



Common solution to generalized mixed equilibrium problem and fixed point problems in Hilbert space

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Received: 14 December 2017 / Accepted: 16 July 2019 / Published online: 27 July 2019
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Abstract

In this paper, we deal with an extra-gradient iterative method for finding a common solution to a generalized mixed equilibrium problem and fixed point problems for a nonexpansive mapping and for a finite family of k -strict pseudo-contraction mappings in Hilbert space. We prove a strong convergence theorem for the extra-gradient iterative method under some mild conditions. Further, we give a numerical example to illustrate the main result.

Keywords Generalized mixed equilibrium problem · Monotone mapping · Lipschitz continuous mapping · k -strict pseudo-contraction mapping · Extra-gradient iterative method

Mathematics Subject Classification 47H09 · 47H10 · 47J20

1 Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $G : C \times C \rightarrow \mathbb{R}$ and $\phi : C \times C \rightarrow \mathbb{R}$ be nonlinear bifunctions, where \mathbb{R} is the set of all real numbers and let $A : C \rightarrow H$ be a nonlinear mapping. In 1994, Blum and Oettli [2] introduced and studied the following equilibrium problem (in short, EP): Find $x \in C$ such that

$$G(x, y) \geq 0, \quad \forall y \in C. \quad (1.1)$$

The solution set of EP(1.1) is denoted by $\text{Sol}(\text{EP}(1.1))$. An important generalization of EP(1.1) is the mixed equilibrium problem (in short, MEP) introduced and studied by Moudafi and Thera [15] which is of finding $x \in C$ such that

$$G(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \quad (1.2)$$

For application of MEP(1.2), see Moudafi and Thera [15].

It is well known that the equilibrium problems have a great impact and influence in the development of several topics of science and engineering. It turned out that many well known

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problems could be fitted into the equilibrium problems. It has been shown that the theory of equilibrium problems provides a natural, novel and unified framework for several problems arising in nonlinear analysis, optimization, economics, finance, game theory and engineering. The equilibrium problem includes many mathematical problems as particular cases, for example, mathematical programming problem, variational inclusion problem, variational inequality problem, complementary problem, saddle point problem, Nash equilibrium problem in noncooperative games, minimax inequality problem, minimization problem and fixed point problem, see [2,5,14].

Now we consider the following generalized mixed equilibrium problem (in short, GMEP): Find $x \in C$ such that

$$G(x, y) + \langle Ax, y - x \rangle + \phi(y, x) - \phi(x, x) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The solution set of GMEP(1.3) is denoted by $Sol(GMEP(1.3))$.

If we set $G(x, y) = 0, \forall x, y \in C$, GMEP(1.3) reduces to the following important class of variational inequalities which represents the boundary value problem arising in the formulation of Signorini problem: Find $x \in C$ such that

$$\langle Ax, y - x \rangle + \phi(y, x) - \phi(x, x) \geq 0, \quad \forall y \in C. \tag{1.4}$$

Problem (1.4) was discussed in Duvaut and Lions [8] and Kikuchi and Oden [11]. For physical and mathematical formulation of the inequality (1.4), see for example Oden and Pires [19]. For related work, see also Baiocchi and Capelo [1].

If we set $G(x, y) = 0$ and $\phi(x, y) = 0, \forall x, y \in C$, GMEP(1.3) reduces to the classical variational inequality problem (in short, VIP): Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C, \tag{1.5}$$

which was introduced and studied by Hartmann and Stampacchia [9]. The solution set of VIP(1.5) is denoted by $Sol(VIP(1.5))$.

Let S be a nonlinear mapping defined on C , the fixed point problem (in short, FPP) for the mapping S is to find $x \in C$ such that

$$x = Sx. \tag{1.6}$$

$F(S)$ denote the fixed point set of S and is given by $\{x \in C | x = Sx\}$.

In 1976, Korpelevich [12] introduced the following iterative algorithm which is known as extra-gradient iterative method for VIP(1.5):

$$\left. \begin{aligned} x_0 &\in C, \\ y_n &= P_C(x_n - \lambda Ax_n), \\ x_{n+1} &= P_C(x_n - \lambda Ay_n), \end{aligned} \right\} \tag{1.7}$$

where $\lambda > 0$ and $n \geq 0, A$ is a monotone and Lipschitz continuous mapping and P_C is the metric projection of H onto C .

In 2006, Nadezhkhina and Takahashi [16] proved that the sequences $\{x_n\}$ and $\{y_n\}$ generated by the following modified version of extra-gradient iterative method (1.7):

$$\left. \begin{aligned} x_0 &\in C, \\ y_n &= P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ay_n), \end{aligned} \right\} \tag{1.8}$$

where $\lambda_n, \alpha_n \in (0, 1)$ for $n \geq 0$, converge weakly to a common solution to VIP(1.5) and FPP(1.6) for a nonexpansive mapping S .

In 2006, by combining a hybrid iterative method [18] with an extra-gradient iterative method (1.8), Nadezhkina and Takahashi [17] introduced the following hybrid extra-gradient iterative method for approximating a common solution of FPP(1.6) for a nonexpansive mapping S and VIP(1.5) for a monotone and Lipschitz continuous mapping A :

$$\left. \begin{aligned} x_0 &\in C, \\ y_n &= P_C(x_n - \lambda_n Ax_n), \\ z_n &= \beta_n x_n + (1 - \beta_n)SP_C(x_n - \lambda_n Ay_n), \\ C_n &= \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2\}, \\ Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \end{aligned} \right\} \tag{1.9}$$

for $n \geq 0$, and proved a strong convergence theorem.

In 2013, Djafari-Rouhani et al. [6] initiated the study of the following system of unrelated mixed equilibrium problems (in short, SUMEP); more precisely, for each $i = 1, 2, \dots, N$, let C_i be a nonempty, closed and convex subset of a real Hilbert space H with $\bigcap_{i=1}^N C_i \neq \emptyset$; let $G_i : C_i \times C_i \rightarrow \mathbb{R}$ be a bifunction such that $G_i(x_i, x_i) = 0, \forall x_i \in C_i$ and let $A_i : H \rightarrow H$ be a monotone and Lipschitz continuous mapping, then SUMEP is to find $x \in \bigcap_{i=1}^N C_i$ such that

$$G_i(x, y_i) + \langle A_i x, y_i - x \rangle \geq 0, \quad \forall y_i \in C_i, \quad i = 1, 2, \dots, N. \tag{1.10}$$

We note that for each $i = 1, 2, \dots, N$, the mixed equilibrium problem (MEP) is to find $x_i \in C_i$ such that

$$G_i(x_i, y_i) + \langle A_i x_i, y_i - x_i \rangle \geq 0, \quad \forall y_i \in C_i, \quad i = 1, 2, \dots, N. \tag{1.11}$$

We denote by $\text{Sol}(\text{MEP}(1.11))$, the solution set of $\text{MEP}(1.11)$ corresponding to the mappings G_i, A_i and the set C_i . Then the solution set of $\text{SUMEP}(1.10)$ is given by $\bigcap_{i=1}^N \text{Sol}(\text{MEP}(1.11))$. If $N = 1$ then $\text{SUMEP}(1.10)$ is the mixed equilibrium problem $\text{MEP}(1.2)$. They proved a strong convergence theorem for the following new hybrid extra-gradient iterative method which can be seen as an important extension of iterative method (1.9) given by Nadezhkina and Takahashi [17], for solving $\text{SUMEP}(1.10)$ under some mild conditions: The iterative sequences $\{x^n\}, \{y_i^n\}$ and $\{z_i^n\}$ be generated by the iterative schemes

$$\left\{ \begin{aligned} x^0 &\in H, \\ y_i^n &= T_{r_i^n}(x^n - r_i^n A_i x^n), \\ z_i^n &= \alpha_i^n x^n + (1 - \alpha_i^n)S_i T_{r_i^n}(x^n - r_i^n A_i y_i^n), \\ C_i^n &= \{z \in H : \|z_i^n - z\|^2 \leq \|x^n - z\|^2\}, \\ C^n &= \bigcap_{i=1}^N C_i^n, \\ Q^n &= \{z \in H : \langle x^n - z, x_0 - x^n \rangle \geq 0\}, \\ x^{n+1} &= P_{C^n \cap Q^n} x_0, \end{aligned} \right. \tag{1.12}$$

for $n \geq 0$ and for each $i = 1, 2, \dots, N$, where $\{r_i^n\}, \{\alpha_i^n\}$ are control sequences. For the further related work, see [10].

It is worth to mention that none of the strong convergence theorems established for the extra-gradient iterative methods presented so far, other than hybrid extra-gradient iterative method (1.12), for approximating a common solution to $\text{MEP}(1.2)$, where A is monotone and Lipschitz continuous mapping, and fixed point problem for nonlinear mappings. Therefore, our main focus is to propose an extra-gradient iterative method which is not hybrid type, for solving $\text{MEP}(1.2)$, where A is monotone and Lipschitz continuous mapping, and fixed point problems for nonlinear mappings and to establish a strong convergence theorem.

Recall that a nonself mapping $T : C \rightarrow H$ is called k -strict pseudo-contraction if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.13}$$

Set $k = 0$ in (1.13), T is said to be nonexpansive and if we set $k = 1$ in (1.13), T is said to be pseudo-contractive. T is said to be strongly pseudo-contractive if there exists a constant $\lambda \in (0, 1)$ such that $T - \lambda I$ is pseudo-contractive. Clearly, the class of k -strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contraction mappings. We note that the class of strongly pseudo-contractive mappings is independent of the class of k -strict pseudo-contraction mappings (see, e.g. [3,4]). In a real Hilbert space H , (1.13) is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - k}{2} \|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in C. \tag{1.14}$$

T is pseudo-contractive if and only if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in C. \tag{1.15}$$

T is strongly pseudo-contractive if and only if there exists a positive constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, x - y \rangle \leq (1 - \lambda)\|x - y\|^2, \quad \forall x, y \in C. \tag{1.16}$$

Further, we note that the iterative methods for strict pseudo-contractions are far less developed than those for nonexpansive mappings though Browder and Petryshyn [4] initiated their work in 1967; the reason is probably that the second term appearing in the right-hand side of (1.13) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strict pseudo-contraction T . However, on the other hand, strict pseudo-contractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see, Scherzer [21]). Therefore it is interesting to develop the iterative methods for finding a common solution to GMEP(1.3) and fixed point problems for a nonexpansive mapping and for a finite family of k -strict pseudo-contraction mappings. For further work, see for example [13,22,25] and the references therein.

Motivated by the recent work [6,10,24], in this paper, we propose an extra-gradient iterative method for approximating a common solution to GMEP(1.3) and fixed point problems for a nonexpansive mapping and for a finite family of k -strict pseudo-contraction mappings in Hilbert space. Further, we prove that the sequences generated by the proposed iterative method converge strongly to the common solution to GMEP(1.3) and fixed point problems for a nonexpansive mapping and for a finite family of k -strict pseudo-contraction mappings. Further, we give a theoretical numerical example to illustrate the strong convergence theorem.

2 Preliminaries

We recall some concepts and results which are required for the presentation of the work. Let symbols \rightarrow and \rightharpoonup denote strong and weak convergence, respectively. It is well known that every Hilbert space satisfies the Opial condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|, \tag{2.1}$$

holds for every $y \in H$ with $y \neq x$.

For every point $x \in H$, there exists a unique nearest point in C denoted by P_Cx such that

$$\|x - P_Cx\| \leq \|x - y\|, \quad \forall y \in C.$$

The mapping P_C is called the *metric projection* of H onto C . It is well known that P_C is nonexpansive and satisfies

$$\langle x - y, P_Cx - P_Cy \rangle \geq \|P_Cx - P_Cy\|^2, \quad \forall x \in H. \tag{2.2}$$

Moreover, P_Cx is characterized by the fact $P_Cx \in C$ and

$$\langle x - P_Cx, y - P_Cx \rangle \leq 0, \quad \forall y \in C \tag{2.3}$$

which implies

$$\|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cx\|^2, \quad \forall x \in H, y \in C. \tag{2.4}$$

Definition 2.1 A mapping $A : H \rightarrow H$ is said to be:

(i) Monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in H;$$

(ii) λ -Lipschitz continuous if there exists a constant $\lambda > 0$ such that

$$\|Ax - Ay\| \leq \lambda \|x - y\|, \quad \forall x, y \in H.$$

Lemma 2.1 [25] *If $T : C \rightarrow H$ is a k strict pseudo-contraction, then T is Lipschitz continuous with Lipschitz constant $\frac{3-k}{1-k}$.*

Lemma 2.2 [25] *If $T : C \rightarrow H$ is a k -strict pseudo-contraction, then the fixed point set $F(T)$ is closed convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.3 [25] *If $T : C \rightarrow H$ is a k -strict pseudo-contraction with $F(T) \neq \emptyset$. Then $F(P_C T) = F(T)$.*

Lemma 2.4 [25] *If $T : C \rightarrow H$ is a k -strict pseudo-contraction and let for $\lambda \in [k, 1)$, define a mapping $S : C \rightarrow H$ by $Sx = \lambda x + (1 - \lambda)Tx$ for all $x \in C$. Then S is nonexpansive mapping such that $F(S) = F(T)$.*

Lemma 2.5 [23] *Given an integer $N \geq 1$, for each $i = 1, 2, \dots, N$, let $T_i : C \rightarrow H$ be a k_i -strictly pseudo-contraction for some $0 \leq k_i < 1$ and $\max_{1 \leq i \leq N} k_i < 1$ such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Assume that $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i^n = 1$. Then $\sum_{i=1}^N \eta_i T_i : C \rightarrow H$ is a k -strictly pseudo-contraction with coefficient $k = \max_{1 \leq i \leq N} k_i$ and $F(\sum_{i=1}^N \eta_i T_i) = \bigcap_{i=1}^N F(T_i)$.*

Lemma 2.6 [20] *For any $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta \|x - y\|^2 - \alpha\gamma \|x - z\|^2 - \beta\gamma \|y - z\|^2.$$

Lemma 2.7 [24] *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - a_n)s_n + a_n b_n + c_n, \quad n \geq 0,$$

where the sequences $\{a_n\}, \{b_n\}, \{c_n\}$ satisfy the conditions: (i) $\{a_n\} \subset [0, 1]$ with $\sum_{n=0}^\infty a_n = \infty$, (ii) $c_n \geq 0$ for all $n \geq 0$ with $\sum_{n=0}^\infty c_n < \infty$, and (iii) $\limsup_{n \rightarrow \infty} b_n \leq 0$. Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 2.8 [24] *Let $\{s_k\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{s_{k_j}\}$ of $\{s_k\}$ such that $s_{k_j} < s_{k_{j+1}}$ for all $j \geq 0$. Define an integer sequence $\{m_k\}_{k \geq k_0}$ as*

$$m_k = \max\{k_0 \leq l \leq k : s_l < s_{l+1}\},$$

then $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and for all $k \geq k_0$ we have $\max\{s_{m_k}, s_k\} \leq s_{m_{k+1}}$.

Assumption 2.1 The bifunctions $G : C \times C \rightarrow \mathbb{R}$ and $\phi : C \times C \rightarrow \mathbb{R}$ satisfy the following assumptions:

- (i) $G(x, x) = 0, \forall x \in C$;
- (ii) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0, \forall x, y \in C$;
- (iii) For each $y \in C, x \rightarrow G(x, y)$ is weakly upper-semicontinuous;
- (iv) For each $x \in C, y \rightarrow G(x, y)$ is convex and lower semicontinuous.
- (v) $\phi(\cdot, \cdot)$ is weakly continuous and $\phi(\cdot, y)$ is convex;
- (vi) ϕ is skew symmetric, i.e.,
 $\phi(x, x) - \phi(x, y) + \phi(y, y) - \phi(y, x) \geq 0, \forall x, y \in C$;
- (vii) for each $z \in H$ and for each $x \in C$, there exists a bounded subset $D_x \subseteq C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,
 $G(y, z_x) + \phi(z_x, y) - \phi(y, y) + \frac{1}{r} \langle z_x - y, y - z \rangle < 0$.

Assumption 2.2 The bifunction $G : C \times C \rightarrow \mathbb{R}$ is 2-monotone, i.e.,

$$G(x, y) + G(y, z) + G(z, x) \leq 0, \forall x, y, z \in C. \tag{2.5}$$

By taking $y = z$, it is clear that 2-monotone bifunction is a monotone bifunction. For example, if $G(x, y) = x(y - x)$, then G is a 2-monotone bifunction.

Now, we give the concept of 2-skew-symmetric bifunction.

Definition 2.2 The bifunction $\phi : C \times C \rightarrow \mathbb{R}$ is said to be 2-skew-symmetric if

$$\phi(x, x) - \phi(x, y) + \phi(y, y) - \phi(y, z) + \phi(z, z) - \phi(z, x) \geq 0, \forall x, y, z \in C. \tag{2.6}$$

We remark that if set $z = x$ or $x = y$ or $y = z$ in (2.6) then 2-skew-symmetric bifunction becomes skew-symmetric bifunction.

Theorem 2.1 [7] *Let C be a nonempty closed convex subset of a real Hilbert space H . Let the bifunctions $G : C \times C \rightarrow \mathbb{R}$ and $\phi : C \times C \rightarrow \mathbb{R}$ satisfying Assumption 2.1. For $r > 0$ and $z \in H$, define a mapping $T_r : H \rightarrow C$ as follows:*

$$T_r(z) = \{x \in C : G(x, y) + \phi(y, x) - \phi(x, x) + \frac{1}{r} \langle y - x, x - z \rangle \geq 0, \forall y \in C\},$$

for all $z \in H$. Then the following conclusions hold:

- (a) $T_r(z)$ is nonempty for each $z \in H$;
- (b) T_r is single valued;
- (c) T_r is firmly nonexpansive mapping, i.e., for all $z_1, z_2 \in H$,

$$\|T_r z_1 - T_r z_2\|^2 \leq \langle T_r z_1 - T_r z_2, z_1 - z_2 \rangle;$$

- (d) $G(T_r) = \text{Sol}(\text{GMEP}(1.3))$;
- (e) $\text{Sol}(\text{GMEP}(1.3))$ is closed and convex.

Remark 2.1 It follows from Theorem 2.1(a)–(b) that

$$rG(T_r x, y) + r\phi(y, T_r(x)) - r\phi(T_r(x), T_r(x)) + \langle T_r(x) - x, y - T_r(x) \rangle \geq 0, \quad \forall y \in C, x \in H. \tag{2.7}$$

Further Theorem 2.1(c) implies the nonexpansivity of T_r , i.e.,

$$\|T_r(x) - T_r(y)\| \leq \|x - y\|, \quad \forall x, y \in H. \tag{2.8}$$

Furthermore (2.7) implies the following inequality

$$\|T_r(x) - y\|^2 \leq \|x - y\|^2 - \|T_r(x) - x\|^2 + 2rG(T_r(x), y) + 2r[\phi(y, T_r(x)) - \phi(T_r(x), T_r(x))], \quad \forall y \in C, x \in H. \tag{2.9}$$

3 Main result

We prove a strong convergence theorem for finding a common solution to GMEP(1.3) and fixed point problems for a nonexpansive mapping and for a finite family of k -strict pseudo-contraction mappings.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let the bifunction $G : C \times C \rightarrow \mathbb{R}$ satisfy Assumption 2.1(i), (iii), (v), (vii) and Assumption 2.2; let the bifunction $\phi : C \times C \rightarrow \mathbb{R}$ be 2-skew-symmetric and satisfy Assumption 2.1 (v), (vii) and let $f : C \rightarrow C$ be a ρ -contraction mapping. Let $S : C \rightarrow H$ be a nonexpansive mapping and let $A : C \rightarrow H$ be a monotone and Lipschitz continuous mapping with Lipschitz constant λ . For each $i = 1, 2, \dots, N$, let $T_i : C \rightarrow H$ be a k_i -strict pseudo-contraction mapping and let $\{\eta_i^n\}_{i=1}^N$ be a finite sequence of positive numbers such that $\sum_{i=1}^N \eta_i^n = 1$ for all $n \geq 0$. Assume that $\Gamma = \text{Sol}(\text{GMEP}(1.3)) \cap F(S) \cap (\bigcap_{i=1}^N F(T_i)) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by the iterative scheme:*

$$\left. \begin{aligned} x_0 &\in C, \\ y_n &= T_{r_n}(x_n - r_n A x_n), \\ x_{n+1} &= \sigma_n f(x_n) + (1 - \sigma_n) P_C \left[\alpha_n x_n + \beta_n S T_{r_n}(x_n - r_n A y_n) + \gamma_n \sum_{i=1}^N \eta_i^n T_i x_n \right], \end{aligned} \right\} \tag{3.1}$$

for $n \geq 0$, where $\{r_n\} \subset [a, b] \subset (0, \lambda^{-1})$ and $\{\sigma_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are the sequences in $(0, 1)$ satisfying the following conditions:

- (i) $\alpha_n + \beta_n + \gamma_n = 1, \liminf_{n \rightarrow \infty} \beta_n > 0$ and $\liminf_{n \rightarrow \infty} \gamma_n > 0$;
- (ii) $0 \leq k_i \leq \alpha_n \leq l < 1, \lim_{n \rightarrow \infty} \alpha_n = l$;
- (iii) $\lim_{n \rightarrow \infty} \sigma_n = 0$ and $\sum_{n=0}^{\infty} \sigma_n = \infty$;
- (iv) $\sum_{n=1}^{\infty} \sum_{i=1}^N |\eta_i^n - \eta_i^{n-1}| < \infty$.

Then $\{x_n\}$ converges strongly to a point $\hat{x} \in \Gamma$, where $\hat{x} = P_{\Gamma} f(\hat{x})$.

Proof Setting $u_n := T_{r_n}(x_n - r_n A y_n)$ and $z_n := \alpha_n x_n + \beta_n S T_{r_n}(x_n - r_n A y_n) + \gamma_n \sum_{i=1}^N \eta_i^n T_i x_n$, then we have $z_n := \alpha_n x_n + \beta_n S u_n + \gamma_n \sum_{i=1}^N \eta_i^n T_i x_n$. Let $p \in \Gamma$, we have

$$\|S u_n - p\|^2 \leq \|u_n - p\|^2. \tag{3.2}$$

Further, using Remark 2.1, we have

$$\begin{aligned}
 \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nAy_n) - p\|^2 \\
 &\leq \|x_n - r_nAy_n - p\|^2 - \|x_n - r_nAy_n - u_n\|^2 \\
 &\quad + 2r_nG(u_n, p) + 2r_n[\phi(p, u_n) - \phi(u_n, u_n)] \\
 &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Ay_n, p - u_n \rangle + 2r_nG(u_n, p) \\
 &\quad + 2r_n[\phi(p, u_n) - \phi(u_n, u_n)] \\
 &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Ay_n - Ap, p - y_n \rangle + 2r_n\langle Ap, p - y_n \rangle \\
 &\quad + 2r_n\langle Ay_n, y_n - u_n \rangle + 2r_nG(u_n, p) + 2r_n[\phi(p, u_n) - \phi(u_n, u_n)]. \tag{3.3}
 \end{aligned}$$

Since A is monotone and Lipschitz continuous. Since $p \in \text{Sol}(\text{GMEP}(1.3))$ and $y_n \in C$, we have

$$G(p, y_n) + \langle Ap, y_n - p \rangle + \phi(y_n, p) - \phi(p, p) \geq 0, \quad \forall y_n \in C,$$

and hence by using above inequality and monotonicity of A in (3.3), we obtain

$$\begin{aligned}
 \|u_n - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Ay_n, y_n - u_n \rangle + 2r_n[G(p, y_n) + G(u_n, p)] \\
 &\quad + 2r_n[\phi(p, u_n) - \phi(u_n, u_n) + \phi(y_n, p) - \phi(p, p)] \\
 &\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - u_n\|^2 - 2\langle x_n - y_n, y_n - u_n \rangle \\
 &\quad + 2r_n\langle Ay_n, y_n - u_n \rangle \\
 &\quad + 2r_n[G(p, y_n) + G(u_n, p)] + 2r_n[\phi(p, u_n) - \phi(u_n, u_n) \\
 &\quad + \phi(y_n, p) - \phi(p, p)] \\
 &\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - u_n\|^2 - 2\langle y_n - (x_n - r_nAx_n), u_n - y_n \rangle \\
 &\quad + 2r_n\langle Ax_n - Ay_n, u_n - y_n \rangle \\
 &\quad + 2r_n[G(p, y_n) + G(u_n, p)] + 2r_n[\phi(p, u_n) - \phi(u_n, u_n) + \phi(y_n, p) - \phi(p, p)] \\
 &\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - u_n\|^2 + 2r_n[G(y_n, u_n) + \phi(u_n, y_n) - \phi(y_n, y_n)] \\
 &\quad + 2r_n\langle Ax_n - Ay_n, u_n - y_n \rangle + 2r_n[G(p, y_n) + G(u_n, p)] \\
 &\quad + 2r_n[\phi(p, u_n) - \phi(u_n, u_n) + \phi(y_n, p) - \phi(p, p)] \\
 &\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - u_n\|^2 + 2r_n\langle Ax_n - Ay_n, u_n - y_n \rangle \\
 &\quad + 2r_n[G(p, y_n) + G(y_n, y_n) + G(u_n, p)] + 2r_n[\phi(p, u_n) - \phi(u_n, u_n) \\
 &\quad + \phi(y_n, p) - \phi(p, p) + \phi(u_n, y_n) - \phi(y_n, y_n)]. \tag{3.4}
 \end{aligned}$$

Since G is 2-monotone and ϕ is 2-skew-symmetric then (3.4) implies that

$$\begin{aligned}
 \|u_n - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - u_n\|^2 + 2r_n\langle Ax_n - Ay_n, u_n - y_n \rangle \\
 &\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - u_n\|^2 + 2r_n\|Ax_n - Ay_n\|\|u_n - y_n\| \\
 &\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - u_n\|^2 + 2r_n\lambda\|x_n - y_n\|\|u_n - y_n\| \\
 &\leq \|x_n - p\|^2 - (1 - r_n\lambda)\|x_n - y_n\|^2 - (1 - r_n\lambda)\|y_n - u_n\|^2. \tag{3.5}
 \end{aligned}$$

Next by using Lemma 2.6, we estimate

$$\begin{aligned}
 \|x_{n+1} - p\| &= \|\sigma_n f(x_n) + (1 - \sigma_n)P_Cz_n - p\| \\
 &= \|\sigma_n(f(x_n) - p) + (1 - \sigma_n)(P_Cz_n - p)\| \\
 &\leq \sigma_n\|f(x_n) - p\| + (1 - \sigma_n)\|P_Cz_n - p\| \\
 &\leq \sigma_n\|f(x_n) - p\| + (1 - \sigma_n)\|z_n - p\|. \tag{3.6}
 \end{aligned}$$

Now,

$$\begin{aligned} \|f(x_n) - p\| &= \|f(x_n) - f(p) + f(p) - p\| \\ &\leq \|f(x_n) - f(p)\| + \|f(p) - p\| \\ &\leq \rho \|x_n - p\| + \|f(p) - p\|. \end{aligned} \tag{3.7}$$

Denote $W_n = \sum_{i=1}^N \eta_i^n T_i$, it follows from Lemma 2.5 that the mapping $W_n : C \rightarrow H$ is k -strict pseudo-contraction with $k = \max_{1 \leq i \leq N} k_i$ and $F(W_n) = \bigcap_{i=1}^N F(T_i)$ and hence using Lemma 2.6 and (3.5), we have

$$\begin{aligned} \|z_n - p\|^2 &= \|\alpha_n x_n + \beta_n Su_n + \gamma_n W_n x_n - p\|^2 \\ &= \|\alpha_n(x_n - p) + \beta_n(Su_n - p) + \gamma_n(W_n x_n - p)\|^2 \\ &= \alpha_n \|x_n - p\|^2 + \beta_n \|Su_n - p\|^2 + \gamma_n \|W_n x_n - p\|^2 - \alpha_n \beta_n \|x_n - Su_n\|^2 \\ &\quad - \alpha_n \gamma_n \|x_n - W_n x_n\|^2 - \beta_n \gamma_n \|Su_n - W_n x_n\|^2 \\ &= \alpha_n \|x_n - p\|^2 + \beta_n \|u_n - p\|^2 + \gamma_n (\|x_n - p\|^2 + k \|x_n - W_n x_n\|^2) \\ &\quad - \alpha_n \beta_n \|x_n - Su_n\|^2 - \alpha_n \gamma_n \|x_n - W_n x_n\|^2 - \beta_n \gamma_n \|Su_n - W_n x_n\|^2 \\ &\leq (\alpha_n + \beta_n + \gamma_n) \|x_n - p\|^2 - (1 - r_n \lambda) \beta_n \|x_n - y_n\|^2 \\ &\quad - (1 - r_n \lambda) \beta_n \|y_n - u_n\|^2 + \gamma_n k \|x_n - W_n x_n\|^2 - \alpha_n \beta_n \|x_n - Su_n\|^2 \\ &\quad - \alpha_n \gamma_n \|x_n - W_n x_n\|^2 - \beta_n \gamma_n \|Su_n - W_n x_n\|^2 \\ &\leq \|x_n - p\|^2 - \gamma_n (\alpha_n - k) \|x_n - W_n x_n\|^2 - (1 - r_n \lambda) \beta_n \|x_n - y_n\|^2 \\ &\quad - (1 - r_n \lambda) \beta_n \|y_n - u_n\|^2 \\ &\quad - \alpha_n \beta_n \|x_n - Su_n\|^2 - \beta_n \gamma_n \|Su_n - W_n x_n\|^2, \end{aligned} \tag{3.8}$$

which implies

$$\|z_n - p\| \leq \|x_n - p\|. \tag{3.9}$$

Hence, it follows from (3.6), (3.7) and (3.9) that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \sigma_n [\rho \|x_n - p\| + \|f(p) - p\|] + (1 - \sigma_n) \|x_n - p\| \\ &\leq [1 - \sigma_n(1 - \rho)] \|x_n - p\| + \sigma_n \|f(p) - p\|. \end{aligned} \tag{3.10}$$

Since $1 - \rho > 0$ for $\rho \in (0, 1)$, it follows from mathematical induction that

$$\|x_{n+1} - p\| \leq \max \left\{ \|x_0 - p\|, \frac{1}{(1 - \rho)} \|f(p) - p\| \right\}, \tag{3.11}$$

for all $n \geq 0$. Further, it follows from (3.11), (3.9) and (3.5) that the sequences $\{x_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded. Again, we estimate $\|x_{n+1} - \hat{x}\|^2$ with $\hat{x} = P_{O_f}(\emptyset)$. Since $\hat{x} \in \Gamma \subset C$, we have

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 &= \|\sigma_n(f(x_n) - \hat{x}) + (1 - \sigma_n)(P_C z_n - \hat{x})\|^2 \\ &\leq (1 - \sigma_n) \|P_C z_n - \hat{x}\|^2 + 2 \langle \sigma_n(f(x_n) - \hat{x}), x_{n+1} - \hat{x} \rangle \\ &\leq (1 - \sigma_n) \|z_n - \hat{x}\|^2 + 2 \sigma_n \langle f(x_n) - \hat{x}, x_{n+1} - \hat{x} \rangle. \end{aligned} \tag{3.12}$$

Now,

$$\begin{aligned} \langle f(x_n) - \hat{x}, x_{n+1} - \hat{x} \rangle &= \langle f(x_n) - \hat{x}, x_n - \hat{x} \rangle + \langle f(x_n) - \hat{x}, x_{n+1} - x_n \rangle \\ &\leq \|f(x_n) - f(\hat{x})\| \|x_n - \hat{x}\| + \frac{K}{2} \|x_{n+1} - x_n\| + \langle f(\hat{x}) - \hat{x}, x_n - \hat{x} \rangle \\ &\leq \rho \|x_n - \hat{x}\|^2 + \frac{K}{2} \|x_{n+1} - x_n\| + \langle f(\hat{x}) - \hat{x}, x_n - \hat{x} \rangle, \end{aligned} \tag{3.13}$$

where $K = \sup_n 2\|f(x_n) - \hat{x}\|$. It follows from (3.12), (3.13) and (3.8) with \hat{x} in the place of p , that

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 &\leq (1 - \sigma_n(1 - 2\rho))\|x_n - \hat{x}\|^2 + \sigma_n K \|x_{n+1} - x_n\| + 2\sigma_n \langle f(\hat{x}) - \hat{x}, x_n - \hat{x} \rangle \\ &\quad - (1 - \sigma_n)\beta_n[(1 - r_n\lambda)(\|x_n - y_n\|^2 + \|y_n - u_n\|^2) + \alpha_n\|x_n - Su_n\|^2] \\ &\quad - (1 - \sigma_n)\gamma_n[(\alpha_n - k)\|x_n - W_nx_n\|^2 + \beta_n\|Su_n - W_nx_n\|^2], \end{aligned} \tag{3.14}$$

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 &\leq (1 - \sigma_n(1 - 2\rho))\|x_n - \hat{x}\|^2 + \sigma_n K \|x_{n+1} - x_n\| \\ &\quad + 2\sigma_n \langle f(\hat{x}) - \hat{x}, x_n - \hat{x} \rangle. \end{aligned} \tag{3.15}$$

Now, we consider two cases on $s_n := \|x_n - \hat{x}\|^2$.

Case 1. Let the sequence $\{s_n\}$ be decreasing for all $n \geq n_0$ ($n_0 \in \mathbb{N}$), then it is convergent. Since $\{r_n\} \subset [a, b] \subset (0, \lambda^{-1})$, $\lim_{n \rightarrow \infty} \sigma_n = 0$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are the sequences in $(0, 1)$ such that $\liminf_{n \rightarrow \infty} \beta_n > 0$ and $\liminf_{n \rightarrow \infty} \gamma_n > 0$ and $k \leq \alpha_n \forall n$, then (3.14) implies

$$0 = \lim_{n \rightarrow \infty} \|Su_n - W_nx_n\| = \lim_{n \rightarrow \infty} \|x_n - y_n\| = \lim_{n \rightarrow \infty} \|y_n - u_n\| = \lim_{n \rightarrow \infty} \|x_n - Su_n\|. \tag{3.16}$$

This implies that

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| \leq \lim_{n \rightarrow \infty} \|x_n - y_n\| + \lim_{n \rightarrow \infty} \|y_n - u_n\| = 0. \tag{3.17}$$

It follows from (3.16), (3.17), inequality

$$\|x_n - W_nx_n\| \leq \|x_n - Su_n\| + \|Su_n - W_nx_n\|$$

and

$$\|x_n - S_nx_n\| \leq \|u_n - x_n\| + \|x_n - Su_n\|$$

that

$$\lim_{n \rightarrow \infty} \|x_n - W_nx_n\| = 0 \tag{3.18}$$

and

$$\lim_{n \rightarrow \infty} \|u_n - Su_n\| = 0. \tag{3.19}$$

Since $\{x_n\} \subset C$ is bounded, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow q$ in C and satisfying

$$\limsup_{n \rightarrow \infty} \langle f(\hat{x}) - \hat{x}, x_n - \hat{x} \rangle = \lim_{k \rightarrow \infty} \langle f(\hat{x}) - \hat{x}, x_{n_k} - \hat{x} \rangle. \tag{3.20}$$

Now, for each n , define a mapping $V_nx = \alpha_nx + (1 - \alpha_n)W_nx$, $\forall x \in C$ and $\alpha_n \in [k, 1)$. Then by Lemma 2.4, $V_n : C \rightarrow H$ is nonexpansive. Further, we have

$$\begin{aligned} \|x_n - V_nx_n\| &= \|x_n - (\alpha_nx_n + (1 - \alpha_n)W_nx_n)\| \\ &= \|(\alpha_n + (1 - \alpha_n)x_n) - (\alpha_nx_n + (1 - \alpha_n)W_nx_n)\| \\ &= (1 - \alpha_n)\|x_n - W_nx_n\|. \end{aligned} \tag{3.21}$$

Taking limit $n \rightarrow \infty$ and using (3.18), we get

$$\lim_{n \rightarrow \infty} \|x_n - V_nx_n\| = 0. \tag{3.22}$$

Now, by Condition (iv), we may assume that $\eta_i^n \rightarrow \eta_i$ as $n \rightarrow \infty$ for every $1 \leq i \leq N$. It is easy to observe that each $\eta_i > 0$ and $\sum_{i=1}^N \eta_i = 1$. It follows from Lemma 2.5 that the mapping $W : C \rightarrow H$ defined by $Wx = (\sum_{i=1}^N \eta_i T_i)x, \forall x \in C$ is a k -strict pseudo-contraction and $F(W) = \bigcap_{i=1}^N F(T_i)$. Since $\{x_n\}$ is bounded, it follows from Lemma 2.2, condition (iv) and

$$\begin{aligned} \|x_n - Wx_n\| &\leq \|x_n - W_n x_n\| + \|W_n x_n - Wx_n\| \\ &\leq \|x_n - W_n x_n\| + \sum_{i=1}^N |\eta_i^n - \eta_i| \|T_i x_n\| \end{aligned} \tag{3.23}$$

that

$$\lim_{n \rightarrow \infty} \|x_n - Wx_n\| = 0. \tag{3.24}$$

Since

$$\|W_n x_n - Wx_n\| \leq \|W_n x_n - x_n\| + \|x_n - Wx_n\|, \tag{3.25}$$

it follows from (3.18) and (3.24) that

$$\lim_{n \rightarrow \infty} \|W_n x_n - Wx_n\| = 0. \tag{3.26}$$

Again, we observe that the mapping $V : C \rightarrow H$ defined by $Vx = lx + (1 - l)Wx$, for all $x \in C$ and $\alpha_n \in [k, 1)$, is nonexpansive and $F(V) = F(W)$. Hence, we have

$$\begin{aligned} \|x_n - Vx_n\| &\leq \|x_n - V_n x_n\| + \|V_n x_n - Vx_n\| \\ &\leq \|x_n - V_n x_n\| + \|\alpha_n x_n + (1 - \alpha_n)W_n x_n - lx_n - (1 - l)Wx_n\| \\ &\leq \|x_n - V_n x_n\| + |\alpha_n - l| \|x_n - Wx_n\| + (1 - \alpha_n) \|W_n x_n - Wx_n\|. \end{aligned} \tag{3.27}$$

It follows from (3.22), (3.24) and (3.26) that

$$\lim_{n \rightarrow \infty} \|x_n - Vx_n\| = 0. \tag{3.28}$$

Now, we prove $q \in F(V) = F(W) = F(W_n) = \bigcap_{i=1}^N F(T_i)$. Assume that $q \notin F(V)$. Since $x_{n_k} \rightharpoonup q$ and $q \neq Vq$, from Opial condition, we have

$$\begin{aligned} \liminf_{k \rightarrow \infty} \|x_{n_k} - q\| &< \liminf_{k \rightarrow \infty} \|x_{n_k} - Vq\| \\ &\leq \liminf_{k \rightarrow \infty} \|x_{n_k} - Vx_{n_k}\| + \|Vx_{n_k} - Vq\| \\ &\leq \liminf_{k \rightarrow \infty} \|x_{n_k} - q\|, \end{aligned} \tag{3.29}$$

which is a contradiction. Thus, we get $q \in F(V) = F(W) = F(W_n) = \bigcap_{i=1}^N F(T_i)$. It follows from (3.17) that the sequences $\{x_n\}$ and $\{u_n\}$ both have the same asymptotic behaviour and hence there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightarrow q$. Further, it follows from (3.17) and opial condition that $q \in F(S)$. Next, we show that $q \in \text{Sol}(\text{GMEP}(1.3))$.

It follows from (3.16) that sequences $\{x_n\}$ and $\{y_n\}$ both have the same asymptotic behaviour. Therefore, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \rightarrow q$. Now, the relation $y_n = T_{r_n}(x_n - r_n Ax_n)$ implies

$$G(y_n, y) + \langle Ax_{n_k}, y_{n_k} - y \rangle + \phi(y, y_n) - \phi(y_n, y_n) + \frac{1}{r_n} \langle y - y_n, y_n - (x_n - r_n Ax_n) \rangle \geq 0, \forall y \in C$$

which implies that

$$\phi(y, y_n) - \phi(y_n, y_n) + \frac{1}{r_n} \langle y - y_n, y_n - (x_n - r_n Ax_n) \rangle \geq G(y, y_n) + \langle Ax_{n_k}, y_{n_k} - y \rangle.$$

Hence,

$$\phi(y, y_{n_k}) - \phi(y_{n_k}, y_{n_k}) + \left\langle y - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle \geq G(y, y_{n_k}) + \langle Ax_{n_k}, y_{n_k} - y \rangle, \forall y \in C.$$

For t , with $0 \leq t \leq 1$, let $y_t := ty + (1 - t)q \in C$ and $r_n \geq a, \forall n$, then we have

$$\begin{aligned} 0 &\geq -\phi(y_t, y_{n_k}) + \phi(y_{n_k}, y_{n_k}) - \left\langle y - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle \\ &\quad + G(y_t, y_{n_k}) + \langle Ax_{n_k}, y_{n_k} - y_t \rangle \\ &= -\langle y - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{r_{n_k}} \rangle - \phi(y_t, y_{n_k}) + \phi(y_{n_k}, y_{n_k}) \\ &\quad + G(y_t, y_{n_k}) + \langle Ax_{n_k}, y_{n_k} - y_t \rangle \\ &= -\phi(y_t, q) + \phi(q, q) + G(y_t, q) + \langle Aq, q - y_t \rangle, \end{aligned} \tag{3.30}$$

which implies, on taking limit $k \rightarrow \infty$, that

$$\phi(y_t, q) - \phi(q, q) \geq G(y_t, q) + \langle Aq, q - y_t \rangle.$$

Now,

$$\begin{aligned} 0 &= G(y_t, y_t) \\ &\leq tG(y_t, y) + (1 - t)G(y_t, q) \\ &\leq tG(y_t, y) + (1 - t)\phi(y_t, q) - (1 - t)\phi(q, q) + (1 - t)\langle Aq, y_t - q \rangle \\ &\leq tG(y_t, y) + (1 - t)[\phi(y_t, q) - \phi(q, q) + \langle Aq, y_t - q \rangle] \\ &\leq tG(y_t, y) + (1 - t)t[\phi(y, q) - \phi(q, q)] + (1 - t)t\langle Aq, y - q \rangle \\ &\leq G(y_t, y) + (1 - t)[\phi(y, q) - \phi(q, q)] + (1 - t)\langle Aq, y - q \rangle. \end{aligned}$$

Letting $t \rightarrow 0^+$ and for each $y \in C$, we have

$$G(q, y) + \phi(y, q) - \phi(q, q) + \langle Aq, y - q \rangle \geq 0,$$

which implies $q \in \text{Sol}(\text{GMEP}(1.3))$. Thus $q \in \Gamma$. Now, it follows from (2.8) and (3.20) that

$$\limsup_{n \rightarrow \infty} \langle f(\hat{x}) - \hat{x}, x_n - \hat{x} \rangle = \langle f(\hat{x}) - \hat{x}, q - \hat{x} \rangle \leq 0. \tag{3.31}$$

Since $x_n \in C$, we have

$$\|x_{n+1} - x_n\| \leq \sigma_n \|f(x_n) - x_n\| + (1 - \sigma_n)[\beta_n \|Su_n - x_n\| + \gamma_n \|x_n - W_n x_n\|],$$

and hence using $\lim_{n \rightarrow \infty} \sigma_n = 0$, (3.16), (3.18), we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.32}$$

Now, it follows from $\sum_{n=0}^\infty \sigma_n = \infty$, (3.15), (3.31), (3.32) and Lemma 2.7 that $\lim_{n \rightarrow \infty} s_n = 0$. Thus $\{x_n\}$ converges strongly to $\hat{x} = P_\Gamma f(\hat{x})$.

Case 2. Let there be a subsequence $\{s_{k_i}\}$ of $\{s_k\}$ such that $s_{k_i} < s_{k_{i+1}} \forall i \geq 0$. Then according to Lemma 2.8, we can define a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$ as $k \rightarrow \infty$ and $\max\{s_{m_k}, s_k\} \leq s_{m_{k+1}} \forall k$. Since $\{r_k\} \in [a, b] \subset (0, \lambda^{-1}), \forall k \geq 0$ and $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ are the sequences in $(0, 1)$ with conditions (i)–(ii), it follows from (3.14) that

$$\begin{aligned} \lim_{k \rightarrow \infty} \|Su_{m_k} - W_{m_k}x_{m_k}\| &= \lim_{k \rightarrow \infty} \|x_{m_k} - y_{m_k}\| = \lim_{k \rightarrow \infty} \|y_{m_k} - u_{m_k}\| \\ &= \lim_{k \rightarrow \infty} \|x_{m_k} - Su_{m_k}\| = 0. \end{aligned} \tag{3.33}$$

Further, following similar steps as in Case 1, we obtain

$$\limsup_{k \rightarrow \infty} \langle f(\hat{x}) - \hat{x}, x_{m_k} - \hat{x} \rangle \leq 0.$$

Since $\{x_k\}$ is bounded and $\lim_{k \rightarrow \infty} \sigma_k = 0$, it follows from (3.17), (3.18) and inequality

$$\|x_{m_{k+1}} - x_{m_k}\| \leq \sigma_{m_k} \|f(x_{m_k}) - x_{m_k}\| + \beta_{m_k} \|u_{m_k} - x_{m_k}\| + \gamma_{m_k} \|x_{m_k} - W_{m_k}x_{m_k}\|,$$

that

$$\lim_{k \rightarrow \infty} \|x_{m_{k+1}} - x_{m_k}\| = 0. \tag{3.34}$$

Since $s_{m_k} \leq s_{m_{k+1}} \forall k$, it follows from (3.15) that

$$(1 - 2\rho)s_{m_{k+1}} \leq K \|x_{m_{k+1}} - x_{m_k}\| + 2 \langle f(\hat{x}) - \hat{x}, x_{m_k} - \hat{x} \rangle.$$

Now taking limits as $k \rightarrow \infty$, we obtain $s_{m_{k+1}} \rightarrow 0$ as $k \rightarrow \infty$. Since $s_k \leq s_{k+1} \forall k$, it follows that $s_k \rightarrow 0$ as $k \rightarrow \infty$. Hence $x_k \rightarrow \hat{x}$ as $k \rightarrow \infty$. Thus, we have shown that the sequence $\{x_n\}$ generated by iterative algorithm (3.1) converges strongly to $\hat{x} = P_\Gamma f(\hat{x})$. \square

We give the following corollary which is an immediate consequence of Theorem 3.1.

Corollary 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let the bifunction $G : C \times C \rightarrow \mathbb{R}$ satisfy Assumption 2.1 (i), (iii), (v), (vii) and Assumption 2.2; let the bifunction $\phi : C \times C \rightarrow \mathbb{R}$ be 2-skew-symmetric and satisfy Assumption 2.1 (v), (vii) and let $f : C \rightarrow C$ be a ρ -contraction mapping. Let $A : C \rightarrow H$ be a monotone and Lipschitz continuous mapping with Lipschitz constant λ . For each $i = 1, 2, \dots, N$, let $T_i : C \rightarrow H$ be a finite family of nonexpansive mappings and let $\{\eta_i^n\}_{i=1}^N$ be a finite sequence of positive numbers such that $\sum_{i=1}^N \eta_i^n = 1$ for all $n \geq 0$. Assume that $\Gamma_1 = \text{Sol}(\text{GMEP}(1.3)) \cap (\bigcap_{i=1}^N F(T_i)) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by the iterative scheme:*

$$\left. \begin{aligned} x_0 &\in C, \\ y_n &= T_{r_n}(x_n - r_n A x_n), \\ x_{n+1} &= \sigma_n f(x_n) + (1 - \sigma_n) P_C [\alpha_n x_n + \beta_n T_{r_n}(x_n - r_n A y_n) \\ &\quad + \gamma_n \sum_{i=1}^N \eta_i^n T_i x_n], \end{aligned} \right\} \tag{3.35}$$

for $n \geq 0$ where $\{r_n\} \subset [a, b] \subset (0, \lambda^{-1})$ and $\{\sigma_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are the sequences in $(0, 1)$ satisfying the following conditions:

- (i) $\alpha_n + \beta_n + \gamma_n = 1, \liminf_{n \rightarrow \infty} \beta_n > 0$ and $\liminf_{n \rightarrow \infty} \gamma_n > 0$;
- (ii) $\lim_{n \rightarrow \infty} \sigma_n = 0$ and $\sum_{n=0}^\infty \sigma_n = \infty$;

$$(iii) \sum_{n=1}^{\infty} \sum_{i=1}^N |\eta_i^n - \eta_i^{n-1}| < \infty.$$

Then $\{x_n\}$ converges strongly to a point $\hat{x} \in \Gamma_1$, where $\hat{x} = P_{\Gamma_1} f(\hat{x})$.

Proof Set $S = I$, the identity mapping on C , and $k_i = 0$ for $i = 1, 2, \dots, N$ in Theorem 3.1, we get the desired result. □

4 Numerical example

We give a theoretical numerical example which justifies Theorem 3.1.

Example 4.1 Let $H = \mathbb{R}$, $C = [-1, 1]$ and $i = 1, 2, 3$. Define $G : C \times C \rightarrow \mathbb{R}$ and $\phi : C \times C \rightarrow \mathbb{R}$ by $G(x, y) = x(y - x)$ and $\phi(x, y) = y - x$; let the mapping $f : C \rightarrow C$ be defined by $f(x) = \frac{x}{5}, \forall x \in C$; let the mapping $A : C \rightarrow H$ be defined by $A(x) = 3x + 1, \forall x \in C$; let the mapping $T_i : C \rightarrow H$ be defined by $T_i x = -(1 + i)x$ for each $i = 1, 2, 3$, and let the mapping $S : C \rightarrow H$ be defined by $Sx = \frac{x}{4}, \forall x \in C$. Setting $\alpha_n = \frac{1}{10n}$ and $r_n = \frac{1}{5}, \forall n \geq 0$, and $\eta_1 = \eta_2 = \eta_3 = \frac{1}{3}$. Then the sequence $\{x_n\}$ in C generated by the iterative schemes:

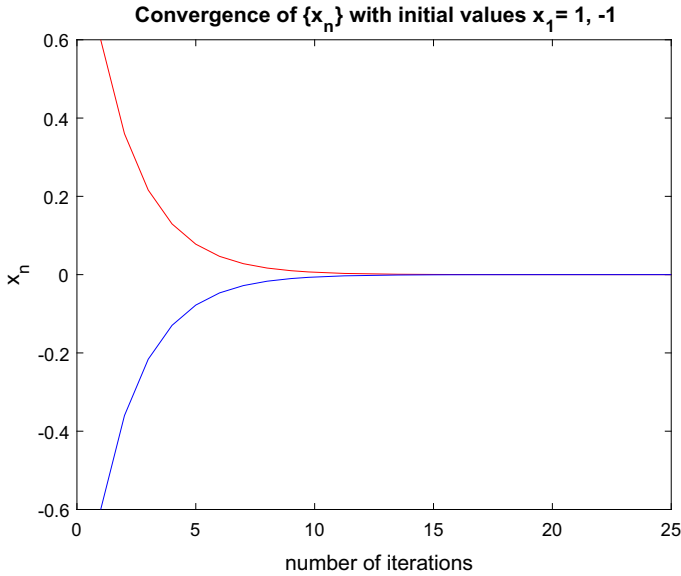
$$\left. \begin{aligned} y_n &= T_{r_n} \left(x_n - \frac{1}{5}(3x + 1)x_n \right); \\ u_n &= T_{r_n} (x_n - r_n A y_n) = \frac{5x_n - 3y_n}{6}; \\ z_n &= \alpha_n x_n + \beta_n \left(\frac{u_n}{4} \right) + \gamma_n [\eta_1 T_1 x_n + \eta_2 T_2 x_n + \eta_3 T_3 x_n]; \\ x_{n+1} &= \frac{x_n}{50n} + \left(1 - \frac{1}{10n} \right) z_n, n \geq 0, \end{aligned} \right\} \tag{4.1}$$

converges to a point $\hat{x} = \{0\} \in \Gamma$.

Proof It is easy to prove that the bifunctions G and ϕ satisfy Assumption 2.1 (i), (iii), (v), (vii) and Assumption 2.2, and Assumption 2.1 (v), (vii) respectively. Choose $\alpha_n = 0.7 + \frac{0.1}{n^2}$, $\beta_n = 0.2 - \frac{0.2}{n^2}$ and $\gamma_n = 0.1 + \frac{0.1}{n^2}$ for all $n \geq 0$, then it is easy to observe that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and satisfy the conditions $\liminf_{n \rightarrow \infty} \beta_n > 0$ and $\liminf_{n \rightarrow \infty} \gamma_n > 0$. Further, for each i , it is easy to prove that T_i are k_i strict pseudo-contraction mappings with $k_1 = \frac{1}{3}, k_2 = \frac{1}{2}$ and $k_3 = \frac{3}{5}$ and $F(T_i) = \{0\}$. Therefore $k = \max\{k_1, k_2, k_3\} = \frac{3}{5}$. Also S is nonexpansive mapping with $F(S) = \{0\}$. Hence $\text{Sol}(\text{GMEP}(1.1)) = \{0\}$. Thus $\Gamma = \text{Sol}(\text{GMEP}(1.3)) \cap F(S) \cap \left(\bigcap_{i=1}^N F(T_i) \right) = \{0\} \neq \emptyset$. After simplification, iterative schemes (4.1) are reduced to the following:

$$\left. \begin{aligned} y_n &= \frac{1}{3}x_n; \\ u_n &= \frac{5x_n - 3y_n}{6}; \\ z_n &= \left(0.7 + \frac{0.1}{n^2} \right) x_n + \left(0.2 - \frac{0.2}{n^2} \right) \frac{u_n}{4} - 3 \left(0.1 + \frac{0.1}{n^2} \right) x_n; \\ x_{n+1} &= \frac{x_n}{50n} + \left(1 - \frac{1}{10n} \right) z_n, n \geq 0. \end{aligned} \right\} \tag{4.2}$$

Next, using the software Matlab 7.8, we have following figure and table which show that $\{x_n\}$ converges to $\hat{x} = \{0\}$.



Convergence of $\{x_n\}$

No. of iterations	x_n $x_1 = -1$	No. of iterations	x_n $x_1 = -1$	No. of iterations	x_n $x_1 = 1$	No. of iterations	x_n $x_1 = 1$
1	-0.600000	14	-0.000784	1	0.600000	14	0.000784
2	-0.360000	15	-0.000470	2	0.360000	15	0.000470
3	-0.216000	16	-0.000282	3	0.216000	16	0.000282
4	-0.129600	17	-0.000169	4	0.129600	17	0.000169
5	-0.077760	18	-0.000102	5	0.077760	18	0.000102
6	-0.046656	19	-0.000061	6	0.046656	19	0.000061
7	-0.027994	20	-0.000037	7	0.027994	20	0.000037
8	-0.016796	21	-0.000022	8	0.016796	21	0.000022
9	-0.010078	22	-0.000013	9	0.010078	22	0.000013
10	-0.006047	23	-0.000008	10	0.006047	23	0.000008
11	-0.003628	24	-0.000005	11	0.003628	24	0.000005
12	-0.002177	25	-0.000003	12	0.002177	25	0.000003
13	-0.001306	26	-0.000002	13	0.001306	26	0.000002

This completes the proof. □

5 Conclusion

We introduced an extra-gradient iterative method for finding a common solution to a generalized mixed equilibrium problem and fixed point problems for a nonexpansive mapping and for a finite family of k -strict pseudo-contraction mappings in Hilbert space and proved the strong convergence of the sequences generated by iterative method. A theoretical numerical example is given to illustrate the Theorem 3.1. It is of further research effort to extend the

iterative method presented in this paper for solving these problems in Banach spaces, and for the case when A is multi-valued mapping.

Acknowledgements Authors are very grateful to the anonymous referees for their critical comments which led to substantial improvements in the original version of the manuscript.

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