : T(x^1, x^2) = (x^1, \pm x^2)\right\}. \end{aligned}$$

2.2 The space of type $\mathcal{A}$ models with rank-one Ricci tensor

If the Ricci tensor has rank 1, we can make a linear change of coordinates to ensure ρ is a multiple of $dx^2 \otimes dx^2$. We first establish Theorem 1.2. We then examine the isotropy groups of the models in Eq. (1.3) to determine the orbits of the Type $\mathcal{A}$ models which are not Type $\mathcal{B}$.

Lemma 2.2 *Let $\mathcal{M}$ be a Type $\mathcal{A}$ model which is not flat. Then ρ is a multiple of $dx^2 \otimes dx^2$ if and only if $b = 0$ and $d = 0$.*

Proof A direct computation shows

$$\rho = \begin{pmatrix} (a-d)d + b(f-c) & cd - be \\ cd - be & c(f-c) + (a-d)e \end{pmatrix}. \tag{2.4}$$

Consequently, if $b = 0$ and if $d = 0$, then ρ is a multiple of $dx^2 \otimes dx^2$. Conversely, assume ρ is a multiple of $dx^2 \otimes dx^2$ or, equivalently, $-bc + ad - d^2 + bf = 0$ and $cd - be = 0$. We wish to show $b = d = 0$.

Case 1. Suppose that $d \neq 0$. The equations are homogeneous so we may assume $d = 1$ and hence $c = be$. Substituting these values yields $\rho_{11} = -1 + a - b^2e + bf = 0$. Thus $a = 1 + b^2e + bf$. This yields $\rho = 0$ so this case is impossible as we assumed $\mathcal{M}$ was not flat.

Case 2. Suppose that $b \neq 0$. Again, we may assume $b = 1$ so $e = cd$. We compute $\rho_{11} = f - c + ad - d^2$. Setting this to zero again yields $\rho = 0$ which is impossible. □

Proof of Theorem 1.2 Let $\mathcal{A}^1_{\pm,0}$ be the space of all Type $\mathcal{A}$ models where the Ricci tensor is a non-zero multiple of $dx^2 \otimes dx^2$ where the $\pm$ refers to whether ρ_{22} is positive or negative. By Lemma 2.2, we set $b = d = 0$ and obtain $\rho_{22} = -c^2 + ae + cf$. We make a change of variables setting

$$a = q + v,\ b = 0,\ c = u + p,\ d = 0,\ e = q - v,\ f = 2p.$$

We then have $\rho_{22} = (p^2 + q^2 - u^2 - v^2)dx^2 \otimes dx^2$ so we may identify

$$\mathcal{A}^1_{+,0} = \{\Gamma(p, q, u, v) : p^2 + q^2 > u^2 + v^2\},$$
$$\mathcal{A}^1_{-,0} = \{\Gamma(p, q, u, v) : p^2 + q^2 < u^2 + v^2\}.$$

We examine $\mathcal{A}^1_{-,0}$ as the analysis of $\mathcal{A}^1_{+,0}$ is the same after interchanging the roles of (p, q) and (u, v). Let $\mathcal{D}^2 := \{(U, V) \in \mathbb{R}^2 : U^2 + V^2 < 1\}$ be the open disk in $\mathbb{R}^2$. Let $-\mathcal{M}$ be the Type $\mathcal{A}$ model $\mathcal{M}(-a, -b, -c, -d, -e, -f)$. We construct a diffeomorphism Φ from $S^1 \times \mathbb{R}^+ \times \mathcal{D}^2$ to $\mathcal{A}^1_{-,0}$ by setting $u = r\cos\theta$, $v = r\sin\theta$, $p = rU$, $q = rV$. For $r > 0$, $\theta \in S^1$, and $U^2 + V^2 < 1$ we have

$$\mathcal{M} = \mathcal{M}(r(\sin(\theta) + V), 0, r(\cos(\theta) + U), 0, r(V - \sin(\theta)), 2rU).$$

It is clear that $-\mathcal{M}(\theta, r, U, V) = \mathcal{M}(\theta + \pi, r, -U, -V)$.

Let $\tilde{\mathcal{M}}$ be an arbitrary Type $\mathcal{A}$ model with $\text{Rank}\{\rho_{\tilde{\mathcal{M}}}\} = 1$ and $\rho_{\tilde{\mathcal{M}}}$ negative semi-definite. We may express

$$\rho_{\tilde{\mathcal{M}}} = \lambda(\cos(\phi)dx^2 - \sin(\phi)dx^1) \otimes (\cos(\phi)dx^2 - \sin(\phi)dx^1)$$

for $\lambda < 0$. Here ϕ is only defined modulo π instead of the usual 2π. Let

$$T_\phi(x^1, x^2) = (\cos(\phi)x^1 + \sin(\phi)x^2, -\sin(\phi)x^1 + \cos(\phi)x^2).$$

be the associated rotation so that $T_\phi^*(dx^2) = -\sin(\phi)dx^1 + \cos(\phi)dx^2$ and thus $(T_\phi)_*\tilde{\mathcal{M}}$ belongs to $\mathcal{A}^1_{-,0}$. We then have

$$\mathcal{A}^1_- = \{\mathbb{R}/(2\pi\mathbb{Z}) \times \mathcal{A}^1_{-,0}\}/(\phi, \mathcal{M}) \sim (\phi+\pi, -\mathcal{M})$$

where the gluing reflects the fact that when $\phi = \pi$ we have replaced (x^1, x^2) by $(-x^1, -x^2)$ and thus changed the sign of the Christoffel symbols. Using our previous parametrization of $\mathcal{A}^1_{-,0}$, this yields

$$\mathcal{A}^1_- = (\mathbb{R}^2/(2\pi\mathbb{Z})^2) \times \mathbb{R}^+ \times \mathcal{D}^2/\{(\phi, \theta, r, U, V) \sim (\phi+\pi, \theta+\pi, r, -U, -V)\}\,.$$

After setting $\tilde{\theta} = \theta + \phi$, we can rewrite this equivalence relation in the form

$$(\phi, \tilde{\theta}, r, U, V) \sim (\phi+\pi, \tilde{\theta}, r, -U, -V)\,.$$

The variable $\tilde{\theta}$ now no longer plays a role in the gluing. After replacing $\mathbb{R}^+$ by $\mathbb{R}$ and $\mathcal{D}^2$ by $\mathbb{R}^2$, we see $\mathcal{A}^1_-$ is diffeomorphic to $S^1 \times S^1 \times \mathbb{R}^3$ modulo the relation

$$(\phi, \tilde{\theta}, x_1, x_2, x_3) \sim (\phi+\pi, \tilde{\theta}, x_1, -x_2, -x_3)\,.$$

These gluing relations define the total space of the bundle $\mathbb{1} \oplus \mathbb{L} \oplus \mathbb{L}$ over (S^1, ϕ). Since $\mathbb{L} \oplus \mathbb{L}$ is diffeomorphic to the trivial 2-plane bundle $\mathbb{1} \oplus \mathbb{1}$, we obtain finally that $\mathcal{A}^1_-$ is diffeomorphic to $S^1 \times S^1 \times \mathbb{R}^3$. □

We adopt the notation of Eq. (1.3) to describe the orbits of the models $\mathcal{M}^1_i(\cdot)$ in the following lemma.

Lemma 2.3 (1) $\mathcal{I}(\mathcal{M}^1_1) = \{\mathrm{id}\}$.

(2) $\mathcal{I}(\mathcal{M}^1_2(c_1)) = \{\mathrm{id}\}$ *if* $c_1 \neq -\frac{1}{2}$.

(3) $\mathcal{I}(\mathcal{M}^1_2(-\frac{1}{2})) = \{\mathrm{id}, T\}$, *where* $T(x^1, x^2) = (x^1 + x^2, -x^2)$.

(4) $\mathcal{I}(\mathcal{M}^1_3(c_1)) = \{T : T(x^1, x^2) = (v^{-1}x^1, x^2)$ *for* $v \in \mathbb{R}\backslash\{0\}\}$.

(5) $\mathcal{I}(\mathcal{M}^1_4(c)) = \{T : T(x^1, x^2) = (x^1 - wx^2, x^2)$ *for* $w \in \mathbb{R}\}$, *if* $c \neq 0$,.

(6) $\mathcal{I}(\mathcal{M}^1_4(0)) = \{T : T(x^1, x^2) = (v^{-1}(x^1 - wx^2), x^2)$ *for* $w \in \mathbb{R}$, $v \in \mathbb{R}\backslash\{0\}\}$.

(7) $\mathcal{I}(\mathcal{M}^1_5(c)) = \{\mathrm{id}\}$, *if* $c \neq 0$.

(8) $\mathcal{I}(\mathcal{M}^1_5(0)) = \{\mathrm{id}, T\}$ *where* $T(x^1, x^2) = (x^1, -x^2)$.

Proof Suppose $T \in \mathcal{I}(\mathcal{M}^1_i(\cdot))$. The Ricci tensor of $\mathcal{M}^1_i(\cdot)$ is a non-zero multiple of $dx^2 \otimes dx^2$. Since T must preserve the Ricci tensor, $T(dx^2) = \pm dx^2$. This implies $(y^1, y^2) = T(x^1, x^2) = (v^{-1}(x^1 - wx^2), \varepsilon x^2)$ for $\varepsilon = \pm 1$. Then

$$\begin{aligned}
dy^1 &= v^{-1}(dx^1 - w dx^2), \quad dy^2 = \varepsilon dx^2, \quad \partial_{y^1} = v\partial_{x^1}, \quad \partial_{y^2} = \varepsilon(w\partial_{x^1} + \partial_{x^2}),\\
{}^y\Gamma_{11}{}^1 &:= v({}^x\Gamma_{11}{}^1 - w\,{}^x\Gamma_{11}{}^2).\\
{}^y\Gamma_{11}{}^2 &:= v^2\varepsilon{}^x\Gamma_{11}{}^2,\\
{}^y\Gamma_{12}{}^1 &:= \varepsilon({}^x\Gamma_{12}{}^1 + w\,({}^x\Gamma_{11}{}^1 - {}^x\Gamma_{12}{}^2 - w\,{}^x\Gamma_{11}{}^2)),\\
{}^y\Gamma_{12}{}^2 &:= v({}^x\Gamma_{12}{}^2 + w\,{}^x\Gamma_{11}{}^2),\\
{}^y\Gamma_{22}{}^1 &:= \frac{1}{v}({}^x\Gamma_{22}{}^1 + w(2\,{}^x\Gamma_{12}{}^1 - {}^x\Gamma_{22}{}^2) + w^2({}^x\Gamma_{11}{}^1 - 2\,{}^x\Gamma_{12}{}^2) - w^3\,{}^x\Gamma_{11}{}^2),\\
{}^y\Gamma_{22}{}^2 &:= \varepsilon({}^x\Gamma_{22}{}^2 + 2w\,{}^x\Gamma_{12}{}^2 + w^2\,{}^x\Gamma_{11}{}^2).
\end{aligned}$$

Case 1. $\mathcal{M}_1^1 = \mathcal{M}(-1, 0, 1, 0, 0, 2)$ and $T^*\mathcal{M}_1^1 = \mathcal{M}(-v, 0, \varepsilon(1-w), 0, -\frac{w^2}{v}, 2\varepsilon)$. Examining ${\Gamma_{11}}^1$ and ${\Gamma_{22}}^2$ yields $\varepsilon = 1$ and $v = 1$. Examining ${\Gamma_{22}}^1$ yields $w = 0$.

Case 2. We have $c \notin \{0, -1\}$, $\mathcal{M}_2^1(c) = \mathcal{M}(-1, 0, c, 0, 0, 1 + 2c)$, and

$$T^*\mathcal{M}_2^1(c) = \mathcal{M}(-v, 0, \varepsilon(c - w), 0, -\tfrac{1}{v}(w + w^2), (1 + 2c)\varepsilon)\,.$$

Examining ${\Gamma_{11}}^1$ yields $v = 1$. Suppose $c \neq -\frac{1}{2}$. Examining ${\Gamma_{22}}^2$ yields $\varepsilon = 1$. Since $\varepsilon = 1$, examining ${\Gamma_{12}}^1$ yields $w = 0$. Suppose $c = -\frac{1}{2}$. Examining ${\Gamma_{12}}^1$ and ${\Gamma_{22}}^1$ yields $(\varepsilon, w) = (1, 0)$ or $(\varepsilon, w) = (-1, -1)$.

Case 3. We have $c \notin \{0, -1\}$, $\mathcal{M}_3^1(c) = \mathcal{M}(0, 0, c, 0, 0, 1 + 2c)$, and

$$T^*\mathcal{M}_3^1(c) = \mathcal{M}(0, 0, c\varepsilon, 0, -\tfrac{w}{v}, (1 + 2c)\varepsilon)\,.$$

Examining ${\Gamma_{12}}^1$ yields $\varepsilon = 1$. Examining ${\Gamma_{22}}^1$ yields $w = 0$. There is then no condition on v.

Case 4. $\mathcal{M}_4^1(c) = \mathcal{M}(0, 0, 1, 0, c, 2)$ and $T^*\mathcal{M}_4^1(c) = \mathcal{M}(0, 0, \varepsilon, 0, \frac{c}{v}, 2\varepsilon)$. Examining ${\Gamma_{22}}^2$ yields $\varepsilon = 1$. There is no condition on w. If $c \neq 0$, examining ${\Gamma_{22}}^1$ yields $v = 1$; if $c = 0$, there is no condition on v.

Case 5. $\mathcal{M}_5^1(c) = \mathcal{M}(1, 0, 0, 0, 1 + c^2, 2c)$ and

$$T^*\mathcal{M}_5^1(c) = \mathcal{M}(v, 0, w\varepsilon, 0, \tfrac{1}{v}(1 + (c - w)^2), 2c\varepsilon)\,.$$

Examining ${\Gamma_{11}}^1$ shows $v = 1$. Examining ${\Gamma_{12}}^1$ shows $w = 0$. If $c \neq 0$, examining ${\Gamma_{22}}^2$ shows $\varepsilon = 1$. If $c = 0$, we obtain $\varepsilon = \pm 1$. □

The general linear group $GL(2, \mathbb{R})$ acts on the space $\mathbb{R}^6$ of all Type $\mathcal{A}$ geometries via change of coordinates. Let $GL_+(2, \mathbb{R})$ be the subgroup of matrices with positive determinant. If $\mathcal{M}$ is a Type $\mathcal{A}$ model with $\text{Rank}\{\rho\}(\mathcal{M}) = 2$, then the associated space of affine Killing vector fields is 2-dimensional and $\mathcal{M}$ does not also admit a Type $\mathcal{B}$ structure [3]. But there are Type $\mathcal{A}$ models with $\text{Rank}\{\rho\} = 1$ which also admit Type $\mathcal{B}$ structures. Let $\mathcal{O}_\pm^1 \subset \mathcal{A}_\pm^1$ be the set of Type $\mathcal{A}$ models with $\text{Rank}\{\rho\} = 1$ and which do not admit Type $\mathcal{B}$ structures.

Theorem 2.4 (1) *$\mathcal{O}_-^1$ is empty; every element of $\mathcal{A}_-^1$ also admits Type $\mathcal{B}$ structure.*
(2) *$GL_+(2, \mathbb{R})$ acts without fixed points on $\mathcal{O}_+^1$. The action admits a section $s : \mathbb{R} \to \mathcal{O}_+^1$ so $\mathcal{O}_+^1 = GL_+(2, \mathbb{R}) \times \mathbb{R}$ is a principal fiber bundle over $\mathbb{R}$.*

Proof Resuts of [3] show that the models $\mathcal{M}_i^1(\cdot)$ for $1 \leq i \leq 4$ also admit Type $\mathcal{B}$ structures while the models $\mathcal{M}_5^1(c)$ do not. The Ricci tensor associated to $\mathcal{M}_i^1(\cdot)$ is given by:

$$\begin{aligned} &\rho^{\mathcal{M}_1^1} = dx^2 \otimes dx^2, && \rho^{\mathcal{M}_2^1} = c_1(1 + c_1)dx^2 \otimes dx^2,\\ &\rho^{\mathcal{M}_3^1} = c_1(1 + c_1)dx^2 \otimes dx^2, && \rho^{\mathcal{M}_4^1} = dx^2 \otimes dx^2,\\ &\rho^{\mathcal{M}_5^1} = (1 + c^2)dx^2 \otimes dx^2. \end{aligned}$$

If $\rho \leq 0$, then it follows that $i = 2$ or $i = 3$ and $c \in (-1, 0)$. Thus any element of $\mathcal{A}_-^1$ admits a Type $\mathcal{B}$ structure which proves Assertion (1).

Let $\mathfrak{M}_5^1 = \cup_c \mathcal{M}_5^1(c)$; this is a smooth curve in $\mathbb{R}^6$. Type $\mathcal{A}$ models which are linearly equivalent to $\mathcal{M}_1^1$, $\mathcal{M}_2^1(c_1)$ for $c_1 + c_1^2 > 0$, $\mathcal{M}_3^1(c_1)$ for $c_1 + c_1^2 > 0$, or $\mathcal{M}_4^1(c)$ all admit Type $\mathcal{B}$ structures and have $\rho \geq 0$. Thus we may identify the structures $\mathcal{O}_{1,+}$ which do not admit Type $\mathcal{B}$ structures with $GL(2, \mathbb{R}) \cdot \mathfrak{M}_5^1$. Let $T(x^1, x^2) := (x^1, -x^2)$. We have $T\mathcal{M}_5^1(c) = \mathcal{M}_5^1(-c)$. Since $\det(T) = -1$, we conclude therefore that $\mathcal{O}_+^1 = GL_+(2, \mathbb{R}) \cdot \mathfrak{M}_5^1$. By Lemma 2.3, the action of $GL_+(2, \mathbb{R})$ on $\mathfrak{M}_5^1$ is fixed point free. Assertion (2) follows. □

3 The space of type $\mathcal{B}$ connections

Let $\mathcal{N}(a, b, c, d, e, f) := (\mathbb{R}^+ \times \mathbb{R}, \nabla)$ where the Christoffel symbols of ∇ are given by (1.4). The Ricci tensor needs not be symmetric in this setting:

$$\rho = (x^1)^{-2} \begin{pmatrix} (a-d+1)d + b(f-c) & cd - be + f \\ c(d-1) - be & -c^2 + fc + (a-d-1)e \end{pmatrix} \tag{3.1}$$

3.1 The space of flat type $\mathcal{B}$ models

The proof of Theorem 1.3 Let $\mathcal{N} = \mathcal{N}(a, b, c, d, e, f)$. We clear denominators in Eq. (3.1) and set $\tilde{\rho}_{ij} = (x^1)^2 \rho_{ij}$. Adopt the notation of Eq. (1.5). A direct computation shows that the structures $\mathcal{U}_i(\cdot)$ are flat. We distinguish cases to establish the converse. We use Eq. (3.1) and set $\tilde{\rho} = 0$. Since $\tilde{\rho}_{12} - \tilde{\rho}_{21} = c + f$, $f = -c$.

Case 1. Assume $e \neq 0$. Set $c = rs$, $e = r$, and $f = -rs$ for $r \neq 0$. Then

$$\tilde{\rho}_{22} = -r(1 - a + d + 2rs^2) \text{ and } \tilde{\rho}_{21} = -r(b + s - ds)\,.$$

We solve these equations to obtain $a = 1 + d + 2rs^2$ and $b = (-1 + d)s$. We have $\tilde{\rho}_{11} = 2(d + rs^2)$. Thus $d = -rs^2$ which gives the parametrization $\mathcal{U}_1$.

Case 2. Suppose $e = 0$. Set $a = u$, $b = v$, and $f = -c$ to obtain

$$\tilde{\rho} = \begin{pmatrix} d(1 + u - d) - 2cv & c(d-1) \\ c(d-1) & -2c^2 \end{pmatrix}.$$

This yields $c = 0$ and $d(1 + u - d) = 0$. If we set $d = 0$, we obtain the parametrization $\mathcal{U}_2$; if we set $d = 1 + u$, we obtain the parametrization $\mathcal{U}_3$. This establishes the first assertion.

The parametrization $\mathcal{U}_2$ and $\mathcal{U}_3$ intersect when $u = -1$; the intersection is transversal along the curve $\mathcal{N}(-1, v, 0, 0, 0, 0)$. We wish to extend the parametrization $\mathcal{U}_1$ to study the limiting behavior as $e \to 0$. We distinguish cases.

Case A. Suppose $\lim_{n\to\infty} \mathcal{U}_1(r_n, s_n) \in \text{Range}\{\mathcal{U}_2\}$. We have

$$\begin{aligned} &\lim_{n\to\infty} 1 + r_n s_n^2 = u, && \lim_{n\to\infty} -s_n(1 + r_n s_n^2) = v, && \lim_{n\to\infty} -r_n s_n = 0, \\ &\lim_{n\to\infty} -r_n s_n^2 = 0, && \lim_{n\to\infty} r_n = 0, && \lim_{n\to\infty} -r_n s_n = 0. \end{aligned}$$

These equations imply $u = 1$, $\lim_{n\to\infty} r_n = 0$, $\lim_{n\to\infty} s_n = -v$. Thus we may simply set $r = 0$ to obtain a transversal intersection along the curve $\mathcal{N}(1, v, 0, 0, 0, 0)$.

Case B. Suppose $\lim_{n\to\infty} \mathcal{U}_1(r_n, s_n) \in \text{Range}\{\mathcal{U}_3\}$. We have

$$\begin{aligned} &\lim_{n\to\infty} 1 + r_n s_n^2 = u, && \lim_{n\to\infty} -s_n(1 + r_n s_n^2) = v, && \lim_{n\to\infty} -r_n s_n = 0, \\ &\lim_{n\to\infty} -r_n s_n^2 = 1 + u, && \lim_{n\to\infty} r_n = 0, && \lim_{n\to\infty} -r_n s_n = 0. \end{aligned}$$

These equations imply $u = 0$, $\lim_{n\to\infty} r_n = 0$, and $\lim_{n\to\infty} r_n s_n^2 = -1$. We change variables setting $r = -t^2$ and $s = \frac{1}{t} + w$ to express

$$\begin{aligned} \mathcal{U}_1(-t^2, \tfrac{1}{t} + w) = \mathcal{N}(\, &-tw(2 + tw),\ w(2 + 3tw + t^2w^2),\ -t(1 + tw), \\ &(1 + tw)^2,\quad -t^2,\quad t(1 + tw))\,. \end{aligned}$$

We may now safely set $t = 0$ to obtain the intersection with Range$\{\mathcal{U}_3\}$ along the curve $\mathcal{N}(0, 2w, 0, 1, 0, 0)$. □

3.2 Type $\mathcal{B}$ models with alternating Ricci tensor

It was shown in [3] that any Type $\mathcal{B}$ model with alternating Ricci tensor is linearly equivalent to one of the following models:

$$\begin{aligned} &\mathcal{N}_1(c) := \mathcal{N}(0, c, 1, 0, 0, 1), && \text{for } c \in \mathbb{R}, \\ &\mathcal{N}_2(c, \pm) := \mathcal{N}(1 \mp c^2), c, 0, \mp c^2, \pm 1, \pm 2c), && \text{for } c > 0. \end{aligned}$$

The proof of Theorem 1.4 Adopt the notation of Eq. (1.6). It is clear that $\mathcal{V}_1$ defines a smooth 3-dimensional submanifold of $\mathbb{R}^6$. To see similarly that $\mathcal{V}_2$ is smooth, we note that we can recover $u = \frac{1}{2}(c+f)$ and $v = e$. If $v \neq 0$, then $w = \frac{1}{v}(f-u)$ while if $v = 0$, $w = \frac{1}{2u}(1-a)$. Thus $\mathcal{V}_2$ is 1-1; it is not difficult to verify that the Jacobian determinant is non-zero. This shows that $\mathcal{V}_2$ also defines a smooth 3-dimensional submanifold of $\mathbb{R}^6$. We set $v = 0$ and $u = r$ to see that $\mathcal{V}_1$ and $\mathcal{V}_2$ intersect along the surface $v = 0$, $u = r$, $s = 1 - 2uw$ and $t = w(1-uw)$. A direct computation shows that the associated Ricci tensors are non-trivial and alternating:

$$\tilde{\rho}_{\mathcal{V}_1} = r\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \tilde{\rho}_{\mathcal{V}_2} = u\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $\mathcal{N}$ be a Type $\mathcal{B}$ model with $\rho_s = 0$ and $\tilde{\rho}_{a,12} = \frac{c+f}{2} \neq 0$. We distinguish two cases.

Case 1. Suppose $e = 0$. Set $c = 2r - f$ for $r \neq 0$. Setting the $\rho_s = 0$ yields

$$\begin{aligned} &\rho_{s,11}: \ 0 = d(1 + a - d) + 2b(f - r), \ \rho_{s,12}: \ 0 = (1-d)f + r(2d-1), \\ &\rho_{s,22}: \ 0 = -2(f^2 - 3fr + 2r^2). \end{aligned}$$

We solve the equation $-2(f^2 - 3fr + 2r^2) = 0$ to obtain $f = r$ or $f = 2r$. Setting $f = 2r$ yields ρ_{s12}: $0 = r$ which is false. Thus $f = r$. We obtain $\rho_{s,12} = 2dr$ so $d = 0$. Set $a = s$ and $b = t$ to obtain the parametrization $\mathcal{V}_1$.

Case 2. Set $c = 2u - f$ and $e = v$ for $u \neq 0$ and $v \neq 0$. We obtain

$$\begin{aligned} \rho_{s,11}&: \ 0 = d(1 + a - d) + 2b(f - u), \\ \rho_{s,12}&: \ 0 = (1-d)f - u + 2du - bv, \\ \rho_{s,22}&: \ 0 = -2f^2 + 6fu - 4u^2 - (1 - a + d)v. \end{aligned}$$

Setting $\rho_{s,12} = 0$ and $\rho_{s,22} = 0$ yields $a = \frac{1}{v}(2f^2 - 6fu + 4u^2 + v + dv)$ and $b = \frac{1}{v}(f - df - u + 2du)$. We obtain $\rho_{s,11} = \frac{1}{v}(2(f^2 - 2fu + u^2 + dv))$. This implies that $d = -\frac{(f-u)^2}{v}$. Setting $f = vw + u$ yields the parametrization $\mathcal{V}_2$. This parametrization can be extended safely to $v = 0$; we require $u \neq 0$ to ensure $\rho_a \neq 0$. □

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