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ORIGINAL PAPER

On positive solutions for a m-point fractional boundary value problem on an infinite interval

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Abstract
In this paper, by using a recent fixed point theorem, we study the existence and uniqueness of positive solutions for the following m-point fractional boundary value problem on an infinite interval

$$\begin{cases} D_{0+}^{\alpha}x(t) + f(t, x(t)) = 0, & 0 < t < \infty, \\ x(0) = x'(0) = 0, & D_{0^+}^{\alpha-1}x(+\infty) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases}$$

where $2 < \alpha < 3$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative,

$$D_{0^+}^{\alpha-1}x(+\infty) = \lim_{t\to\infty} D_{0^+}^{\alpha-1}x(t),$$
$$0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \infty \quad \text{and} \quad \beta_i \geq 0 \quad for \quad i = 1, 2, \ldots, m-2.$$

Moreover, we present an example illustrating our results.

Keywords m-point fractional boundary value problem · Fixed point theorem · Positive solution

Mathematics Subject Classification 47H10 · 49L20

1 Introduction

Fractional differential equations arise from a variety of applications including in various fields of science and engineering (see [1–3] and the references therein).

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The existence and multiplicity of solutions to fractional boundary value problem on an infinite interval and the applications of this type of problems have been studied in recent years [4–13].

In this paper, we study the existence and uniqueness of solutions for the following fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha} x(t) + f(t, x(t)) = 0, & 0 < t < \infty, \\ x(0) = x'(0) = 0, & D_{0^+}^{\alpha-1} x(+\infty) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases} \tag{1}$$

where $2 < \alpha < 3$, $D_{0^+}^{\alpha}$ denotes the standard Riemann-Liouville fractional derivative, $D_{0^+}^{\alpha-1} x(+\infty) = \lim_{t\to\infty} D_{0^+}^{\alpha-1} x(t)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \infty$ and $\beta_i \geq 0$ for $i = 1, 2, \ldots, m-2$.

The main tool in our study is a recent fixed point theorem.

2 Preliminaries

For convenience of the reader, the material from fractional calculus theory can be seen in [2].

In order to transform Problem (1) in an integral equation, we need the following lemma which appears in [10].

Lemma 1 *Suppose that $h \in \mathcal{C}[0, \infty)$ then the fractional boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} x(t) + h(t) = 0, & 0 < t < \infty, \\ x(0) = x'(0) = 0, & D_{0^+}^{\alpha-1} x(+\infty) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases}$$

where $2 < \alpha < 3$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \infty$, $\beta_i \geq 0$ for $i = 1, 2, \ldots, m-2$ and $0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} < \Gamma(\alpha)$, has as unique solution

$$x(t) = \int_0^{+\infty} G(t, s) h(s) ds,$$

where

$$G(t, s) = G_1(t, s) + G_2(t, s)$$

and

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t < +\infty \\ t^{\alpha-1}, & 0 \leq t \leq s < +\infty, \end{cases}$$

$$G_2(t, s) = \frac{\sum_{i=1}^{m-2} \beta_i G_1(\xi_i, s)}{\Gamma(\alpha) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}} \cdot t^{\alpha-1}.$$

Remark 1 In [10], the authors proved that $G_1(t, s)$ satisfies the following conditions:

(i) G_1 is continuous on $[0, +\infty) \times [0, +\infty)$ and $G_1(t, s) \geq 0$ for $t, s \in [0, \infty)$.
(ii) $G_1(t, s)$ is strictly increasing respect to the first variable.

In order to present the fixed point theorem which we will use in our study, we need to introduce the following class of functions $\mathcal{F}$.

By $\mathcal{F}$, we denote the class of functions $\varphi : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(i) φ is strictly increasing.
(ii) For any sequence $(t_n) \subset (0, \infty)$,

$$\lim_{n\to\infty} t_n = 0 \iff \lim_{n\to\infty} \varphi(t_n) = -\infty.$$

(iii) There exists $\alpha \in (0, 1)$ such that

$$\lim_{t\to 0^+} t^\alpha \varphi(t) = 0.$$

Examples of functions φ belonging to class $\mathcal{F}$ are $\varphi(t) = -\dfrac{1}{\sqrt{t}}$, $\varphi(t) = \ln t$, $\varphi(t) = \ln t + t$ and $\varphi(t) = \ln(t^2 + t)$.

Next, we present the above announced fixed point theorem which appears in [14].

Theorem 1 *Let (X, d) be a complete metric space and $T : X \to X$ a mapping such that there exist $\tau > 0$ and $\varphi \in \mathcal{F}$ satisfying, for any $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\tau + \varphi(d(Tx, Ty)) \leq \varphi(d(x, y)).$$

Then T has a unique fixed point.

3 Main result

Our starting point in this section is the following estimate for the function G appearing in Lemma 1 which appears in [10].

In order to that the paper is selfcontained, we give a proof.

Lemma 2 *Under assumptions of* Lemma 1*, we have that, for any $t, s \in [0, \infty)$*

$$\frac{G(t,s)}{1 + t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)} + \frac{\xi_{m-2}^{\alpha-1} \sum_{i=1}^{m-2} \beta_i}{\Gamma(\alpha)\left(\Gamma(\alpha) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}\right)}.$$

Proof Since

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t < +\infty, \\ t^{\alpha-1}, & 0 \leq t \leq s < +\infty, \end{cases}$$

it is clear that $G_1(t, s) \leq \dfrac{1}{\Gamma(\alpha)} t^{\alpha-1}$, for any $t, s \in [0, \infty)$.

Therefore, for any $t, s \in [0, \infty)$

$$\frac{G_1(t,s)}{1 + t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)} \frac{t^{\alpha-1}}{1 + t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)}.$$

On the other hand, since

$$G_2(t,s) = \frac{\sum_{i=1}^{m-2} \beta_i G_1(\xi_i, s)}{\Gamma(\alpha) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}} \cdot t^{\alpha-1},$$

we infer,

$$\begin{aligned}\frac{G_2(t,s)}{1+t^{\alpha-1}} &= \frac{\sum_{i=1}^{m-2}\beta_i G_1(\xi_i,s)}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\cdot\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\\ &\le \frac{\sum_{i=1}^{m-2}\beta_i G_1(\xi_i,s)}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\\ &\le \frac{G_1(\xi_{m-2},s)\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\\ &\le \frac{\frac{1}{\Gamma(\alpha)}\xi_{m-2}^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\\ &= \frac{\xi_{m-2}^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)\left(\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}\right)},\end{aligned}$$

where we have used that $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<\infty$, the strictly increasing character respect to the first variable of the function $G_1(t,s)$ (see Remark 1) and that $G_1(\xi_{m-2},s)\le \frac{1}{\Gamma(\alpha)}\xi_{m-2}^{\alpha-1}$ for any $s\in[0,\infty)$.

Now, since $G(t,s)=G_1(t,s)+G_2(t)$, we infer

$$\begin{aligned}\frac{G(t,s)}{1+t^{\alpha-1}} &\le \frac{G_1(t,s)}{1+t^{\alpha-1}}+\frac{G_2(t,s)}{1+t^{\alpha-1}}\\ &\le \frac{1}{\Gamma(\alpha)}+\frac{\xi_{m-2}^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)\left(\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}\right)}.\end{aligned}$$

This finishes the proof. □

Now, we recall a well known result in order to present a selfcontained paper.

Lemma 3 *Suppose that $\varphi\colon[0,\infty)\to[0,\infty)$ is a concave function and $\varphi(0)=0$. Then φ is subadditive, this is,*

$$\varphi(x+y)\le\varphi(x)+\varphi(y)\,,\quad \textit{for any } x,y\in[0,\infty).$$

Proof Since φ is concave and $\varphi(0)=0$, for any $x,y\in[0,\infty)$, we have

$$\begin{aligned}\varphi(x) &= \varphi\left(\frac{x}{x+y}(x+y)+\frac{y}{x+y}0\right)\\ &\ge \frac{x}{x+y}\varphi(x+y)+\frac{y}{x+y}\varphi(0)\\ &= \frac{x}{x+y}\varphi(x+y)\\ \varphi(y) &= \varphi\left(\frac{x}{x+y}0+\frac{y}{x+y}(x+y)\right)\\ &\ge \frac{x}{x+y}\varphi(0)+\frac{y}{x+y}\varphi(x+y)\\ &= \frac{y}{x+y}\varphi(x+y).\end{aligned}$$

Adding these inequalities, we get

$$\varphi(x)+\varphi(y) \geq \frac{x}{x+y}\varphi(x+y)+\frac{y}{x+y}\varphi(x+y)=\varphi(x+y).$$

This proves the subadditivity of φ. □

Before to present the main result of the paper, we need the following technical result.

Lemma 4 *Suppose that $p > 1$ and $\tau > 0$.*
Consider the function $\Phi_p^\tau : [0,\infty) \to [0,\infty)$ given by

$$\Phi_p^\tau(t) = \frac{t}{(1+\tau \cdot t^{1/p})^p}, \qquad for \quad t \in [0,\infty).$$

Then:

(i) Φ_p^τ *is increasing.*
(ii) Φ_p^τ *is subadditive.*
(iii) *For any $t, s \in [0,\infty)$*

$$|\Phi_p^\tau(t) - \Phi_p^\tau(s)| \leq \Phi_p^\tau(|t-s|).$$

Proof (i) It is clear since $\Phi_p^\tau(t)' = \dfrac{1}{(1+\tau \cdot t^{1/p})^{p+1}} \geq 0$ for $t \in [0,\infty)$.
(ii) Since $\Phi_p^\tau(0) = 0$ and

$$(\Phi_p^\tau(t))'' = -\frac{(p+1)\dfrac{\tau}{p}}{t^{1-1/p}(1+\tau \cdot t^{1/p})^{p+2}} \leq 0, \quad for \quad t \in [0,\infty),$$

Φ_p^τ is concave.
By using Lemma 3, Φ_p^τ is subadditive.
(iii) Suppose, without loss of generality, that $s < t$ and $s, t \in [0,\infty)$. Then, by (ii),

$$\Phi_p^\tau(t) = \Phi_p^\tau(s+t-s) \leq \Phi_p^\tau(s) + \Phi_p^\tau(t-s).$$

From this inequality, we infer

$$\Phi_p^\tau(t) - \Phi_p^\tau(s) \leq \Phi_p^\tau(t-s).$$

Taking into account (ii), we get

$$\begin{aligned} |\Phi_p^\tau(t) - \Phi_p^\tau(s)| &= \Phi_p^\tau(t) - \Phi_p^\tau(s) \\ &\leq \Phi_p^\tau(t-s) \\ &= \Phi_p^\tau(|t-s|). \end{aligned}$$

This proves our claim. □

In the proof of the main result of the paper, we will use the following space E defined by

$$E = \left\{ x \in \mathcal{C}[0,\infty) : \sup\left\{ \frac{|x(t)|}{1+t^{\alpha-1}} : t \in [0,\infty) \right\} < \infty \right\}$$

This space E equipped with the norm

$$\|x\| = \sup_{0\le t<+\infty} \frac{|x(t)|}{1+t^{\alpha-1}}$$

is a Banach space.

Now, we are ready to present the main result of the paper.

Theorem 2 *Consider the following assumptions:*

(H1) $f:[0,\infty)\times[0,\infty)\to[0,\infty)$ *is continuous and such that* $\int_0^\infty f(s,0)ds<\infty$.

(H2) *There exist* $\tau>0$ *and* $p>1$ *such that, for any* $t\in[0,\infty)$ *and* $x,y\in[0,\infty)$,

$$|f(t,x)-f(t,y)| \le a(t)\frac{|x-y|}{(1+\tau|x-y|^{1/p})^p},$$

where $a:[0,\infty)\to[0,\infty)$ *and it satisfies* $\int_0^\infty (1+s^{\alpha-1})a(s)ds<\infty$.

(H3) $\int_0^\infty (1+s^{\alpha-1})a(s)ds \le \frac{1}{L}$, *where* $L=\frac{1}{\Gamma(\alpha)}+\frac{\xi_{m-2}^{\alpha-1}\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)\left(\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}\right)}$.

Then Problem (1) *has a unique nonnegative solution in the space* E.

Proof Consider the cone $P=\{x\in E : x\ge 0\}$. Notice that P is closed subset of E and, consequently, (P,d) is a complete metric space, where d is the distance given by

$$d(x,y)=\sup\left\{\frac{|x(t)-y(t)|}{1+t^{\alpha-1}} : t\in[0,\infty)\right\}, \quad for\ any \quad x,y\in P.$$

Next, we consider the operator T defined on P as

$$(Tx)(t)=\int_0^\infty G(t,s)f(s,x(s))ds, \quad for \quad x\in P.$$

Since $G(t,s)\ge 0$ (see Remark 1) and $(H1)$ it is clear that for $x\in P$, $Tx\ge 0$.

Now, we will prove that if $x\in E$ then $Tx\in\mathcal{C}[0,\infty)$.

In fact, we take $t_0\in[0,\infty)$ and $(t_n)\subset[0,\infty)$ such that $t_n\to t_0$. We have to prove that $(Tx)(t_n)\to(Tx)(t_0)$.

In fact

$$\begin{aligned}
|(Tx)(t_n)-(Tx)(t_0)| &= \left|\int_0^\infty G(t_n,s)f(s,x(s))ds-\int_0^\infty G(t_0,s)f(s,x(s))ds\right| \\
&= \left|\int_0^\infty (G(t_n,s)-G(t_0,s))f(s,x(s))ds\right| \\
&\le \int_0^\infty |G(t_n,s)-G(t_0,s)||f(s,x(s))|ds \\
&\le \int_0^\infty |G(t_n,s)-G(t_0,s)|[|f(s,x(s))-f(s,0)|+|f(s,0)|]ds \\
&\le \int_0^\infty |G(t_n,s)-G(t_0,s)|\left[a(s)\frac{|x(s)|}{(1+\tau|x(s)|^{1/p})^p)}+|f(s,0)|\right]ds \\
&= \int_0^\infty |G(t_n,s)-G(t_0,s)|\left[a(s)\frac{x(s)}{(1+\tau x(s)^{1/p})^p}+f(s,0)\right]ds
\end{aligned}$$

$$= \int_0^\infty |G(t_n,s)-G(t_0,s)| \left[(1+s^{\alpha-1}) \frac{a(s)\dfrac{x(s)}{1+s^{\alpha-1}}}{(1+\tau(x(s))^{1/p})^p} + f(s,0) \right] ds$$

$$\leq \int_0^\infty |G(t_n,s)-G(t_0,s)| \left[(1+s^{\alpha-1}) \frac{a(s)\dfrac{x(s)}{1+s^{\alpha-1}}}{\left(1+\tau\left(\dfrac{x(s)}{1+s^{\alpha-1}}\right)^{1/p}\right)^p} + f(s,0) \right] ds$$

$$\leq \int_0^\infty |G(t_n,s)-G(t_0,s)| \left[(1+s^{\alpha-1})a(s)\frac{\|x\|}{(1+\tau\|x\|^{1/p})^p} + f(s,0) \right] ds$$

where we have used Lemma 4.

Since $G(t,s)$ is continuous on $[0,\infty)\times[0,\infty)$ (see Remark 1), and taking into account our assumptions, for $\varepsilon>0$ given, we can find $n_0\in\mathbb{N}$ such that, for $n\geq n_0$,

$$|G(t_n,s)-G(t_0,s)| < \frac{\varepsilon}{\left(\displaystyle\int_0^\infty (1+s^{\alpha-1})(a(s))ds\right)\dfrac{\|x\|}{(1+\tau\|x\|^{1/p})^p} + \displaystyle\int_0^\infty f(s,0)ds}.$$

Therefore, from this fact and the last inequality, we have, for $n\geq n_0$, $|(Tx)(t_n)-(Tx)(t_0)|\leq\varepsilon$.

This proves that $Tx\in\mathcal{C}[0,\infty)$.

Next, we wil prove that

$$\sup_{0\leq t\leq+\infty} \frac{(Tx)(t))}{1+t^{\alpha-1}} < \infty.$$

In fact, taking into account our assumptions and Lemmas 2 and 4 , we get, for $x\in P$, and $t\in[0,\infty)$

$$\begin{aligned}
\frac{(Tx)(t)}{1+t^{\alpha-1}} &= \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f(s,x(s))ds \\
&\leq L\int_0^\infty f(s,x(s))ds \\
&= L\int_0^\infty (|f(s,x(s))-f(s,0)|+f(s,0))ds \\
&\leq L\int_0^\infty \left(a(s)\frac{x(s)}{(1+\tau(x(s))^{1/p})^p}+f(s,0)\right)ds \\
&\leq L\int_0^\infty \left[(1+s^{\alpha-1})a(s)\frac{\left(\dfrac{x(s)}{1+s^{\alpha-1}}\right)}{\left(1+\tau\left(\dfrac{x(s)}{1+s^{\alpha-1}}\right)^{1/p}\right)^p}+f(s,0)\right]ds \\
&\leq L\int_0^\infty \left[(1+s^{\alpha-1})a(s)\frac{\|x\|}{(1+\tau\|x\|^{1/p})^p}+f(s,0)\right]ds \\
&\leq L\left[\frac{\|x\|}{(1+\tau\|x\|^{1/p})^p}\cdot\int_0^\infty (1+s^{\alpha-1})a(s)ds+\int_0^\infty f(s,0)ds\right] \\
&< \infty.
\end{aligned}$$

This proves that T applies P into itself.

Next, we check that T satisfies the contractivity condition appearing in Theorem 1.

In fact, for $x, y \in P$, with $d(Tx, Ty) > 0$, we have

$$\begin{aligned} d(Tx, Ty) &= \sup_{0\le t<+\infty} \left\{ \frac{|(Tx)(t) - (Ty)(t)|}{1+t^{\alpha-1}} \right\} \\ &= \sup_{0\le t<+\infty} \left\{ \frac{\left| \int_0^\infty G(t,s) f(s, x(s)) - f(s, y(s)) \right|}{1+t^{\alpha-1}} \right\} \\ &\le \sup_{0\le t<+\infty} \left\{ \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} |f(s, x(s)) - f(s, y(s))| ds \right\} \\ &\le L \int_0^\infty |f(s, x(s)) - f(s, y(s))| ds \\ &\le L \int_0^\infty a(s) \cdot \frac{|x(s) - y(s)|}{(1+\tau |x(s) - y(s)|^{1/p})^p} ds \\ &\le L \int_0^\infty (1+s^{\alpha-1}) a(s) \cdot \frac{\left| \frac{x(s)}{1+s^{\alpha-1}} - \frac{y(s)}{1+s^{\alpha-1}} \right|}{\left(1+\tau \left(\left| \frac{x(s)}{1+s^{\alpha-1}} - \frac{y(s)}{1+s^{\alpha-1}} \right|^{1/p} \right)\right)^p} ds \\ &\le L \int_0^\infty (1+s^{\alpha-1}) a(s) \cdot \frac{d(x,y)}{(1+\tau d(x,y)^{1/p})^p} ds \\ &\le \frac{d(x,y)}{(1+\tau d(x,y)^{1/p})^p}, \end{aligned}$$

where we have used $(H3)$.

Summarizing, we have, for $x, y \in P$ with $d(Tx, Ty) > 0$,

$$d(Tx, Ty) \le \frac{d(x,y)}{(1+\tau d(x,y)^{1/p})^p}.$$

From this, it follows

$$d(Tx, Ty)^{1/p} \le \frac{d(x,y)^{1/p}}{(1+\tau d(x,y)^{1/p})},$$

or, equivalently,

$$\frac{1}{d(x,y)^{1/p}} + \tau \le \frac{1}{d(Tx, Ty)^{1/p}}.$$

This gives us

$$\tau - \frac{1}{d(Tx, Ty)^{1/p}} \le -\frac{1}{d(x,y)^{1/p}}$$

and, therefore, the contractivity condition appearing in Theorem 1 is satisfied with the function $\varphi(t) = -\frac{1}{t^{1/p}}$ and it is easily checked that $\varphi \in \mathcal{F}$.

Finally, Theorem 1 says us that the operator T has a unique fixed point in P. This means that our Problem (1) has a unique nonnegative solution in E.

This finishes the proof. □

An interesting question from a practical point of view is that the solution to Problem (1) given by Theorem 2 is positive, this is, $x(t) > 0$ for $t \in (0, \infty)$.

In the following result, we present a sufficient condition for this fact holds.

Theorem 3 *If to assumptions of* Theorem 2, *we add the following one:*

(H4) $f(t, x)$ *is increasing respect to the variable* x *and there exists* $t_0 \in [0, \infty)$ *such that* $f(t_0, 0) > 0$

then the nonnegative solution given by Theorem 2 *is positive.*

Proof Since the solution $x(t)$ given by Theorem 2 is a fixed point of the operator T we take

$$x(t) = \int_0^\infty G(t, s) f(s, x(s)) ds, \quad for \quad t \in [0, \infty).$$

Suppose in contrary case, that $x(t)$ is not positive.

This means that there exists $t^* \in (0, \infty)$ such that $x(t^*) = 0$, and therefore,

$$0 = x(t^*) = \int_0^\infty G(t^*, s) f(s, x(s)) ds.$$

Since f is increasing respect to the second variable and $x(t)$ is nonnegative, we infer

$$0 = x(t^*) = \int_0^\infty G(t^*, s) f(s, x(s)) ds \geq \int_0^\infty G(t^*, s) f(s, 0) ds \geq 0.$$

This gives us

$$\int_0^\infty G(t^*, s) f(s, 0) ds = 0.$$

As the integrand is nonnegative, it follows that

$$G(t^*, s) f(s, 0) = 0, a.e(s).$$

Since $G(t^*, s) \geq G_1(t^*, s)$ and it is clear that $G_1(t^*, s) \neq 0 \quad a.e(s)$, we infer

$$f(s, 0) = 0, a.e(s).$$

Now, taking into account our assumption $(H4)$, there exists $t_0 \in [0, \infty)$ such that $f(t_0, 0) > 0$. Now, by the continuity of f, we can find a set A such that $t_0 \in A$ and $\mu(A) > 0$, where μ is the Lebesgue measure and such that $f(s, 0) > 0$ for $s \in A$. This says us that $f(s, 0) \neq 0$ a.e (s).

This gives us a contradiction and, consequently, $x(t) > 0$ for $t \in (0, \infty)$.

This completes the proof. □

4 An example

In this section, we present an example illustrating our results.

Consider the following m-point boundary value problem

$$\begin{cases} D_{0^+}^{5/2}x(t)+\dfrac{\lambda}{(t+1)^2(1+\sqrt{t^3})}\left(\dfrac{x(t)}{(1+6\sqrt[4]{x(t)})^4}+\dfrac{1}{(t+1)^3}\right)=0, \\ x(0)=x'(0), \qquad D_{0^+}^{3/2}x(+\infty)=\dfrac{1}{4}x\left(\dfrac{1}{9}\right)+\dfrac{1}{2}x\left(\dfrac{1}{4}\right), \end{cases} \tag{2}$$

where $\lambda > 0$ and $t \in (0, \infty)$.

Notice that Problem (2) is a particular case of Problem (1), where $\alpha = \frac{5}{2}$, $\beta_1 = \frac{1}{4}$, $\beta_2 = \frac{1}{2}$, $\xi_1 = \frac{1}{9}$, $\xi_2 = \frac{1}{4}$, $m = 4$ and

$$f(t,x)=\frac{\lambda}{(t+1)^2(1+\sqrt{t^3})}\left(\frac{x(t)}{(1+6\sqrt[4]{x(t)})^4}+\frac{1}{(t+1)^3}\right).$$

It is clear that $f : [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous and, moreover,

$$\int_0^\infty f(s,0)ds=\int_0^\infty \frac{\lambda}{(s+1)^5(1+\sqrt{s^3})}ds \le \int_0^\infty \frac{\lambda}{(s+1)^5}ds=\frac{\lambda}{4}<\infty.$$

This proves that assumption $(H1)$ of Theorem 2 is satisfied.

On the other hand, for any $t \in [0, \infty)$ and $x, y \in [0, \infty)$, we have

$$\begin{aligned} |f(t,x)-f(t,y)| &\le \frac{\lambda}{(t+1)^2(1+\sqrt{t^3})}\left|\frac{x}{(1+6\sqrt[4]{x})^4}-\frac{y}{(1+6\sqrt[4]{y})^4}\right| \\ &= \frac{\lambda}{(t+1)^2(1+\sqrt{t^3})}\left|\Phi_4^6(x)-\Phi_4^6(y)\right| \\ &\le \frac{\lambda}{(t+1)^2(1+\sqrt{t^3})}\Phi_4^6(|x-y|) \\ &= \frac{\lambda}{(t+1)^2(1+\sqrt{t^3})}\frac{|x-y|}{(1+6\sqrt[4]{|x-y|})^4}, \end{aligned}$$

where we have used Lemma 4.

Therefore, assumption $(H2)$ of Theorem 2 is satisfied with $a(t) = \dfrac{\lambda}{(t+1)^2(1+\sqrt{t^3})}$, $\tau = 6$ and $p = 4$, since

$$\begin{aligned} \int_0^\infty (1+s^{3/2})a(s)ds &= \int_0^\infty (1+s^{3/2})\frac{\lambda}{(s+1)^2(1+s^{3/2})}ds \\ &= \int_0^\infty \frac{\lambda}{(s+1)^2}ds \\ &= \lambda < \infty. \end{aligned}$$

Moreover, taking into account that, in our case,

$$L=\frac{1}{\Gamma(5/2)}+\frac{(1/4)^{3/2}(1/4+1/2)}{\Gamma(5/2)[\Gamma(5/2)-(1/4(1/9)^{3/2}+1/2(1/4)^{3/2})]}\cong 0'8087,$$

and, consequently,

$$\int_0^\infty (1+s^{3/2})a(s)ds = \lambda.$$

Therefore, if $\lambda \leq \frac{1}{L} \cong 1'2365$, Theorem 2 says us that Problem (2) has a unique nonegative solution in the space E.

As the function $f(t,x)$ is increasing respect to the variable x and it is clear that, since $f(t,0) = \dfrac{\lambda}{(t+1)^5(1+\sqrt{t^3})}$, assumption $(H4)$ of Theorem 3 is satisfied. By Theorem 3, Problem (2) has a unique positive solution in the space E.

Notice that our example cannot be treated by using the Banach's contraction principle, since that, in our case, the operator T defined on the cone $P = \{x \in E : x \geq 0\}$, where $E = \left\{ x \in C[0,\infty) : \sup \left\{ \dfrac{|x(t)|}{1+t^{3/2}} : t \in [0,\infty) \right\} < \infty \right\}$ equipped with the distance

$$d(x,y) = \sup_{0 \leq t < +\infty} \frac{|x(t)-y(t)|}{1+t^{3/2}},$$

is given by,

$$(Tx)(t) = \int_0^\infty G(t,s) \frac{\lambda}{(s+1)^2(1+\sqrt{s^3})} \left(\frac{x(s)}{(1+6\sqrt[4]{x(s)})^4} + \frac{1}{(s+1)^3} \right) ds,$$

for $x \in P$, where $G(t,s)$ is the function appearing in Lemma 1.

Moreover, for $\lambda = \frac{1}{L} \cong 1'2365$, we have

$$d(Tx,Ty) \leq \frac{d(x,y)}{(1+6d(x,y)^{1/4})^4}, \qquad for \quad x,y \in P.$$

Therefore,

$$\frac{d(Tx,Ty)}{d(x,y)} \leq \frac{1}{(1+6d(x,y)^{1/4})^4}, \text{ for } x,y \in P \text{ with } x \neq y.$$

Notice that, when $d(x,y) \to 0$ (for example, for $x \in P$ fixed and $y_n = x + \dfrac{1}{n}, n \in \mathbb{N}$) we have $\dfrac{d(Tx,Ty)}{d(x,y)} \to 1$ and, this proves that the Banach's contraction principle doesn't work in our example.

5 Comparison with other result

In [13], the authors studied the following m-point fractional boundary value problem on the half-line

$$\begin{cases} D_{0+}^{\alpha} x(t) + \lambda a(t) f(t,x(t)) = 0, & 0 < t < \infty, \\ x(0) = x'(0) = 0, & D_{0+}^{\alpha-1} x(+\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases} \tag{3}$$

where $2 < \alpha < 3$, $\lambda > 0$, $a : [0,\infty) \to [0,\infty)$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ are continuous functions and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \infty$, $\beta_i \geq 0$ $(i = 1,2,\ldots,m-2)$ and $0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} < \Gamma(\alpha)$.

Notice that Problem (3) is our Problem (1), where the role of $f(t,x(t))$ is played by $\lambda a(t) f(t,x(t))$.

Part of the main result of [13] is the following.

Theorem 4 [13] *Under the following assumptions:*

(i) $f : [0, \infty) \times [0, \infty) \to [0, \infty)$ *is continuous,* $f(t, 0) \not\equiv 0$, $t \in [0, \infty)$.
(ii) $f(t, x)$ *is increasing in* $x \in [0, \infty)$ *and when* x *is bounded,* $f(t, (1+t^{\alpha-1})x)$ *is bounded on* $[0, \infty)$.
(iii) *For* $r \in (0, 1)$, *there exists* $\varphi(r) \in (r, 1)$ *such that* $f(t, rx) \geq \varphi(r) f(t, x)$, $t, x \in [0, \infty)$.
(iv) $a(t)$ *is continuous with* $0 < \int_0^{+\infty} a(s)ds < \infty$.

Then Problem (3) *has a unique positive solution.*

Notice that our Problem (2) is a particular case of Problem (3), with $a(t) = \frac{1}{(t+1)^2(1+\sqrt{t^3})}$ and $f(t, x) = \frac{x}{(1+6\sqrt[4]{x})^4} + \frac{1}{(t+1)^3}$.

In the sequel, we will show that Problem (2) cannot be treated by Theorem 4.

In fact, we will prove that assumption (iii) of Theorem 4 is not satisfied.

In fact, for $r \in (0, 1)$ fixed, we have that

$$\frac{f(t, rx)}{f(t, x)} = \frac{\frac{rx}{(1+6\sqrt[4]{rx})^4} + \frac{1}{(t+1)^3}}{\frac{x}{(1+y\sqrt[4]{x})^4} + \frac{1}{(t+1)^3}}, \quad for \quad t, x \in [0, \infty).$$

When $t \to \infty$, we have that

$$\frac{f(t, rx)}{f(t, x)} \to \frac{r(1+6\sqrt[4]{x})^4}{(1+6\sqrt[4]{rx})^4} \geq r,$$

and when $x \to 0$ and $t \to \infty$

$$\frac{f(t, rx)}{f(t, x)} \to r.$$

From this, it follows that the condition $\frac{f(t, rx)}{f(t, x)} \geq \varphi(r) > r$ for certain $\varphi(r) \in (r, 1)$, appearing in assumption (iii) of Theorem 4 cannot be satisfied.

This proves that Problem (2) cannot be treated by Theorem 4.

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