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ORIGINAL PAPER

Natural approximation of Masjed-Jamei's inequality

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Abstract
In this paper, we obtain a general result on the natural approximation of the function $(\arctan x)^2 - (x \operatorname{arcsinh} x)/\sqrt{1+x^2}$, and prove a conjecture raised by Zhu and Malešević (J Inequal Appl 2019:93, 2019).

Keywords Conjecture · Inverse tangent function · Inverse hyperbolic sine function

Mathematics Subject Classification Primary 26D05 · 26D07 · 26D15; Secondary 33B10 · 41A58

1 Introduction

Masjed-Jamei [1] obtained the following inequality

$$(\arctan x)^2 \leq \frac{x \ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}},\ |x|<1, \tag{1.1}$$

where $\ln(x+\sqrt{1+x^2})$ is the inverse hyperbolic sinefunction $\operatorname{arcsinh} x = \sinh^{-1} x$. In [1] the author conjectured that the above inequality is established in a larger interval $(-\infty, \infty)$. Recently, the authors of this paper [2] first affirmed Masjed-Jamei's conjecture, obtained some natural generalizations of this inequality, and pose a conjecture about a natural approach of Masjed-Jamei's inequality inspired by [3–9].

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Proposition 1.1 ([2], Theorem 1.1]) *The inequality*

$$(\arctan x)^2 \leq \frac{x \ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} \tag{1.2}$$

holds for all $x \in (-\infty, \infty)$*, and the power number* 2 *is the best in* (1.2).

Proposition 1.2 ([2], Theorem 1.3]) *Let* $-\infty < x < \infty$. *Then we have*

$$-\frac{1}{45}x^6 \leq (\arctan x)^2 - \frac{x \ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} \leq -\frac{1}{45}x^6 + \frac{4}{105}x^8, \tag{1.3}$$

$$\begin{aligned} -\frac{1}{45}x^6 + \frac{4}{105}x^8 - \frac{11}{225}x^{10} &\leq (\arctan x)^2 - \frac{x \ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} \\ &\leq -\frac{1}{45}x^6 + \frac{4}{105}x^8 - \frac{11}{225}x^{10} + \frac{586}{10\,395}x^{12}. \end{aligned} \tag{1.4}$$

Conjecture 1.1 ([2], Conjecture 8.1]) *Let* $x \in \mathbf{R}$, $m \geq 1$, *and* v_n *be defined by*

$$v_n = \frac{1}{n}\left(\frac{n!2^{n-1}}{(2n-1)!!} - \sum_{i=1}^{n}\frac{1}{2i-1}\right), n \geq 3. \tag{1.5}$$

Then the double inequality

$$\sum_{n=3}^{2m+1} (-1)^n v_n x^{2n} \leq (\arctan x)^2 - \frac{x \ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} \leq \sum_{n=3}^{2m+2} (-1)^n v_n x^{2n} \tag{1.6}$$

holds.

Using flexible analysis tools this paper obtains a more general conclusion on the natural approximation of the function $(\arctan x)^2 - (x \operatorname{arcsinh} x)/\sqrt{1+x^2}$, and proves the above conjecture.

Theorem 1.1 *Let* $k \geq 3$ *and* v_n *be defined by* (1.5). *Then the function*

$$G_k(x) = \frac{\sqrt{1+x^2}}{x}\left((\arctan x)^2 - \frac{x \ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} - \sum_{n=3}^{k}(-1)^n v_n x^{2n}\right) \tag{1.7}$$

is decreasing and negative on $(0, +\infty)$ *when* k *is an even number, and is increasing and positive on* $(0, +\infty)$ *when* k *is an odd number. In particular, for* $m \geq 1$,

(i) the inequality

$$(\arctan x)^2 - \frac{x \ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} \leq \sum_{n=3}^{2m+2} (-1)^n v_n x^{2n} \tag{1.8}$$

holds for all $x \in [0, +\infty)$*, the constant* v_{2m+2} *is best possible in (1.8);*

(ii) the inequality

$$\sum_{n=3}^{2m+1} (-1)^n v_n x^{2n} \leq (\arctan x)^2 - \frac{x \ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} \tag{1.9}$$

holds for all $x \in [0, +\infty)$, *the constant* $-v_{2m+1}$ *is best possible in (1.9).*

Obviously, the Conjecture 1.1 is from Theorem 1.1 immediately.

2 Lemmas

Lemma 2.1 *Let* $|x| < 1$. *Then*

$$\frac{\operatorname{arcsinh} x}{\sqrt{1+x^2}} = \frac{\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!2^{n-1}}{n\,(2n-1)!!}x^{2n-1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2x)^{2n-1}}{n\binom{2n}{n}}. \tag{2.1}$$

Lemma 2.2 *Let* $|x| < 1$, *and* v_n *be defined by* (1.5). *Then*

$$(\arctan x)^2 - \frac{x\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} = \sum_{n=3}^{\infty}(-1)^n v_n x^{2n}, \tag{2.2}$$

and $v_n > 0$ *for* $n \geq 3$.

Lemma 2.3 *Let* v_n *be defined by* (1.5), *and*

$$\begin{aligned}\gamma_3 &= 45v_3 - 1,\\ \gamma_k &= k\,(2k-1)\,(2k-3)\,v_k - 2\,(k-1)\,(3k-4)\,(2k-3)\,v_{k-1}\\ &\quad + (k-2)\left(-44k+12k^2+41\right)v_{k-2} - 2\,(k-3)\,(2k-5)\,(k-2)\,v_{k-3}.\end{aligned}$$

Then $\gamma_k = 0$ *for* $k \geq 3$.

Lemma 2.4 *Let* v_n *be defined by (1.5). For* $k \geq 3$,

$$\begin{aligned}\mu &= \left(1+4k+12k^2\right)v_k - \left(4k^2-6k+2\right)v_{k-1} > 0,\\ \zeta &= 2k\,(3k-1)\,(2k-1)\,v_k - (k-1)\left(12k^2-20k+9\right)v_{k-1}\\ &\quad +2\,(2k-3)\,(k-1)\,(k-2)\,v_{k-2} > 0.\end{aligned}$$

3 Proof of Lemma 2.1

Let

$$p(x) = \frac{\operatorname{arcsinh} x}{\sqrt{1+x^2}}, \quad |x| < 1. \tag{3.1}$$

Then $p(0) = 0$, and

$$\sqrt{1+x^2}\,p(x) = \operatorname{arcsinh} x. \tag{3.2}$$

Differentiating (3.2) gives

$$\left(1+x^2\right)p'(x) + xp(x) = 1. \tag{3.3}$$

Since $p(x)$ is an odd function, we can express it as a power series

$$p(x)=\sum_{n=0}^{\infty}a_nx^{2n+1}. \tag{3.4}$$

Differentiation of (3.4) yields

$$p'(x)=\sum_{n=0}^{\infty}(2n+1)\,a_nx^{2n},$$

Substituting the series of $p'(x)$ and $p(x)$ into (3.3) yields

$$a_0+\sum_{n=0}^{\infty}\left((2n+3)\,a_{n+1}+2\,(n+1)\,a_n\right)x^{2n+2}=1.$$

Equating coefficients of x^{2n+2}, we can obtain

$$\begin{aligned}a_0&=1,\\ a_{n+1}&=-2\frac{n+1}{2n+3}a_n,\, n\geq 0.\end{aligned}$$

Then

$$a_n=(-2)^n\,\frac{n}{2n+1}\,\frac{n-1}{2n-1}\cdots\frac{2}{5}\frac{1}{3}1=(-2)^n\,\frac{n!}{(2n+1)!!},\, n\geq 0,$$

and

$$\begin{aligned}p(x)=\frac{\operatorname{arcsinh}x}{\sqrt{1+x^2}}&=\sum_{n=0}^{\infty}(-2)^n\,\frac{n!}{(2n+1)!!}x^{2n+1}=\sum_{n=1}^{\infty}(-2)^{n-1}\,\frac{(n-1)!}{(2n-1)!!}x^{2n-1}\\ &=\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{n!2^{n-1}}{n\,(2n-1)!!}x^{2n-1}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,(2x)^{2n-1}}{n\binom{2n}{n}}.\end{aligned}$$

The last equation is true due to

$$\binom{2n}{n}=\frac{2^n\,(2n-1)!!}{n!}.$$

The proof of Lemma 2.1 is complete.

4 Proof of Lemma 2.2

First, by (2.1) we have

$$\frac{x\operatorname{arcsinh}x}{\sqrt{1+x^2}}=\frac{x\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}=\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{n!2^{n-1}}{n\,(2n-1)!!}x^{2n},\quad |x|<1. \tag{4.1}$$

Second, we have

$$\frac{d}{dx}\ (\arctan x)^2 = \frac{2}{1+x^2}\arctan x = 2\left(\sum_{n=0}^{\infty}(-1)^n\, x^{2n}\right)\left(\sum_{n=0}^{\infty}(-1)^n\,\frac{x^{2n+1}}{2n+1}\right)$$
$$= 2\sum_{n=1}^{\infty}(-1)^{n+1}\sum_{i=1}^{n}\frac{1}{2i-1}x^{2n-1}. \tag{4.2}$$

Integrating two sides of (4.2) on $[0, x]$ we can obtain

$$(\arctan x)^2 = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2i-1}\right)x^{2n}. \tag{4.3}$$

From (4.1) and (4.3) we have

$$(\arctan x)^2 - \frac{x\operatorname{arcsinh} x}{\sqrt{1+x^2}} = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2i-1} - \frac{n!2^{n-1}}{n\,(2n-1)!!}x^{2n}\right)x^{2n}$$
$$= \sum_{n=3}^{\infty}\frac{(-1)^n}{n}\left(\frac{n!2^{n-1}}{(2n-1)!!} - \sum_{i=1}^{n}\frac{1}{2i-1}\right)x^{2n}$$
$$= \sum_{n=3}^{\infty}(-1)^n\, v_n x^{2n},$$

where v_n is defined by (1.5). The proof of $v_n > 0$ $(n \geq 3)$ can be found in [2].

5 Proof of Lemma 2.3

The fact $v_1 = v_2 = 0$ and $v_3 = 1/45$ is directly derived from (1.5). Then $\gamma_3 = 45v_3 - 1 = 0$.

Below we assume $k \geq 4$. For convenience, we can order

$$\sum_{i=1}^{k-3}\frac{1}{2i-1} := \theta.$$

Substituting the expression of v_k defined by (1.5) to the left-hand side of the formula (1.6), we can easy verify that

$$\begin{aligned}\gamma_k = &\ (2k-1)\,(2k-3)\left(\frac{k!2^{k-1}}{(2k-1)!!} - \left(\theta + \frac{1}{2k-5} + \frac{1}{2k-3} + \frac{1}{2k-1}\right)\right)\\ &-2\,(3k-4)\,(2k-3)\left(\frac{(k-1)!2^{k-2}}{(2k-3)!!} - \left(\theta + \frac{1}{2k-5} + \frac{1}{2k-3}\right)\right)\\ &+\left(-44k + 12k^2 + 41\right)\left(\frac{(k-2)!2^{k-3}}{(2k-5)!!} - \left(\theta + \frac{1}{2k-5}\right)\right)\\ &-2\,(2k-5)\,(k-2)\left(\frac{(k-3)!2^{k-4}}{(2k-7)!!} - \theta\right)\\ = &\ 0\cdot\theta = 0.\end{aligned}$$

6 Proof of Lemma 2.4

We first verify the first inequality, which is equivalent to

$$\begin{aligned}&\left(4k+12k^2+1\right)\frac{1}{k}\left(\frac{k!2^{k-1}}{(2k-1)!!}-\sum_{i=1}^{k}\frac{1}{2i-1}\right)\\&>2\left(2k-1\right)\left(\frac{(k-1)!2^{k-2}}{(2k-3)!!}-\sum_{i=1}^{k-1}\frac{1}{2i-1}\right),\end{aligned}$$

that is,

$$\frac{8\left(k+1\right)!2^{k-1}}{(2k-1)!!}>\frac{(2k+1)\left(4k+1\right)}{k}\sum_{i=1}^{k-1}\frac{1}{2i-1}+\frac{4k+12k^2+1}{k\left(2k-1\right)}. \tag{6.1}$$

We use mathematical induction to prove (6.1). Obviously, the formula (6.1) holds for $k=3$. Assuming that (6.1) holds for $k=m$, we have

$$\frac{8\left(m+1\right)!2^{m-1}}{(2m-1)!!}>\frac{(2m+1)\left(4m+1\right)}{m}\sum_{i=1}^{m-1}\frac{1}{2i-1}+\frac{4m+12m^2+1}{m\left(2m-1\right)}. \tag{6.2}$$

Next, we prove that (6.1) is valid for $k=m+1$. By (6.2) we have

$$\begin{aligned}\frac{8\left(m+2\right)!2^{m}}{(2m+1)!!}&=\frac{2\left(m+2\right)}{2m+1}\frac{8\left(m+1\right)!2^{m-1}}{(2m-1)!!}\\&>\frac{2\left(m+2\right)}{2m+1}\left(\frac{(2m+1)\left(4m+1\right)}{m}\sum_{i=1}^{m-1}\frac{1}{2i-1}+\frac{4m+12m^2+1}{m\left(2m-1\right)}\right).\end{aligned}$$

In order to complete the proof of (6.1) it suffices to show that

$$\begin{aligned}&\frac{2\left(m+2\right)}{2m+1}\left(\frac{(2m+1)\left(4m+1\right)}{m}\sum_{i=1}^{m-1}\frac{1}{2i-1}+\frac{4m+12m^2+1}{m\left(2m-1\right)}\right)\\&>\frac{(2m+3)\left(4m+5\right)}{m+1}\sum_{i=1}^{m}\frac{1}{2i-1}+\frac{28m+12m^2+17}{(2m+1)\left(m+1\right)},\end{aligned}$$

which is

$$\sum_{i=1}^{m-1}\frac{1}{2i-1}>\frac{4\left(4m^4+4m^3-4m^2-6m-1\right)}{(2m-1)\left(2m+1\right)\left(7m+4m^2+4\right)}. \tag{6.3}$$

Using mathematical induction again to prove (6.3). First, we can see that (6.3) holds for $m=3$. Second, assuming that (6.3) holds for $m=k$, we have

$$\sum_{i=1}^{k-1}\frac{1}{2i-1}>\frac{4\left(4k^4+4k^3-4k^2-6k-1\right)}{(2k-1)\left(2k+1\right)\left(7k+4k^2+4\right)}.$$

Via the inequality above, we have

$$\begin{aligned}\sum_{i=1}^{k}\frac{1}{2i-1}&=\sum_{i=1}^{k-1}\frac{1}{2i-1}+\frac{1}{2k-1}\\&>\frac{4\left(4k^4+4k^3-4k^2-6k-1\right)}{(2k-1)(2k+1)\left(7k+4k^2+4\right)}+\frac{1}{2k-1}.\end{aligned}$$

In order to prove that (6.3) is true for $m=k+1$ it suffices to show that

$$\begin{aligned}&\frac{4\left(4k^4+4k^3-4k^2-6k-1\right)}{(2k-1)(2k+1)\left(7k+4k^2+4\right)}+\frac{1}{2k-1}\\&>\frac{4\left(14k+32k^2+20k^3+4k^4-3\right)}{(2k+1)(2k+3)\left(15k+4k^2+15\right)},\end{aligned}$$

that is,

$$\frac{1}{2k-1}>\frac{4\left(172k+184k^2+80k^3+12k^4+57\right)}{(2k+3)(2k-1)\left(7k+4k^2+4\right)\left(15k+4k^2+15\right)}.$$

The last inequality holds due to

$$\begin{aligned}&(2k+3)(2k-1)\left(7k+4k^2+4\right)\left(15k+4k^2+15\right)\\&\quad-4(2k-1)\left(172k+184k^2+80k^3+12k^4+57\right)\\&=(k+1)\left(121k+80k^2+16k^3+48\right)(2k-1)^2>0.\end{aligned}$$

Then we turn to the proof of second inequality. The desired one is equivalent to that

$$\begin{aligned}&2k(3k-1)(2k-1)\frac{1}{k}\left(\frac{k!2^{k-1}}{(2k-1)!!}-\sum_{i=1}^{k}\frac{1}{2i-1}\right)\\&\quad-(k-1)\left(12k^2-20k+9\right)\frac{1}{(k-1)}\left(\frac{(k-1)!2^{k-2}}{(2k-3)!!}-\sum_{i=1}^{k-1}\frac{1}{2i-1}\right)\\&\quad+2(2k-3)(k-1)(k-2)\frac{1}{(k-2)}\left(\frac{(k-2)!2^{k-3}}{(2k-5)!!}-\sum_{i=1}^{k-2}\frac{1}{2i-1}\right)\\&>0,\end{aligned}$$

that is,

$$\frac{(k+1)!2^{k}}{(2k-3)!!}>(2k-1)(2k+1)\sum_{i=1}^{k-2}\frac{1}{2i-1}+\frac{12k^2-12k-1}{2k-3},$$

or

$$\frac{(k+2)!2^{k+1}}{(2k-1)!!}>(2k+1)(2k+3)\sum_{i=1}^{k-1}\frac{1}{2i-1}+\frac{12k+12k^2-1}{2k-1}. \tag{6.4}$$

By (6.1) we have

$$\begin{aligned}\frac{(k+2)!2^{k+1}}{(2k-1)!!} &= \frac{k+2}{2}\frac{8\,(k+1)!2^{k-1}}{(2k-1)!!}\\ &> \frac{k+2}{2}\left(\frac{(2k+1)\,(4k+1)}{k}\sum_{i=1}^{k-1}\frac{1}{2i-1}++\frac{4k+12k^2+1}{k\,(2k-1)}\right).\end{aligned}$$

In order to complete the proof of (6.4) it suffices to show that

$$\sum_{i=1}^{k-1}\frac{1}{2i-1} > 1-\frac{4k}{4k^2-1},$$

which is true due to

$$\sum_{i=1}^{k-1}\frac{1}{2i-1} > 1.$$

7 Proof of Theorem 1.1

Let $\arctan x = t, x \in (-\infty, \infty)$. Then $x = \tan t, t \in (-\pi/2, \pi/2)$. Because the functions involved in this section are all even functions, we only assume $x > 0$, which is corresponding to $t \in (0, \pi/2)$. Since

$$\begin{aligned}G_k(x) &= \frac{\sqrt{1+x^2}}{x}\left((\arctan x)^2 - \frac{x\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} - \sum_{n=3}^{k}(-1)^n\,v_n x^{2n}\right)\\ &= \frac{t^2-(\sin t)\ln(\tan t+\sec t)-\sum_{n=3}^{k}(-1)^n\,v_n\tan^{2n}t}{\sin t}\\ &= \frac{t^2}{\sin t}-\ln\frac{1+\sin t}{\cos t}-\frac{1}{\sin t}\sum_{n=3}^{k}(-1)^n\,v_n\tan^{2n}t\\ &:= F_k(t).\end{aligned}$$

we examine the properties of the function $F_k(t)$ $(k \geq 3)$ as follows.

Computing to give

$$\begin{aligned}F_k'(t) &= -\frac{1}{\sin^2 t}\left(t^2\cos t-2t\sin t\right)-\frac{1}{\cos t}+\frac{\cos t}{\sin^2 t}\sum_{n=3}^{k}(-1)^n\,v_n\tan^{2n}t\\ &\quad -\frac{1}{\sin t}\sum_{n=3}^{k}(-1)^n\,2nv_n\left(\tan^{2n-1}t+\tan^{2n+1}t\right)\\ &:= \frac{\cos t}{\sin^2 t}g_k(t),\end{aligned}$$

where

$$
\begin{aligned}
g_k(t) &= -(t-\tan t)^2 - 2\sum_{n=3}^{k}(-1)^n\, n v_n \tan^{2n+2} t - \sum_{n=3}^{k}(-1)^n\, v_n\,(2n-1)\tan^{2n} t \\
&= -(t-\tan t)^2 - 2\left((-1)^k\, k v_k \tan^{2k+2} t + \sum_{n=3}^{k-1}(-1)^n\, n v_n \tan^{2n+2} t\right) \\
&\quad -\left((-1)^3\, 5 v_3 \tan^6 t + \sum_{n=4}^{k}(-1)^n\, v_n\,(2n-1)\tan^{2n} t\right) \\
&= -(t-\tan t)^2 - 2\,(-1)^k\, k v_k \tan^{2k+2} t - 2\sum_{n=3}^{k-1}(-1)^n\, n v_n \tan^{2n+2} t \\
&\quad +5 v_3 \tan^6 t - \sum_{n=4}^{k}(-1)^n\, v_n\,(2n-1)\tan^{2n} t \\
&= -(t-\tan t)^2 + 5 v_3 \tan^6 t - 2\sum_{n=3}^{k-1}(-1)^n\, n v_n \tan^{2n+2} t \\
&\quad -\sum_{n=4}^{k}(-1)^n\, v_n\,(2n-1)\tan^{2n} t - 2\,(-1)^k\, k v_k \tan^{2k+2} t \\
&= -(t-\tan t)^2 + 5 v_3 \tan^6 t - 2\sum_{n=4}^{k}(-1)^{n-1}\,(n-1)\, v_{n-1} \tan^{2n} t \\
&\quad -\sum_{n=4}^{k}(-1)^n\, v_n\,(2n-1)\tan^{2n} t - 2\,(-1)^k\, k v_k \tan^{2k+2} t \\
&= -(t-\tan t)^2 + 5 v_3 \tan^6 t \\
&\quad -\sum_{n=4}^{k}(-1)^{n-1}\,\left(2\,(n-1)\, v_{n-1} - (2n-1)\, v_n\right)\tan^{2n} t \\
&\quad -2\,(-1)^k\, k v_k \tan^{2k+2} t.
\end{aligned}
$$

Then

$$
\begin{aligned}
g_k'(t) &= 2\left(t\tan^2 t - \tan^3 t\right) + 30 v_3\left(\tan^7 t + \tan^5 t\right) \\
&\quad -\sum_{n=4}^{k}(-1)^{n-1}\, 2n\,\left(2\,(n-1)\, v_{n-1} - (2n-1)\, v_n\right)\left(\tan^{2n+1} t + \tan^{2n-1} t\right) \\
&\quad -2\,(-1)^k\, k\,(2k+2)\, v_k\left(\tan^{2k+3} t + \tan^{2k+1} t\right) \\
&:= \left(\tan^2 t\right) f_k(t),
\end{aligned}
$$

where

$$
\begin{aligned}
f_k(t) &= 2(t - \tan t) \\
&\quad - \sum_{i=3}^{k} (-1)^{i-1} 2i \left[2(i-1) v_{i-1} - (2i-1) v_i\right] \left(\tan^{2i-1} t + \tan^{2i-3} t\right) \\
&\quad -2(-1)^k k(2k+2) v_k \left(\tan^{2k+1} t + \tan^{2k-1} t\right).
\end{aligned}
$$

Computing to get

$$
\begin{aligned}
f_k'(t) &= -2\tan^2 t \\
&\quad + \sum_{i=3}^{k} (-1)^{i-1} 2i \left[2(i-1) v_{i-1} - (2i-1) v_i\right] \cdot \\
&\quad \left[(1-2i)\tan^{2i} t + 4(1-i)\tan^{2i-2} t + (3-2i)\tan^{2i-4} t\right] \\
&\quad -2(-1)^k k(2k+2) v_k \left[(2k+1)\tan^{2k+2} t + 4k\tan^{2k} t + (2k-1)\tan^{2k-2} t\right] \\
&:= -2(1-45v_3)\tan^2 t + \sum_{i=4}^{k} (-1)^{i-1} 2\gamma_i \tan^{2i-4} t \\
&\quad +2(-1)^{k+1}\left(\tan^{2k-2} t\right)\left(\lambda \tan^4 t + k\mu \tan^2 t + \zeta\right) \\
&= \sum_{i=3}^{k} (-1)^{i-1} 2\gamma_i \tan^{2i-4} t + 2(-1)^{k+1}\left(\tan^{2k-2} t\right)\left(\lambda \tan^4 t + k\mu \tan^2 t + \zeta\right), \\
&= 2(-1)^{k+1}\left(\tan^{2k-2} t\right)\left(\lambda \tan^4 t + k\mu \tan^2 t + \zeta\right)
\end{aligned}
$$

by Lemm 2.3, where

$$
\begin{aligned}
\lambda &= k(2k+2)(2k+1) v_k, \\
\mu &= \left(1 + 4k + 12k^2\right) v_k - \left(4k^2 - 6k + 2\right) v_{k-1}, \\
\zeta &= 2k(3k-1)(2k-1) v_k - (k-1)\left(12k^2 - 20k + 9\right) v_{k-1} \\
&\quad +2(2k-3)(k-1)(k-2) v_{k-2}.
\end{aligned}
$$

By Lemmas 2.2 and 2.4 respectively, we have $\lambda > 0$, and $\mu, \zeta > 0$. So we have

$$
\lambda \tan^4 t + k\mu \tan^2 t + \zeta > 0,
$$

which leads to that $f_k'(t) < 0$ for all $t \in (0, \pi/2)$ when k is an even number and $f_k'(t) > 0$ for all $t \in (0, \pi/2)$ when k is an odd number. Noticing these facts $F_k(0^+) = g_k(0) = f_k(0) = 0$, by differential method we get the conclusion that $F_k(t)$ is decreasing and negative on $(0, \pi/2)$ when k is an even number, and is increasing and positive on $(0, \pi/2)$ when k is an odd number. Because the transformation $x = \tan t$ increases monotonically on the interval $(0, \pi/2)$, the function $G_k(x)$ involved in Theorem 1.1 has the same properties as $F_k(t)$, that is to say, $G_k(x)$ is decreasing and negative on $(0, +\infty)$ when k is an even number and is increasing and positive on $(0, +\infty)$ when k is an odd number.

In view of

$$\lim_{x\to 0}\frac{(\arctan x)^2-\frac{x\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}-\sum_{n=3}^{k-1}(-1)^n\,v_n x^{2n}}{x^{2k}}=(-1)^k\,v_k \tag{7.1}$$

by Lemma 2.2, the proof of Theorem 1.1 is complete.

Remark 7.1 We know that the inequality (1.1) can be traced back to the generalization of the famous Cauchy-Schwarz inequality, which can be found in [10] and the references cited therein.

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