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Fekete-Szegö inequality for classes of (p, q)-Starlike and (p, q)-convex functions

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Abstract
In this paper, the new generalized classes of (p, q)-starlike and (p, q)-convex functions are introduced by using the (p, q)-derivative operator. Also, the (p, q)-Bernardi integral operator for analytic function is defined in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Our aim for these classes is to investigate the Fekete-Szegö inequalities. Moreover, Some special cases of the established results are discussed. Further, certain applications of the main results are obtained by applying the (p, q)-Bernardi integral operator.

Keywords (p, q)-starlike functions · (p, q)-convex functions · Fekete-Szegö inequality · (p, q)-Bernardi integral operator

Mathematics Subject Classification 30C45 · 30C50

1 Introduction

The q-analysis is a generalization of the ordinary analysis without using the limit notation. The first application and usage of the q-calculus was introduced by Jackson in [11] and [12]. Moreover, several applications in various fields of Mathematics and Physics (see for details [22,26]). Recently, there is an extension of q-calculus, denoted by (p, q)-calculus which is obtained by substituting q by q/p in q-calculus. The (p, q)-integer was considered by Chakrabarti and Jagannathan in [5]. There are further results to this also in [2,3,20] . The two important geometric properties of analytic functions are starlikeness and convexity. We have seen many publications in Geometric Function Theory by using the q-differential operator. A generalization of starlike functions $\mathcal{S}^*$ was investigated by Ismail et al. in [10]. Furthermore, close-to-convexity of certain families of q-Mittag-Leffler functions were studied in [27]. We have also seen the coefficient inequality of q-starlike functions discussed by [30]. More recently, coefficient estimates of q-starlike and q-convex functions were studied in [21]. There has also been a new subclasses of analytic functions associated with q-differential operators introduced and discussed in many works [1,9,16,17,23,24,30]. Motivated by an

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emerging idea of (p,q)-analysis as a generalization of q-analysis, in this paper, we extend the idea of q-starlikeness and q-convexity to (p,q)-starlikeness and (p,q)-convexity. From this, we will obtain the Fekete-Szegö inequalities for these classes, we also apply these results on the newly introduced (p,q)-Bernardi integral operator as given applications of our results here.

1.1 Background

We recall some basic notations and definitions from (p,q)-calculus, which are predominantly in this paper.

The (p,q)-derivative of the function f is defined as in [29]:

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z} \qquad (z \neq 0;\ 0 < q < p \leq 1)\,; \tag{1.1}$$

From Eq. 1.1, it is clear that if f and g are the two functions, then

$$D_{p,q}\left(f(z) + g(z)\right) = D_{p,q}f(z) + D_{p,q}g(z) \tag{1.2}$$

and

$$D_{p,q}\left(cf(z)\right) = cD_{p,q}f(z), \tag{1.3}$$

where c is constant.

We note that $D_{p,q}f(z) \longrightarrow f'(z)$ as $p = 1$ and $q \longrightarrow 1-$, where f' is the ordinary derivative of the function f.

In particular, using Eq. 1.1, the (p,q)-derivative of the function $h(z) = z^n$ is as follows:

$$D_{p,q}h(z) = [n]_{p,q}z^{n-1}, \tag{1.4}$$

where $[n]_{p,q}$ denotes the (p,q)-number and is given as:

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} \qquad (0 < q < p \leq 1)\,. \tag{1.5}$$

Since, we note that $[n]_{p,q} \longrightarrow n$ as $p = 1$ and $q \longrightarrow 1-$, therefore in view of Eq. 1.4, $D_{p,q}h(z) \longrightarrow h'(z)$ as $p = 1$ and $q \longrightarrow 1-$, where $h'(z)$ denotes the ordinary derivative of the function $h(z)$ with respect to z.

Also, the (p,q)-integral of the function f on $[0,z]$ is defined as in [14] :

$$\int_0^z f(t)d_{p,d}t = (p-q)z\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}z\right),$$

where $\left|\frac{q}{p}\right| < 1$ and $0 < q < p \leq 1$.

In particular, the (p,q)-integral of the function $h(z) = z^n$ is given by

$$\int_0^z h(t)d_{p,d}t = \frac{z^{n+1}}{[n+1]_{p,q}}, \tag{1.6}$$

where $n \neq -1$ and $[.]_{p,q}$ is given by Eq. 1.5.

Again, since $[n+1]_{p,q} \longrightarrow n+1$ as $p = 1$ and $q \longrightarrow 1-$, therefore for the same choices of p and q, Eq. 1.6 reduces to $\int_0^z h(t)dt = \frac{z^{n+1}}{n+1}$, which is the ordinary integral of the function $h(z)$ on $[0,z]$.

In this paper, we consider the class $\mathcal{A}$ consisting of functions of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.7}$$

and analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Also, using Eqs. 1.2, 1.3 and 1.4, we get the (p, q)-derivative of the function f, given by Eq. 1.7 as:

$$D_{p,q} f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1} \qquad (0 < q < p \leq 1) \tag{1.8}$$

where $[n]_{p,q}$ is given by Eq. 1.5.

For the analytic functions f and g in $\mathbb{U}$, we say that the function g is subordinate to f in $\mathbb{U}$ [18], and write

$$g(z) \prec f(z) \text{ or } g \prec f,$$

if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \text{ and } |w(z)| < 1,$$

such that

$$g(z) = f(w(z)) \qquad (z \in \mathbb{U}). \tag{1.9}$$

Ma-Minda defined the classes of starlike and convex functions, denoted by $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$, respectively, by using the subordination principle between certain analytic functions [15] These subclasses are defined as follows:

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\} \tag{1.10}$$

and

$$\mathcal{C}(\phi) = \left\{ f \in \mathcal{A} : \left(1 + \frac{zf''(z)}{f'(z)} \right) \prec \phi(z) \right\}, \tag{1.11}$$

where the function $\phi(z)$ is analytic in $\mathbb{U}$ with $\Re(\phi(z)) > 0$, $\phi(0) = 1$ and $\phi'(0) > 0$. It is clear that $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ are the subclasses of $\mathcal{A}$.

The classes of q-starlike and q-convex functions, denoted by $\mathcal{S}_q^*(\phi)$ and $\mathcal{C}_q(\phi)$, respectively, are defined by using the subordination principle as in [4]:

$$\mathcal{S}_q^*(\phi) = \left\{ f \in \mathcal{A} : z \frac{D_q f(z)}{f(z)} \prec \phi(z) \right\} \tag{1.12}$$

and

$$\mathcal{C}_q(\phi) = \left\{ f \in \mathcal{A} : \frac{D_q(zD_q f(z))}{D_q f(z)} \prec \phi(z) \right\}, \tag{1.13}$$

where the function $\phi(z)$ is analytic in $\mathbb{U}$ with $\Re(\phi(z)) > 0$, $\phi(0) = 1$ and $\phi'(0) > 0$. These classes are the subclasses of $\mathcal{A}$.

The Feteke-Szegö problem is to find the coefficients estimates for second and third coefficients of functions in any class of analytic function having a specified geometric property [7]. In this paper, we introduce the classes of (p, q)-starlike and (p, q)-convex functions by using the (p, q)-derivative in terms of the subordination principle. Also, we find the Fekete-Szegö inequalities which is obtained by the maximizing the absolute value of the coefficient $|a_3 - a_2^2|$

for the functions belonging to these classes, as in [6,8,13,25,28]. Furthermore, the (p,q)-Bernardi integral operator for analytic functions, is defined in the open unit disc $\mathbb{U}$ to discuss the application of the results established in this paper.

2 Main results

First, we define the classes of (p,q)-starlike functions and (p,q)-convex functions, denoted by $\mathcal{S}^*_{p,q}(\phi)$ and $\mathcal{C}_{p,q}(\phi)$, respectively, in terms of the subordination principle by taking the (p,q)-derivative in place of q-derivative in the respective definitions of the classes of q-starlike and q-convex functions.

The respective definitions of the classes $\mathcal{S}^*_{p,q}(\phi)$ and $\mathcal{C}_{p,q}(\phi)$ are as follows:

Definition 2.1 The function $f \in \mathcal{A}$ is said to be (p,q)-starlike if it satisfies the following subordination:

$$\frac{zD_{p,q}f(z)}{f(z)} \prec \phi(z) \qquad (0 < q < p \leq 1), \tag{2.1}$$

where the function $\phi(z)$ is analytic in $\mathbb{U}$ with $\Re(\phi(z)) > 0$, $\phi(0) = 1$ and $\phi'(0) > 0$.

Definition 2.2 The function $f \in \mathcal{A}$ is said to be (p,q)-convex if it satisfies the following subordination:

$$\frac{D_{p,q}\left(zD_{p,q}f(z)\right)}{D_{p,q}f(z)} \prec \phi(z) \qquad (0 < q < p \leq 1), \tag{2.2}$$

where the function $\phi(z)$ is analytic in $\mathbb{U}$ with $\Re(\phi(z)) > 0$, $\phi(0) = 1$ and $\phi'(0) > 0$ (Figs. 1, 2).

Remark 2.1 We note that, for $p = 1$ the classes $\mathcal{S}^*_{p,q}(\phi)$ and $\mathcal{C}_{p,q}(\phi)$, reduce to the classes $\mathcal{S}^*_q(\phi)$ and $\mathcal{C}_q(\phi)$, which are defined by Eqs. 1.12 and 1.13, respectively. Again, for $p = 1$ and $q \longrightarrow 1-$, the classes $\mathcal{S}^*_{p,q}(\phi)$ and $\mathcal{C}_{p,q}(\phi)$ reduce to the classes $\mathcal{S}^*(\phi)$, defined by Eq. 1.10 and $\mathcal{C}(\phi)$, defined by Eq. 1.11, respectively.

First of all, we need to mention the following lemma originally defined in [15]:

Lemma 2.1 *If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with $\Re(p(z)) > 0$ and $\mu \in \mathbb{C}$, then*

$$|c_2 - \mu c_1^2| \leq 2 \max\{1; |2\mu - 1|\}.$$

The result is sharp for giving two choices of the function $p(z)$ as follows:

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}.$$

Now, we investigate the Feteke-Szegö inequality of the class $\mathcal{S}^*_{p,q}(\phi)$ in the following result:

Theorem 2.1 *Let $\phi(z) = 1 + b_1 z + b_2 z^2 \cdots$, with $b_1 \neq 0$. If f, given by Eq. 1.7, belongs to the class $\mathcal{S}^*_{p,q}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{|b_1|}{[3]_{p,q} - 1} \max\left\{1; \left|\frac{b_2}{b_1} + \frac{b_1}{[2]_{p,q} - 1}\left(1 - \frac{[3]_{p,q} - 1}{[2]_{p,q} - 1}\right)\mu\right|\right\}, \tag{2.3}$$

where $b_1, b_2, \cdots \in \mathbb{R}$, $\mu \in \mathbb{C}$ and $0 < q < p \leq 1$. The result is sharp.

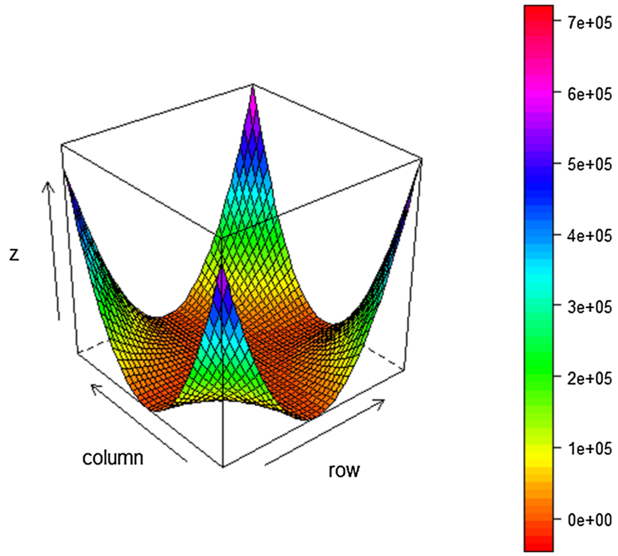

Fig. 1 The class $\mathcal{S}^*_{0.2,0.5}\left(\frac{1+z}{1-z}\right)$ for the complex number $z = x + iy, \quad x, y \in \mathbb{R}$

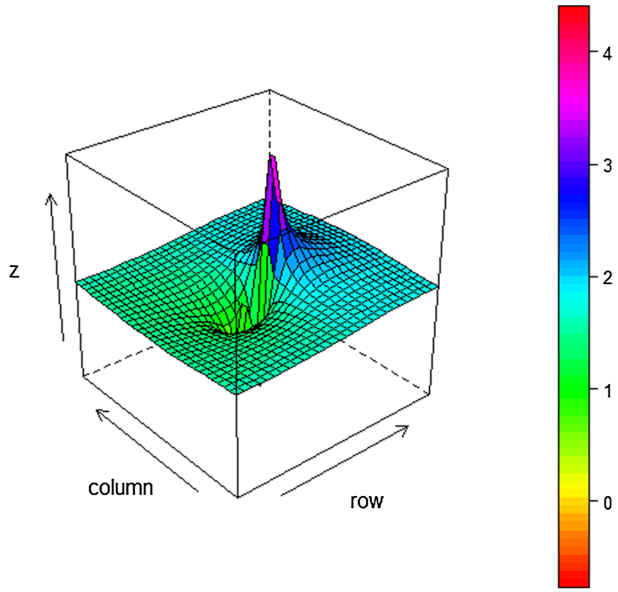

Fig. 2 The class $\mathcal{C}_{0.2,0.5}\left(\frac{1+z}{1-z}\right)$ for the complex number $z = x + iy, \quad x, y \in \mathbb{R}$

Proof Let $f \in \mathcal{S}^*_{p,q}(\phi)$, then in view of Definition 2.1, the function f satisfies the Subordination 2.1. Thus, by using Eq. 1.9, there is a Schwarz function w such that

$$\frac{zD_{p,q}f(z)}{f(z)} = \phi(w(z)). \tag{2.4}$$

We define the function

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{2.5}$$

in terms of the function $w(z)$ as :

$$p(z) = \frac{1 + w(z)}{1 - w(z)},$$

which gives

$$w(z) = \frac{p(z) - 1}{p(z) + 1}. \tag{2.6}$$

Using Eqs. 2.5 and 2.6, we get

$$\begin{aligned} \phi(w(z)) &= \phi\left(\frac{c_1 z + c_2 z^2 + \cdots}{2 + c_1 z + c_2 z^2 + \cdots}\right) \\ &= \phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{1}{2}c_1^2\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right]\right). \end{aligned} \tag{2.7}$$

Since $\phi(z) = 1 + b_1 z + b_2 z^2 \cdots$, therefore, Eq. 2.7 gives

$$\phi(w(z)) = 1 + \frac{b_1 c_1}{2} z + \left[\frac{b_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{b_2 c_1^2}{4}\right] z^2 + \cdots. \tag{2.8}$$

Now, using Eqs. 1.7 and 1.8, we get

$$\begin{aligned} \frac{z D_{p,q} f(z)}{f(z)} &= \frac{z + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^n}{z + \sum_{n=2}^{\infty} a_n z^n} = 1 + ([2]_{p,q} - 1) a_2 z \\ &\quad + \left(([3]_{p,q} - 1) a_3 - ([2]_{p,q} - 1) a_2^2\right) z^2 + \cdots. \end{aligned} \tag{2.9}$$

Using Eqs. 2.8 and 2.9 in Eq. 2.4, then comparing the coefficients of z and z^2 from the both sides of the resultant equation and simplifying, we get

$$a_2 = \frac{b_1 c_1}{2([2]_{p,q} - 1)} \tag{2.10}$$

and

$$a_3 = \frac{b_1}{2([3]_{p,q} - 1)}\left[c_2 - \frac{1}{2}\left(1 - \frac{b_2}{b_1} - \frac{b_1}{[2]_{p,q} - 1}\right)c_1^2\right]. \tag{2.11}$$

Next, for $\mu \in \mathbb{C}$, using Eqs. 2.10 and 2.11, we have

$$a_3 - \mu a_2^2 = \frac{b_1}{2([3]_{p,q} - 1)}\left[c_2 - \frac{1}{2}\left(1 - \frac{b_2}{b_1} - \frac{b_1}{[2]_{p,q} - 1}\left(1 - \frac{[3]_{p,q} - 1}{[2]_{p,q} - 1}\mu\right)\right)c_1^2\right]. \tag{2.12}$$

If we take

$$v = \frac{1}{2}\left(1 - \frac{b_2}{b_1} - \frac{b_1}{[2]_{p,q} - 1}\left(1 - \frac{[3]_{p,q} - 1}{[2]_{p,q} - 1}\mu\right)\right), \tag{2.13}$$

then, from Eq. 2.12, we get

$$|a_3 - \mu a_2^2| = \frac{|b_1|}{2([3]_{p,q} - 1)}|c_2 - v c_1^2|. \tag{2.14}$$

Hence, by applying Lemma 2.1, Eq. 2.14, gives the Feteke-Szegö inequality, given by Eq. 2.3, for the class $\mathcal{S}^*_{p,q}(\phi)$.

Further, our result is sharp. That is, the equality holds, when $p(z) = p_1(z) = \dfrac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots$ and Eq. 2.4, gives

$$\frac{zD_{p,q}f(z)}{f(z)} = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = \phi(z) = 1 + b_1z + b_2z^2\cdots. \tag{2.15}$$

Then, by comparing Eqs. 2.8 and 2.15, we have $c_1 = 2$ and $c_2 = 2$, then Eq. 2.12 gives the equality sign in the place of inequality in Assertion 2.3.

Similarly, for $p(z) = p_2(z) = \dfrac{1+z^2}{1-z^2} = 1 + 2z^2 + \cdots$, Eq. 2.4 gives

$$\frac{zD_{p,q}f(z)}{f(z)} = \phi\left(\frac{p_2(z)-1}{p_2(z)+1}\right) = \phi(z^2) = 1 + b_1z^2 + \cdots. \tag{2.16}$$

Then, by comparing Eqs. 2.8 and 2.16, we have $c_1 = 0$ and $c_2 = 2$ and hence Eq.2.12 gives the equality sign in the place of inequality in Assertion 2.3. □

Taking $p = 1$ and $q \longrightarrow 1-$ in Theorem 2.1, we get the following corollary originally shown in [4]:

Corollary 2.1 *Let $\phi(z) = 1 + b_1z + b_2z^2\cdots$, with $b_1 \neq 0$. If f given by Eq. 1.7 belongs to the class $\mathcal{S}^*(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{|b_1|}{2}\max\left\{1; \left|\frac{b_2}{b_1} + b_1(1-2\mu)\right|\right\},$$

where $b_1, b_2, \cdots \in \mathbb{R}$ and $\mu \in \mathbb{C}$. The result is sharp.

Remark 2.2 For $p = 1$, Inequality 2.3, gives the Feteke-Szegö inequality from [4] for the class $\mathcal{S}^*_q(\phi)$.

Next, we investigate the Feteke-Szegö inequality for the class $\mathcal{C}_{p,q}(\phi)$ in the following result:

Theorem 2.2 *Let $\phi(z) = 1 + b_1z + b_2z^2\ldots$ with $b_1 \neq 0$. If f, given by Eq. 1.7, belongs to the class $\mathcal{C}_{p,q}(\phi)$, then*

$$|a_3-\mu a_2^2|\le\frac{|b_1|}{[3]_{p,q}([3]_{p,q}-1)}\max\left\{1;\left|\frac{b_2}{b_1}+\frac{b_1}{[2]_{p,q}-1}\left(1-\frac{[3]_{p,q}([3]_{p,q}-1)}{[2]_{p,q}^2([2]_{p,q}-1)}\right)\mu\right|\right\},\tag{2.17}$$

where $b_1, b_2, \ldots \in \mathbb{R}$, $\mu \in \mathbb{C}$ *and* $0 < q < p \le 1$. *The result is sharp.*

Proof Let $f \in \mathcal{C}_{p,q}(\phi)$, then in view of Definition 2.2 the function f satisfies the Subordination 2.2, thus, by using Eq. 1.9, there exists a Schwarz function w such that

$$\frac{D_{p,q}(zD_{p,q}f(z))}{D_{p,q}f(z)}=\phi(w(z)),\tag{2.18}$$

where w is given by Eq. 2.6 and $\phi(w(z))$ is given by Eq. 2.8.

Using Eqs. 1.7 and 1.8, we obtain

$$\begin{aligned}\frac{D_{p,q}(zD_{p,q}f(z))}{D_{p,q}f(z)}&=\frac{z+\sum_{n=2}^{\infty}[n]_{p,q}^2a_nz^n}{z+\sum_{n=2}^{\infty}[n]_{p,q}a_nz^n}\\&=1+[2]_{p,q}([2]_{p,q}-1)a_2z+\left([3]_{p,q}([3]_{p,q}-1)a_3-[2]_{p,q}^2([2]_{p,q}-1)a_2^2\right)z^2+\cdots.\end{aligned}\tag{2.19}$$

Comparing the coefficients of z and z^2 in Eqs. 2.8 and 2.19 and simplifying them, we obtain

$$a_2=\frac{b_1c_1}{2[2]_{p,q}([2]_{p,q}-1)}\tag{2.20}$$

and

$$a_3=\frac{b_1}{2[3]_{p,q}([3]_{p,q}-1)}\left[c_2-\frac{1}{2}\left(1-\frac{b_2}{b_1}-\frac{b_1}{[2]_{p,q}-1}\right)c_1^2\right].\tag{2.21}$$

Next, for $\mu \in \mathbb{C}$, Eqs. 2.20 and 2.21, gives

$$\begin{aligned}a_3-\mu a_2^2=\frac{b_1}{2[3]_{p,q}([3]_{p,q}-1)}&\left[c_2-\frac{1}{2}\left(1-\frac{b_2}{b_1}\right.\right.\\&\left.\left.-\frac{b_1}{[2]_{p,q}-1}\left(1-\frac{[3]_{p,q}([3]_{p,q}-1)}{[2]_{p,q}^2([2]_{p,q}-1)}\mu\right)\right)c_1^2\right].\end{aligned}\tag{2.22}$$

If we take

$$v=\frac{1}{2}\left(1-\frac{b_2}{b_1}-\frac{b_1}{[2]_{p,q}-1}\left(1-\frac{[3]_{p,q}([3]_{p,q}-1)}{[2]_{p,q}^2([2]_{p,q}-1)}\mu\right)\right),\tag{2.23}$$

then using Eqs. 2.22 and 2.23, we get

$$|a_3-\mu a_2^2|=\frac{|b_1|}{2[3]_{p,q}([3]_{p,q}-1)}|c_2-vc_1^2|.\tag{2.24}$$

Now, by applying Lemma 2.1, Eq. 2.24 gives the Feteke-Szegö inequality, given by Eq. 2.17 for the class $\mathcal{C}_{p,q}(\phi)$.

Further, our result is sharp, when $p(z)=p_1(z)=\frac{1+z}{1-z}=1+2z+2z^2+\cdots$ and Eq. 2.18, gives

$$\frac{D_{p,q}(zD_{p,q}f(z))}{D_{p,q}f(z)}=\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)=\phi(z)=1+b_1z+b_2z^2\ldots.\tag{2.25}$$

Then, by comparing Eqs. 2.8 and 2.25, we have $c_1 = 2$ and $c_2 = 2$ and hence Eq. 2.22 gives the equality sign in the place of inequality in Assertion 2.17.

Similarly, when $p(z) = p_2(z) = \dfrac{1+z^2}{1-z^2} = 1 + 2z^2 + \cdots$, Eq. 2.18 gives

$$\frac{D_{p,q}(zD_{p,q}f(z))}{D_{p,q}f(z)} = \phi\left(\frac{p_2(z)-1}{p_2(z)+1}\right) = \phi(z^2) = 1 + b_1z^2 + \cdots, \tag{2.26}$$

then, by comparing Eqs. 2.26 and 2.8, we have $c_1 = 0$ and $c_2 = 2$ and hence Eq. 2.22 gives the equality sign in the place of inequality in Assertion 2.17. □

Taking $p = 1$ and $q \longrightarrow 1-$ in Theorem 2.2, we get the following corollary [4]:

Corollary 2.2 *Let $\phi(z) = 1 + b_1z + b_2z^2 \ldots$, with $b_1 \neq 0$. If f given by Eq. 1.7) belongs to the class $\mathcal{C}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{|b_1|}{6} \max\left\{1; \left|\frac{b_2}{b_1} + b_1\left(1 - \frac{3}{2}\mu\right)\right|\right\}, \tag{2.27}$$

where $b_1, b_2, \cdots \in \mathbb{R}$ and $\mu \in \mathbb{C}$. The result is sharp.

Remark 2.3 For $p = 1$, Inequality 2.17 gives the Feteke-Szegö inequality for the class $\mathcal{C}_q(\phi)$ from [4].

In the next section, we discuss the coefficient bounds of the first and third coefficients of the functions belonging to the classes $\mathcal{S}^*_{p,q}(\phi)$ and $\mathcal{C}_{p,q}(\phi)$.

3 Coefficient bounds

In this section, we estimate the coefficient bounds for the coefficients of z and z^2 of (p, q)-starlike and (p, q)-convex functions.

First, we need to mention the following lemma originally given in [15]:

Lemma 3.1 *If $p(z) = 1 + c_1z + c_2z^2 + \ldots$ is a function with $\Re(p(z)) > 0$, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2, & if \quad v \leq 0; \\ 2, & if \quad 0 \leq v \leq 1; \\ 4v - 2, & if \quad v \geq 1. \end{cases} \tag{3.1}$$

Also, the above upper bound is sharp, and it can be improved as follows when $0 < v < 1$:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \quad \left(0 < v \leq \frac{1}{2}\right) \tag{3.2}$$

and

$$|c_2 - vc_1^2| + (1 - v)|c_1|^2 \leq 2 \quad \left(\frac{1}{2} \leq v < 1\right). \tag{3.3}$$

Now, we establish the following result for estimation of the coefficient bound for the functions belonging to the class $\mathcal{S}^*_{p,q}(\phi)$:

Theorem 3.1 *Let $\phi(z) = 1 + b_1 z + b_2 z^2 \ldots$ with $b_1 > 0$ and $b_2 \geq 0$. Let*

$$\sigma_1 = \frac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2(b_2 - b_1)}{([3]_{p,q} - 1)b_1^2}, \tag{3.4}$$

$$\sigma_2 = \frac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2(b_2 + b_1)}{([3]_{p,q} - 1)b_1^2}, \tag{3.5}$$

$$\sigma_3 = \frac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2 b_2}{([3]_{p,q} - 1)b_1^2}. \tag{3.6}$$

If f, given by Eq. 1.7, belongs to the class $\mathcal{S}^*_{p,q}(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{b_2}{[3]_{p,q} - 1} + \dfrac{b_1^2}{[2]_{p,q} - 1}\left(\dfrac{1}{[3]_{p,q} - 1} - \dfrac{\mu}{[2]_{p,q} - 1}\right), & if \quad \mu \leq \sigma_1; \\[2ex] \dfrac{b_1}{[3]_{p,q} - 1}, & if \quad \sigma_1 \leq \mu \leq \sigma_2; \\[2ex] \dfrac{-b_2}{[3]_{p,q} - 1} - \dfrac{b_1^2}{[2]_{p,q} - 1}\left(\dfrac{1}{[3]_{p,q} - 1} - \dfrac{\mu}{[2]_{p,q} - 1}\right), & if \quad \mu \geq \sigma_2. \end{cases} \tag{3.7}$$

Further, if $\sigma_1 < \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{([2]_{p,q} - 1)^2}{([3]_{p,q} - 1)b_1^2}\left[b_1 - b_2 - \frac{b_1^2}{[2]_{p,q} - 1}\left(1 - \frac{[3]_{p,q} - 1}{[2]_{p,q} - 1}\mu\right)|a_2|^2\right] \leq \frac{b_1}{[3]_{p,q} - 1} \tag{3.8}$$

and if $\sigma_3 \leq \mu < \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{([2]_{p,q} - 1)^2}{([3]_{p,q} - 1)b_1^2}\left[b_1 + b_2 + \frac{b_1^2}{[2]_{p,q} - 1}\left(1 - \frac{[3]_{p,q} - 1}{[2]_{p,q} - 1}\mu\right)|a_2|^2\right] \leq \frac{b_1}{[3]_{p,q} - 1}. \tag{3.9}$$

Proof For $v \leq 0$, Eq. (2.13) gives

$$\mu \leq \frac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2(b_2 - b_1)}{([3]_{p,q} - 1)b_1^2}.$$

Let $\dfrac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2(b_2 - b_1)}{([3]_{p,q} - 1)b_1^2} = \sigma_1$, then from the above relation, we have $\mu \leq \sigma_1$.

Let $p(z)$ be a function, given by Eq. 2.5, with $\Re\,(p(z)) > 0$ and $f(z)$, given by Eq. 1.7, be a member of the class $\mathcal{S}^*_{p,q}(\phi)$, then Eq. 2.14 holds. Thus using Lemma 3.1 for $v \leq 0$ in Eq. 2.14, we get

$$|a_3 - \mu a_2^2| \leq \frac{b_1}{2([3]_{p,q} - 1)}(-4v + 2),$$

which on making use of Eq. 2.13, gives

$$|a_3 - \mu a_2^2| \leq \frac{b_1}{[3]_{p,q} - 1}\left(\frac{b_2}{b_1} + \frac{b_1}{[2]_{p,q} - 1}\left(1 - \frac{[3]_{p,q} - 1}{[2]_{p,q} - 1}\mu\right)\right), \tag{3.10}$$

where $\mu \leq \sigma_1$.

Simplifying the right hand side of Inequality 3.10, we get the first inequality of Assertion 3.7.

Again, if we take $0 \leq v \leq 1$, then Eq. 2.13, gives

$$\sigma_1 \leq \mu \leq \frac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2(b_2 + b_1)}{([3]_{p,q} - 1)b_1^2},$$

where σ_1 is given by Eq. 3.4.

Let $\frac{\left([2]_{p,q} - 1\right) b_1^2 + ([2]_{p,q} - 1)^2(b_2 - b_1)}{([3]_{p,q} - 1)b_1^2} = \sigma_2$, then from the above relation, we have $\sigma_1 \leq \mu \leq \sigma_2$.

Now, using Lemma 3.1 for $0 \leq v \leq 1$ in Eq. 2.14, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{b_1}{[3]_{p,q} - 1},$$

which gives the second inequality of Assertion 3.7.

Next, if we take $v \geq 1$, then Eq. 2.13, gives that $\mu \geq \sigma_2$.

Now, using Lemma 3.1, for $v \geq 1$ in Eq. 2.14, we get

$$|a_3 - \mu a_2^2| \leq \frac{b_1}{2([3]_{p,q} - 1)}(4v - 2),$$

which on using Eq. 2.13, gives

$$|a_3 - \mu a_2^2| \leq \frac{b_1}{[3]_{p,q} - 1}\left(-\frac{b_2}{b_1} - \frac{b_1}{[2]_{p,q} - 1}\left(1 - \frac{[3]_{p,q} - 1}{[2]_{p,q} - 1}\mu\right)\right). \tag{3.11}$$

Inequality 3.11 gives the third inequality of Assertion 3.7.

Further, if $0 < v \leq \frac{1}{2}$, then using Eq. 2.13, we have

$$0 < \frac{1}{2}\left(1 - \frac{b_2}{b_1} - \frac{b_1}{[2]_{p,q} - 1}\left(1 - \frac{[3]_{p,q} - 1}{[2]_{p,q} - 1}\mu\right)\right) \leq \frac{1}{2},$$

which on simplifying, gives

$$\sigma_1 < \mu \leq \frac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2 b_2}{([3]_{p,q} - 1)b_1^2}, \tag{3.12}$$

where σ_1 is given by Eq. 3.4.

Let $\frac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2 b_2}{([3]_{p,q} - 1)b_1^2} = \sigma_3$, then from Relation 3.12, we have $\sigma_1 < \mu \leq \sigma_3$.

Now, using Eqs. 2.10 and 3.4, we get

$$\begin{aligned}|a_3 - \mu a_2^2| + (\mu - \sigma_1)|a_2|^2 &= |a_3 - \mu a_2^2| \\ &+ \left(\mu - \frac{([2]_{p,q} - 1)b_1^2 + ([2]_{p,q} - 1)^2(b_2 - b_1)}{([3]_{p,q} - 1)b_1^2}\right)\frac{b_1^2|c_1|^2}{4([2]_{p,q} - 1)^2},\end{aligned} \tag{3.13}$$

which on using Eq. 2.14, we get

$$|a_3-\mu a_2^2|+(\mu-\sigma_1)|a_2|^2=\frac{b_1}{2([3]_{p,q}-1)}\left(|c_2-vc_1^2|+\frac{1}{2}\left(1-\frac{b_2}{b_1}\right.\right.$$
$$\left.\left.-\frac{b_1}{[2]_{p,q}-1}\left(1-\frac{[3]_{p,q}-1}{[2]_{p,q}-1}\mu\right)\right)|c_1|^2\right). \tag{3.14}$$

Using Eq. 2.13 in equation Eq. 3.14, we obtain

$$|a_3-\mu a_2^2|+(\mu-\sigma_1)|a_2|^2=\frac{b_1}{[3]_{p,q}-1}\left(\frac{1}{2}\left(|c_2-vc_1^2|+v|c_1|^2\right)\right),$$

which in view of Inequality 3.2, gives

$$|a_3-\mu a_2^2|+(\mu-\sigma_1)|a_2|^2\le\frac{b_1}{[3]_{p,q}-1}. \tag{3.15}$$

Now, using inequality 3.15 in Eq. 3.13, we get

$$|a_3-\mu a_2^2|+\left(\mu-\frac{([2]_{p,q}-1)b_1^2+([2]_{p,q}-1)^2(b_2-b_1)}{([3]_{p,q}-1)b_1^2}\right)|a_2|^2\le\frac{b_1}{[3]_{p,q}-1},$$

where $\sigma_1<\mu\le\sigma_3$.

Simplifying the above inequality, we obtain the Assertion 3.8.

Similarly, if $\frac{1}{2}\le v<1$, then using Eq. 2.13, we get $\sigma_3\le\mu<\sigma_2$, where σ_2 and σ_3 are given by Eqs. 3.5 and 3.6, respectively.

Now, using Eqs. 2.10 and 3.5, we get

$$|a_3-\mu a_2^2|+(\sigma_2-\mu)|a_2|^2=|a_3-\mu a_2^2|$$
$$+\left(\frac{([2]_{p,q}-1)b_1^2+([2]_{p,q}-1)^2(b_2+b_1)}{([3]_{p,q}-1)b_1^2}-\mu\right)\frac{b_1^2|c_1|^2}{4([2]_{p,q}-1)^2}, \tag{3.16}$$

Using Eqs. 2.14 in 3.16, we obtain

$$|a_3-\mu a_2^2|+(\sigma_2-\mu)|a_2|^2=\frac{b_1}{2([3]_{p,q}-1)}\left(|c_2-vc_1^2|+\frac{1}{2}\left(1+\frac{b_2}{b_1}\right.\right.$$
$$\left.\left.+\frac{b_1}{[2]_{p,q}-1}\left(1-\frac{[3]_{p,q}-1}{[2]_{p,q}-1}\mu\right)\right)|c_1|^2\right), \tag{3.17}$$

which, on using Eq. 2.13 gives

$$|a_3-\mu a_2^2|+(\sigma_2-\mu)|a_2|^2=\frac{b_1}{[3]_{p,q}-1}\left(\frac{1}{2}\left(|c_2-vc_1^2|+(1-v)|c_1|^2\right)\right). \tag{3.18}$$

Now, since $\frac{1}{2}\le v<1$, therefore using Inequality 3.3 of Lemma 3.1, Eq. 3.18 gives

$$|a_3-\mu a_2^2|+(\sigma_2-\mu)|a_2|^2\le\frac{b_1}{[3]_{p,q}-1}. \tag{3.19}$$

Using Inequality 3.19 in Eq. 3.16, we get

$$|a_3-\mu a_2^2|+\left(\frac{([2]_{p,q}-1)b_1^2+([2]_{p,q}-1)^2(b_2+b_1)}{([3]_{p,q}-1)b_1^2}-\mu\right)|a_2|^2\le\frac{b_1}{[3]_{p,q}-1},$$

where $\sigma_3 \leq \mu < \sigma_2$.

Finally, on simplifying the above inequality, we obtain the Assertion 3.9. □

Taking $p = 1$ in Theorem 3.1, we get the following corollary for the class $\mathcal{S}_q^*(\phi)$:

Corollary 3.1 *Let $\phi(z) = 1 + b_1 z + b_2 z^2 \ldots$ with $b_1 > 0$ and $b_2 \geq 0$. Let*

$$\sigma_1 = \frac{([2]_q - 1)b_1^2 + ([2]_q - 1)^2(b_2 - b_1)}{([3]_q - 1)b_1^2}, \tag{3.20}$$

$$\sigma_2 = \frac{([2]_q - 1)b_1^2 + ([2]_q - 1)^2(b_2 + b_1)}{([3]_q - 1)b_1^2}, \tag{3.21}$$

$$\sigma_3 = \frac{([2]_q - 1)b_1^2 + ([2]_q - 1)^2 b_2}{([3]_q - 1)b_1^2}. \tag{3.22}$$

If f, given by Eq. 1.7, belongs to the class $S_q^*(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{b_2}{[3]_q - 1} + \frac{b_1^2}{[2]_q - 1}\left(\frac{1}{[3]_q - 1} - \frac{\mu}{[2]_q - 1}\right), & if \ \ \mu \leq \sigma_1; \\ \frac{b_1}{[3]_q - 1}, & if \ \ \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{-b_2}{[3]_q - 1} - \frac{b_1^2}{[2]_q - 1}\left(\frac{1}{[3]_q - 1} - \frac{\mu}{[2]_q - 1}\right), & if \ \ \mu \geq \sigma_2. \end{cases} \tag{3.23}$$

Further, if $\sigma_1 < \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{([2]_q - 1)^2}{([3]_q - 1)b_1^2}\left[b_1 - b_2 - \frac{b_1^2}{[2]_q - 1}\left(1 - \frac{[3]_q - 1}{[2]_q - 1}\mu\right)|a_2|^2\right] \leq \frac{b_1}{[3]_q - 1} \tag{3.24}$$

and if $\sigma_3 \leq \mu < \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{([2]_q - 1)^2}{([3]_q - 1)b_1^2}\left[b_1 + b_2 + \frac{b_1^2}{[2]_q - 1}\left(1 - \frac{[3]_q - 1}{[2]_q - 1}\mu\right)|a_2|^2\right] \leq \frac{b_1}{[3]_q - 1}. \tag{3.25}$$

Next, we obtain the coefficient bound for the functions belonging to the class $\mathcal{C}_{p,q}(\phi)$:

Theorem 3.2 *Let $\phi(z) = 1 + b_1 z + b_2 z^2 \ldots$ with $b_1 > 0$ and $b_2 \geq 0$. Let*

$$\rho_1 = \frac{[2]_{p,q}^2([2]_{p,q} - 1)b_1^2 + ([2]_{p,q}[2]_{p,q} - 1)^2(b_2 - b_1)}{[3]_{p,q}([3]_{p,q} - 1)b_1^2}, \tag{3.26}$$

$$\rho_2 = \frac{[2]_{p,q}^2([2]_{p,q} - 1)b_1^2 + ([2]_{p,q}[2]_{p,q} - 1)^2(b_2 + b_1)}{[3]_{p,q}([3]_{p,q} - 1)b_1^2}, \tag{3.27}$$

$$\rho_3 = \frac{[2]_{p,q}^2([2]_{p,q} - 1)b_1^2 + ([2]_{p,q}[2]_{p,q} - 1)^2 b_2}{[3]_{p,q}([3]_{p,q} - 1)b_1^2}. \tag{3.28}$$

If f, given by Eq. 1.7, belongs to the class $\mathcal{C}_{p,q}(\phi)$, then

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{b_2}{[3]_{p,q}([3]_{p,q}-1)}+\dfrac{b_1^2}{[2]_{p,q}-1}\\ \quad\left(\dfrac{1}{[3]_{p,q}([3]_{p,q}-1)}-\dfrac{\mu}{[2]_{p,q}^2([2]_{p,q}-1)}\right), & if\ \ \mu\le\rho_1;\\ \dfrac{b_1}{[3]_{p,q}([3]_{p,q}-1)}, & if\ \ \rho_1\le\mu\le\rho_2;\\ \dfrac{-b_2}{[3]_{p,q}([3]_{p,q}-1)}-\dfrac{b_1^2}{[2]_{p,q}-1}\\ \quad\left(\dfrac{1}{[3]_{p,q}([3]_{p,q}-1)}-\dfrac{\mu}{[2]_{p,q}^2([2]_{p,q}-1)}\right), & if\ \ \mu\ge\rho_2.\end{cases} \tag{3.29}$$

Further, if $\rho_1<\mu\le\rho_3$, then

$$|a_3-\mu a_2^2|+\frac{[2]_{p,q}^2([2]_{p,q}-1)^2}{[3]_{p,q}([3]_{p,q}-1)b_1^2}\left[b_1-b_2 1\right.$$
$$\left.-\frac{b_1^2}{[2]_{p,q}-1}\left(-\frac{[3]_{p,q}([3]_{p,q}-1)}{[2]_{p,q}^2([2]_{p,q}-1)}\mu\right)|a_2|^2\right]\le\frac{b_1}{[3]_{p,q}([3]_{p,q}-1)} \tag{3.30}$$

and if $\rho_3\le\mu<\rho_2$, then

$$|a_3-\mu a_2^2|+\frac{[2]_{p,q}^2([2]_{p,q}-1)^2}{[3]_{p,q}([3]_{p,q}-1)b_1^2}\left[b_1+b_2\ 1\right.$$
$$\left.-+\frac{b_1^2}{[2]_{p,q}-1}\left(\frac{[3]_{p,q}([3]_{p,q}-1)}{[2]_{p,q}^2([2]_{p,q}-1)}\mu\right)|a_2|^2\right]\le\frac{b_1}{[3]_{p,q}([3]_{p,q}-1)}. \tag{3.31}$$

Proof For $v\le 0$, Eq. 2.23 gives

$$\mu\le\frac{[2]_{p,q}^2([2]_{p,q}-1)b_1^2+([2]_{p,q}[2]_{p,q}-1)^2(b_2-b_1)}{[3]_{p,q}([3]_{p,q}-1)b_1^2}.$$

Let $\dfrac{[2]_{p,q}^2([2]_{p,q}-1)b_1^2+([2]_{p,q}[2]_{p,q}-1)^2(b_2-b_1)}{[3]_{p,q}([3]_{p,q}-1)b_1^2}=\rho_1$, then from the above relation we have $\mu\le\rho_1$.

Let $p(z)$ be a function given by Eq. 2.5 with $\Re(p(z))>0$ and $f(z)$, given by Eq. 1.7, be a member of the class $\mathcal{C}_{p,q}(\phi)$, from this Eq. 2.24 holds. Thus, using Lemma 3.1, for $v\le 0$, in Eq. 2.24, we get

$$|a_3-\mu a_2^2|\le\frac{b_1}{2[3]_{p,q}([3]_{p,q}-1)}(-4v+2),$$

which on using Eq. 2.23, gives

$$|a_3-\mu a_2^2|\le\frac{b_1}{[3]_{p,q}([3]_{p,q}-1)}\left(\frac{b_2}{b_1}+\frac{b_1}{[2]_{p,q}-1}\left(1-\frac{[3]_{p,q}([3]_{p,q}-1)}{[2]_{p,q}^2([2]_{p,q}-1)}\mu\right)\right), \tag{3.32}$$

where $\mu\le\rho_1$.

Inequality 3.32 gives the first inequality of Assertion 3.29.

Again, if we take $0 \leq v \leq 1$, then Eq. 2.23 gives

$$\rho_1 \leq \mu \leq \frac{[2]_{p,q}^2([2]_{p,q}-1)b_1^2+([2]_{p,q}[2]_{p,q}-1)^2(b_2+b_1)}{[3]_{p,q}([3]_{p,q}-1)b_1^2}.$$

Let $\frac{[2]_{p,q}^2([2]_{p,q}-1)b_1^2+([2]_{p,q}[2]_{p,q}-1)^2(b_2+b_1)}{[3]_{p,q}([3]_{p,q}-1)b_1^2} = \rho_2$, then $\rho_1 \leq \mu \leq \rho_2$, where ρ_1 is given by Eq. 3.26.

Now, using Lemma 3.1, for $0 \leq v \leq 1$, in Eq. 2.24, we get

$$|a_3-\mu a_2^2| \leq \frac{b_1}{[3]_{p,q}([3]_{p,q}-1)},$$

which gives the second inequality of Assertion 3.29.

Next, if we take $v \geq 1$, then Eq. 2.23 gives that $\mu \geq \rho_2$.

Now, using Lemma 3.1, for $v \geq 1$ in Eq. 2.24, we get

$$|a_3-\mu a_2^2| \leq \frac{|b_1|}{2[3]_{p,q}([3]_{p,q}-1)}(4v-2),$$

which on using Eq. 2.23 gives

$$|a_3-\mu a_2^2| \leq \frac{b_1}{[3]_{p,q}([3]_{p,q}-1)}\left(-\frac{b_2}{b_1}-\frac{b_1}{[2]_{p,q}-1}\left(1-\frac{[3]_{p,q}([3]_{p,q}-1)}{[2]_{p,q}^2([2]_{p,q}-1)}\mu\right)\right), \tag{3.33}$$

where $\mu \geq \rho_2$.

Simplifying the right hand side of Inequality 3.33, we get the third inequality of Assertion 3.29.

Further, if $0 < v \leq \frac{1}{2}$, then using Eq. 2.23, we have

$$0 < \frac{1}{2}\left(1-\frac{b_2}{b_1}-\frac{b_1}{[2]_{p,q}-1}\left(1-\frac{[3]_{p,q}([3]_{p,q}-1)}{[2]_{p,q}^2([2]_{p,q}-1)}\mu\right)\right) \leq \frac{1}{2},$$

which on simplifying, gives

$$\rho_1 < \mu \leq \frac{[2]_{p,q}^2([2]_{p,q}-1)b_1^2+([2]_{p,q}[2]_{p,q}-1)^2b_2}{[3]_{p,q}([3]_{p,q}-1)b_1^2}. \tag{3.34}$$

Let $\frac{[2]_{p,q}^2([2]_{p,q}-1)b_1^2+([2]_{p,q}[2]_{p,q}-1)^2b_2}{[3]_{p,q}([3]_{p,q}-1)b_1^2} = \rho_3$, then from Inequality 3.34, we have $\rho_1 < \mu \leq \rho_3$, where ρ_1 is given by Eq. 3.26.

Now, using Eqs. 2.20 and 3.26, we get

$$|a_3-\mu a_2^2|+(\mu-\rho_1)|a_2|^2 = |a_3-\mu a_2^2| + \left(\mu-\frac{[2]_{p,q}^2([2]_{p,q}-1)b_1^2+([2]_{p,q}[2]_{p,q}-1)^2(b_2-b_1)}{[3]_{p,q}([3]_{p,q}-1)b_1^2}\right)\frac{b_1^2|c_1|^2}{4[2]_{p,q}^2([2]_{p,q}-1)^2}, \tag{3.35}$$

which on using Eq. 2.24, we obtain

$$
\begin{aligned}
&|a_3-\mu a_2^2|+(\mu-\rho_1)|a_2|^2\\
&\quad=\frac{b_1}{2[3]_{p,q}([3]_{p,q}-1)}\left(|c_2-vc_1^2|+\frac{1}{2}\,(1\right.\\
&\quad\left.-\frac{b_2}{b_1}-\frac{b_1}{[2]_{p,q}-1}\left(1-\frac{[3]_{p,q}([3]_{p,q}-1)}{[2]_{p,q}^2([2]_{p,q}-1)}\mu\right)\Bigg)\,|c_1|^2\right),
\end{aligned}
\tag{3.36}
$$

Again, using Eqs. 2.23 in 3.36, we have

$$|a_3-\mu a_2^2|+(\mu-\rho_1)|a_2|^2=\frac{b_1}{[3]_{p,q}([3]_{p,q}-1)}\left(\frac{1}{2}(|c_2-vc_1^2|+v|c_1|^2)\right),$$

which in view of Inequality 3.2 gives

$$|a_3-\mu a_2^2|+(\mu-\rho_1)|a_2|^2\leq\frac{b_1}{[3]_{p,q}([3]_{p,q}-1)}. \tag{3.37}$$

Now, using Eq. 2.20 and Inequality 3.37 in Eq. 3.35, we get

$$
\begin{aligned}
&|a_3-\mu a_2^2|+\left(\mu-\frac{[2]_{p,q}^2([2]_{p,q}-1)b_1^2+([2]_{p,q}[2]_{p,q}-1)^2(b_2-b_1)}{[3]_{p,q}([3]_{p,q}-1)b_1^2}\right)|a_2|^2\\
&\leq\frac{b_1}{[3]_{p,q}([3]_{p,q}-1)}.
\end{aligned}
$$

Simplifying the above inequality, we obtain the Assertion 3.30.

Similarly, if $\frac{1}{2}\leq v<1$, then using Eq. 2.23, we get $\rho_3\leq\mu<\rho_2$.

Now, using Eqs. 2.20 and 3.27, we get

$$
\begin{aligned}
&|a_3-\mu a_2^2|+(\rho_2-\mu)|a_2|^2\\
&=|a_3-\mu a_2^2|+\left(\frac{[2]_{p,q}^2([2]_{p,q}-1)b_1^2+([2]_{p,q}[2]_{p,q}-1)^2(b_2+b_1)}{[3]_{p,q}([3]_{p,q}-1)b_1^2}-\mu\right)\\
&\frac{b_1^2|c_1|^2}{4[2]_{p,q}^2([2]_{p,q}-1)^2},
\end{aligned}
\tag{3.38}
$$

Using Eqs. 2.24 in 3.38 and then simplifying, we obtain

$$
\begin{aligned}
&|a_3-\mu a_2^2|+(\rho_2-\mu)|a_2|^2\\
&\quad=\frac{b_1}{2[3]_{p,q}([3]_{p,q}-1)}\left(|c_2-vc_1^2|+\frac{1}{2}\left(1+\frac{b_2}{b_1}\right.\right.\\
&\quad\left.\left.+\frac{b_1}{[2]_{p,q}-1}\left(1-\frac{[3]_{p,q}([3]_{p,q}-1)}{[2]_{p,q}^2([2]_{p,q}-1)}\mu\right)|c_1|^2\right)\right),
\end{aligned}
$$

which on using Eqs. 2.23, gives

$$|a_3-\mu a_2^2|+(\rho_2-\mu)|a_2|^2=\frac{|b_1|}{[3]_{p,q}([3]_{p,q}-1)}\left(\frac{1}{2}\left(|c_2-vc_1^2|+(1-v)|c_1|^2\right)\right). \tag{3.39}$$

Now, since $\frac{1}{2} \leq v < 1$, therefore using Inequality 3.3 of Lemma 3.1 in Eq. 3.39, we get

$$|a_3 - \mu a_2^2| + (\rho_2 - \mu)|a_2|^2 \leq \frac{b_1}{[3]_{p,q}([3]_{p,q} - 1)}. \tag{3.40}$$

Using Inequality 3.40 in Eq. 3.38, gives

$$|a_3 - \mu a_2^2| + \left(\frac{[2]_{p,q}^2([2]_{p,q} - 1)b_1^2 + ([2]_{p,q}[2]_{p,q} - 1)^2(b_2 + b_1)}{[3]_{p,q}([3]_{p,q} - 1)b_1^2} - \mu\right)|a_2|^2$$
$$\leq \frac{b_1}{[3]_{p,q}([3]_{p,q} - 1)},$$

where $\rho_3 \leq \mu < \rho_2$.

Finally, on simplifying the above inequality, we obtain Assertion 3.31. □

For $p = 1$, Theorem 2.2, gives the following corollary for the class $\mathcal{C}_q(\phi)$:

Corollary 3.2 *Let $\phi(z) = 1 + b_1 z + b_2 z^2 \ldots$ with $b_1 > 0$ and $b_2 \geq 0$. Let*

$$\rho_1 = \frac{[2]_q^2([2]_q - 1)b_1^2 + ([2]_q[2]_q - 1)^2(b_2 - b_1)}{[3]_q([3]_q - 1)b_1^2}, \tag{3.41}$$

$$\rho_2 = \frac{[2]_q^2([2]_q - 1)b_1^2 + ([2]_q[2]_q - 1)^2(b_2 + b_1)}{[3]_q([3]_q - 1)b_1^2}, \tag{3.42}$$

$$\rho_3 = \frac{[2]_q^2([2]_q - 1)b_1^2 + ([2]_q[2]_q - 1)^2 b_2}{[3]_{p,q}([3]_q - 1)b_1^2}. \tag{3.43}$$

If f, given by Eq. 1.7, belongs to the class $\mathcal{C}_q(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{b_2}{[3]_q([3]_q - 1)} + \frac{b_1^2}{[2]_q - 1} \left(\frac{1}{[3]_q([3]_q - 1)} - \frac{\mu}{[2]_q^2([2]_q - 1)}\right), & if \;\; \mu \leq \rho_1; \\ \frac{b_1}{[3]_q([3]_q - 1)}, & if \;\; \rho_1 \leq \mu \leq \rho_2; \\ \frac{-b_2}{[3]_q([3]_q - 1)} - \frac{b_1^2}{[2]_q - 1} \left(\frac{1}{[3]_q([3]_q - 1)} - \frac{\mu}{[2]_q^2([2]_q - 1)}\right), & if \;\; \mu \geq \rho_2. \end{cases} \tag{3.44}$$

Further, if $\rho_1 < \mu \leq \rho_3$, then

$$|a_3 - \mu a_2^2| + \frac{[2]_q^2([2]_q - 1)^2}{[3]_q([3]_q - 1)b_1^2}\left[b_1 - b_2 - \frac{b_1^2}{[2]_q - 1}\left(1 - \frac{[3]_q([3]_q - 1)}{[2]_q^2([2]_q - 1)}\mu\right)|a_2|^2\right]$$
$$\leq \frac{b_1}{[3]_q([3]_q - 1)}. \tag{3.45}$$

and if $\rho_3 \leq \mu < \rho_2$, *then*

$$|a_3 - \mu a_2^2| + \frac{[2]_q^2([2]_q - 1)^2}{[3]_q([3]_q - 1)b_1^2}\left[b_1 + b_2 + \frac{b_1^2}{[2]_q - 1}\left(1 - \frac{[3]_q([3]_q - 1)}{[2]_q^2([2]_q - 1)}\mu\right)|a_2|^2\right]$$
$$\leq \frac{b_1}{[3]_q([3]_q - 1)}. \tag{3.46}$$

In the next section, we discuss some applications of the results, established in Sects. 1 and 2.

4 Applications

We recall that the Bernardi integral operator $\mathcal{F}_c$ is given in [2] as:

$$\mathcal{F}_c(f(z)) = \frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt \qquad (f \in \mathcal{A},\ c > -1).$$

Now, in view of above equation, we introduce the (p, q)-Bernardi integral operator $\mathcal{L}(z)$ as:

$$\mathcal{L}(z) := \mathcal{F}_{c,p,q}(f(z)) = \frac{[1+c]_{p,q}}{z^\beta}\int_0^z t^{c-1}f(t)d_{p,q}t \qquad c = 0, 1, 2, 3, \ldots. \tag{4.1}$$

Let $f \in \mathcal{A}$, then using Eqs. 1.6 and 1.8, we obtain the following power series for the function $\mathcal{L}$ in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$:

$$\mathcal{L}(z) = z + \sum_{n=2}^{\infty}\frac{[1+c]_{p,q}}{[n+c]_{p,q}}a_n z^n \qquad (c = 1, 2, 3, \ldots;\ 0 < q < p \leq 1;\ f \in \mathcal{A}). \tag{4.2}$$

It is clear that $\mathcal{L}(z)$ is analytic in open disc $\mathbb{U}$.

We note that, by taking $p = 1$ in Eq. 4.1, we get q-Bernardi integral operator as originally prescribed in [19] .

Let

$$L_n = \frac{[1+c]_{p,q}}{[n+c]_{p,q}}, \qquad n \geq 1. \tag{4.3}$$

Now, applying Theorem 2.1 to the function $\mathcal{L}(z)$, defined by Eq. 4.2, we get the following application of the theorem itself:

Let $\phi(z) = 1 + b_1 z + b_2 z^2 \ldots$, with $b_1 \neq 0$. If $\mathcal{L}$, given by Eq. 4.2, belongs to the class $\mathcal{S}^*_{p,q}(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \frac{|b_1|}{[3]_{p,q}L_3 - 1}\max\left\{1; \left|\frac{b_2}{b_1} + \frac{b_1}{[2]_{p,q}L_2 - 1}\left(1 - \frac{[3]_{p,q}L_3 - 1}{[2]_{p,q}L_2 - 1}\right)\mu\right|\right\},$$

where L_2 and L_3 are given by Eq. 4.3, $b_1, b_2, \cdots \in \mathbb{R}$, $\mu \in \mathbb{C}$, $0 < q < p \leq 1$.

Next, applying the Theorem 2.2 to the function $\mathcal{L}(z)$, defined by Eq. 4.2, we get the following application of the theorem:

Let $\phi(z) = 1 + b_1 z + b_2 z^2 \ldots$, with $b_1 \neq 0$. If $\mathcal{L}$, given by Eq. 4.2, belongs to the class $\mathcal{C}_{p,q}(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \frac{|b_1|}{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)}\max\left\{1; \left|\frac{b_2}{b_1}\right.\right.$$
$$\left.\left. + \frac{b_1}{[2]_{p,q} - 1}L_2\left(1 - \frac{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)}{[2]_{p,q}^2 L_2([2]_{p,q}L_2 - 1)}\right)\mu\right|\right\},$$

where L_2 and L_3 are given by Eq. 4.3, $b_1, b_2, \ldots \in \mathbb{R}$, $\mu \in \mathbb{C}$, $0 < q < p \leq 1$.

Further, applying the Theorem 3.1 to the function $\mathcal{L}(z)$, defined by Eq. 4.2, we get the following application of the theorem:

Let $\phi(z) = 1 + b_1 z + b_2 z^2 \ldots$ with $b_1 > 0$ and $b_2 \geq 0$. Let

$$\sigma_1 = \frac{([2]_{p,q}L_2 - 1)b_1^2 + ([2]_{p,q}L_2 - 1)^2(b_2 - b_1)}{([3]_{p,q}L_3 - 1)b_1^2},$$
$$\sigma_2 = \frac{([2]_{p,q}L_2 - 1)b_1^2 + ([2]_{p,q}L_2 - 1)^2(b_2 + b_1)}{([3]_{p,q}L_3 - 1)b_1^2},$$
$$\sigma_3 = \frac{([2]_{p,q}L_2 - 1)b_1^2 + ([2]_{p,q}L_2 - 1)^2 b_2}{([3]_{p,q}L_3 - 1)b_1^2}.$$

If $\mathcal{L}$, given by Eq. 4.2, belongs to the class $\mathcal{S}^*_{p,q}(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{b_2}{[3]_{p,q} - 1} + \dfrac{b_1^2}{[2]_{p,q} - 1}\left(\dfrac{1}{[3]_{p,q} - 1} - \dfrac{\mu}{[2]_{p,q} - 1}\right), & if \;\; \mu \leq \sigma_1; \\[2ex] \dfrac{b_1}{[3]_{p,q} - 1}, & if \;\; \sigma_1 \leq \mu \leq \sigma_2; \\[2ex] \dfrac{-b_2}{[3]_{p,q} - 1} - \dfrac{b_1^2}{[2]_{p,q} - 1}\left(\dfrac{1}{[3]_{p,q} - 1} - \dfrac{\mu}{[2]_{p,q} - 1}\right), & if \;\; \mu \geq \sigma_2. \end{cases} \tag{4.4}$$

Further, if $\sigma_1 < \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{([2]_{p,q}L_2 - 1)^2}{([3]_{p,q}L_3 - 1)b_1^2}\left[b_1 - b_2 - \frac{b_1^2}{[2]_{p,q}L_2 - 1}\left(1 - \frac{[3]_{p,q}L_3 - 1}{[2]_{p,q}L_2 - 1}\mu\right)|a_2|^2\right]$$
$$\leq \frac{b_1}{[3]_{p,q}L_3 - 1}.$$

and if $\sigma_3 \leq \mu < \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{([2]_{p,q}L_2 - 1)^2}{([3]_{p,q}L_3 - 1)b_1^2}\left[b_1 + b_2 + \frac{b_1^2}{[2]_{p,q}L_2 - 1}\left(1 - \frac{[3]_{p,q}L_3 - 1}{[2]_{p,q}L_2 - 1}\mu\right)|a_2|^2\right]$$
$$\leq \frac{b_1}{[3]_{p,q}L_3 - 1},$$

where L_2 and L_3 are given by Eq. 4.3.

Finally, applying Theorem 3.1 to the function $\mathcal{L}(z)$, defined by Eq. 4.2, we get the following application of the theorem:

Let $\phi(z) = 1 + b_1 z + b_2 z^2 \dots$ with $b_1 > 0$ and $b_2 \geq 0$. Let

$$\rho_1 = \frac{[2]_{p,q}^2([2]_{p,q}L_2 - 1)b_1^2 + ([2]_{p,q}L_2[2]_{p,q}L_2 - 1)^2(b_2 - b_1)}{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)b_1^2},$$

$$\rho_2 = \frac{[2]_{p,q}^2 L_2([2]_{p,q}L_2 - 1)b_1^2 + [2]_{p,q}^2 L_2([2]_{p,q}L_2 - 1)^2(b_2 + b_1)}{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)b_1^2},$$

$$\rho_3 = \frac{[2]_{p,q}^2 L_2([2]_{p,q}L_2 - 1)b_1^2 + [2]_{p,q}^2 L_2([2]_{p,q}L_2 - 1)^2 b_2}{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)b_1^2}.$$

If $\mathcal{L}$, given by Eq. 4.2, belongs to the class $\mathcal{C}_{p,q}(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{b_2}{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)} + \frac{b_1^2}{[2]_{p,q}L_2 - 1} \left(\frac{1}{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)} - \frac{\mu}{[2]_{p,q}^2 L_2([2]_{p,q}L_2 - 1)} \right), & if \quad \mu \leq \rho_1; \\ \frac{b_1}{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)}, & if \quad \rho_1 \leq \mu \leq \rho_2; \\ \frac{-b_2}{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)} - \frac{b_1^2}{[2]_{p,q}L_2 - 1} \left(\frac{1}{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)} - \frac{\mu}{[2]_{p,q}^2 L_2([2]_{p,q}L_2 - 1)} \right), & if \quad \mu \geq \rho_2. \end{cases} \tag{4.5}$$

Further, if $\rho_1 < \mu \leq \rho_3$, then

$$|a_3 - \mu a_2^2| + \frac{[2]_{p,q}^2 L_2([2]_{p,q}L_2 - 1)^2}{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)b_1^2} \left[b_1 - b_2 - \frac{b_1^2}{[2]_{p,q}L_2 - 1} \left(1 - \frac{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)}{[2]_{p,q}^2 L_2([2]_{p,q}L_2 - 1)} \mu \right) |a_2|^2 \right] \leq \frac{b_1}{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)}.$$

and if $\rho_3 \leq \mu < \rho_2$, then

$$|a_3 - \mu a_2^2| + \frac{[2]_{p,q}^2 L_2([2]_{p,q}L_2 - 1)^2}{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)b_1^2} \left[b_1 + b_2 + \frac{b_1^2}{[2]_{p,q}L_2 - 1} \left(1 - \frac{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)}{[2]_{p,q}^2 L_2([2]_{p,q}L_2 - 1)} \mu \right) |a_2|^2 \right] \leq \frac{b_1}{[3]_{p,q}L_3([3]_{p,q}L_3 - 1)},$$

where L_2 and L_3 are given by Eq. 4.3.

5 Conclusion

In our results, by using the (p, q)-derivative operator, the generalized classes of (p, q)-starlike and (p, q)-convex functions were introduced which are a generalization of the known starlike and convex functions, respectively. Moreover, the Fekete-Szegö inequalities of the analytic function belonging to these introduced classes were investigated. We also defined the (p, q)-Bernardi integral operator for analytic functions in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, the validity of our results can be applicable for the (p, q)-Bernardi integral operator we introduce here in this paper. There are some special cases of the results that we were also able to show. Lastly, certain applications of the main results for the (p, q)-starlike and (p, q)-convex functions were obtained by applying the (p, q)-Bernardi integral operator.

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