RACSAM (2019) 113:3115–3124
https://doi.org/10.1007/s13398-019-00680-x

ORIGINAL PAPER

Simpson type integral inequalities for generalized fractional integral

Fatma Ertuğral[1] · Mehmet Zeki Sarikaya[1]

Received: 27 September 2018 / Accepted: 22 April 2019 / Published online: 2 May 2019

Abstract
In this paper, we have established some generalized Simpson type integral inequalities for generalized fractional integral. The results presented here would provide some fractional inequalities involving k-fractional integral and Riemann–Liouville type fractional operators.

Keywords Simpson type inequalities · Convex functions · Integral inequalities

Mathematics Subject Classification 26D07 · 26D10 · 26D15 · 26A33

1 Introduction

The following inequality is well known in the literature as Simpson's inequality.

Theorem 1 *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty=\sup\left|f^{(4)}(x)\right|<\infty$. Then, the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see [1–8], [10–14], [16–24].

In [3], Dragomir et. al. proved the following some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

✉ Fatma Ertuğral
fatmaertugral14@gmail.com

Mehmet Zeki Sarikaya
sarikayamz@gmail.com

1 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

Theorem 2 *Suppose $f:[a;b]\to\mathbb{R}$ is an absolutely continuous mapping on $[a,b]$ whose derivative belongs to $L_p[a,b]$. Then, the following inequality holds,*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\|f'\right\|_p$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In [16], Sarikaya et. al. obtained inequalities for differentiable convex mappings which are connected with Simpson's inequality, and they used the following lemma to prove it.

Lemma 1 *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I° such that $f'\in L_1[a,b]$, where $a,b\in I^\circ$ with $a<b$, then the following equality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx$$
$$=\frac{b-a}{2}\int_0^1\left[\left(\frac{t}{2}-\frac{1}{3}\right)f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right.$$
$$\left.+\left(\frac{1}{3}-\frac{t}{2}\right)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt. \tag{1.1}$$

The main inequality in [16], pointed out for $s=1$, as follows:

Theorem 3 *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L_1[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $\left|f'\right|^q$ is a convex on $[a,b]$, $q>1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{(b-a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{3\left|f'(b)\right|^q+\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|f'(b)\right|^q+3\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Also, the following inequality was obtained by using the following identity which is given by Chen et. al in [2].

Lemma 2 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I° where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\begin{aligned} &\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}+f(b)\right)\right] \\ &\quad -\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{b-}f\left(\frac{a+b}{2}\right)+J^\alpha_{a+}f\left(\frac{a+b}{2}\right)\right] \\ &= \frac{b-a}{2}\left[\int_0^1\left(\frac{t^\alpha}{2}-\frac{1}{3}\right)f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt\right. \\ &\quad \left.+\int_0^1\left(\frac{1}{3}-\frac{t^\alpha}{2}\right)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt.\right] \end{aligned} \tag{1.2}$$

The aim of this paper is to establish new Simpson's type inequalities for the class of functions whose derivatives in absolute value at certain powers are convex functions via generalized fractional integral operators.

2 New generalized fractional integral operators

In this section we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [15].

Let's define a function $\varphi : [0,\infty) \to [0,\infty)$ satisfiying the following conditions :

$$\int_0^1 \frac{\varphi(t)}{t}dt < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{a^+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t}f(t)dt, \quad x > a, \tag{2.1}$$

$$_{b^-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x}f(t)dt, \quad x < b. \tag{2.2}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (2.1) and (2.2) are mentioned below.

(i) If we take $\varphi(t) = t$, the operator (2.1) and (2.2) reduce to the Riemann integral as follows:

$$I_{a^+}f(x) = \int_a^x f(t)dt, \quad x > a,$$

$$I_{b^-}f(x) = \int_x^b f(t)dt, \quad x < b.$$

(ii) If we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, the operator (2.1) and (2.2) reduce to the Riemann-Liouville fractional integral as follows:

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$
$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b.$$

(iii) If we take $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)} t^{\frac{\alpha}{k}}$, the operator (2.1) and (2.2) reduce to the k-Riemann-Liouville fractional integral as follows:

$$I_{a^+,k}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a,$$
$$I_{b^-,k}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b$$

where

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \quad \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0; k > 0$$

are given by Mubeen and Habibullah in [9].

3 Main results

Throughout this study, for brevity, we define

$$\Lambda(y) = \int_0^y \frac{\varphi\left(\frac{(b-a)}{2}u\right)}{u} du < \infty, \ \Delta(y) = \int_y^1 \frac{\varphi\left(\frac{(b-a)}{2}u\right)}{u} du < \infty.$$

In this section, using generalized fractional integral operators, we begin by the following theorem:

Lemma 3 *Let $f : I \to \mathbb{R}$ be an absoltely continuous mapping on I° such that $f' \in L_1([a, b])$, where $a, b \in I^\circ$ with $a < b$. Then the following equality holds:*

$$\begin{aligned}
&\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \\
&\quad - \frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi f\left(\frac{a+b}{2}\right) + {}_{b^-}I_\varphi f\left(\frac{a+b}{2}\right)\right] \\
&= \frac{b-a}{2\Lambda(1)} \int_0^1 \left(\frac{\Lambda(t)}{2} - \frac{\Lambda(1)}{3}\right) f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \\
&\quad + \frac{b-a}{2\Lambda(1)} \int_0^1 \left(\frac{\Lambda(1)}{3} - \frac{\Lambda(t)}{2}\right) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt. \qquad (3.1)
\end{aligned}$$

Proof It suffices to note that

$$\begin{aligned} I &= \int_0^1 \left(\frac{\Lambda(t)}{2} - \frac{\Lambda(1)}{3} \right) f' \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) dt \\ &\quad + \int_0^1 \left(\frac{\Lambda(1)}{3} - \frac{\Lambda(t)}{2} \right) f' \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) dt \\ &= I_1 + I_2. \end{aligned} \tag{3.2}$$

Integrating by parts, we obtain

$$\begin{aligned} I_1 &= \int_0^1 \left(\frac{\Lambda(t)}{2} - \frac{\Lambda(1)}{3} \right) f' \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) dt \\ &= \left(\frac{\Lambda(t)}{2} - \frac{\Lambda(1)}{3} \right) f \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) \frac{2}{b-a} \Big|_0^1 \\ &\quad - \frac{1}{b-a} \int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) dt \\ &= \frac{2}{b-a} \left[\frac{\Lambda(1)}{6} f(b) + \frac{\Lambda(1)}{3} f\left(\frac{a+b}{2}\right) \right] \\ &\quad - \frac{1}{b-a} \int_{\frac{a+b}{2}}^b \frac{\varphi\left(x - \frac{a+b}{2}\right)}{x - \frac{a+b}{2}} f(x)\, dx \\ &= \frac{\Lambda(1)}{6(b-a)} \left[2f(b) + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \left({}_{b^-}I_\varphi \right) f\left(\frac{a+b}{2}\right), \end{aligned} \tag{3.3}$$

and similarly we get,

$$\begin{aligned} I_2 &= \int_0^1 \left(\frac{\Lambda(1)}{3} - \frac{\Lambda(t)}{2} \right) f' \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) dt \\ &= \frac{\Lambda(1)}{6(b-a)} \left[2f(a) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \left({}_{a^+}I_\varphi \right) f\left(\frac{a+b}{2}\right). \end{aligned} \tag{3.4}$$

By adding Eq. (3.3) and (3.4), we have

$$\begin{aligned} \frac{b-a}{2\Lambda(1)}(I_1 + I_2) = \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{2\Lambda(1)} \Bigg[{}_{a^+}I_\varphi f\left(\frac{a+b}{2}\right) \\ + {}_{b^-}I_\varphi f\left(\frac{a+b}{2}\right) \Bigg] \end{aligned}$$

that is desired result. □

Remark 1 In Lemma 3 if we take $\varphi(t) = t$, then we have obtain the identity (1.1).

Remark 2 In Lemma 3, if we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then we obtain the equality (1.2).

Corollary 1 *Under assumption of Lemma 3 with* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *we have the following equality*

$$\begin{aligned}
&\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\
&\quad-\frac{2^{1-\frac{\alpha}{k}}(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\left[I^{\alpha}_{a^+,k}\,f\left(\frac{a+b}{2}\right)-I^{\alpha}_{b^-,k}\,f\left(\frac{a+b}{2}\right)\right] \\
&=\frac{b-a}{2}\int_0^1\left[\left(\frac{t^{\frac{\alpha}{k}}}{2}-\frac{1}{3}\right)f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right. \\
&\quad\left.+\left(\frac{1}{3}-\frac{t^{\frac{\alpha}{k}}}{2}\right)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.
\end{aligned}$$

Theorem 4 *Let* $f: I=[a,b]\subset R\to R$ *be an absolutely continuous mapping on* I° *such that* $f'\in L_1([a,b])$, *where* $a,b\in I^{\circ}$ *with* $a<b$. *If the mapping* $|f'|$ *is convex on* $[a,b]$, *then we have the following inequality*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right. \\
&\quad\left.-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b^-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]\right| \\
&\le\frac{(b-a)}{2\Lambda(1)}K(t)\left[|f'(a)|+|f'(b)|\right]
\end{aligned}\tag{3.5}$$

where

$$K(t)=\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|dt. \tag{3.6}$$

Proof From Lemma 3 and $|f'|$ is convex on $[a,b]$, we get

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_{\varphi}f\left(\frac{a+b}{2}\right)+{}_{b^-}I_{\varphi}f\left(\frac{a+b}{2}\right)\right]\right| \\
&\le\frac{b-a}{2\Lambda(1)}\int_0^1\left[\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|\right. \\
&\quad\left.+\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt \\
&\le\frac{b-a}{2\Lambda(1)}\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left(\frac{1-t}{2}f\left|'(a)\right|+\frac{1+t}{2}|f'(b)|\right)dt \\
&\quad+\frac{b-a}{2\Lambda(1)}\int_0^1\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|\left(\frac{1+t}{2}|f'(a)|+\frac{1-t}{2}|f'(b)|\right)dt \\
&\le\frac{b-a}{2\Lambda(1)}\left[|f'(a)|+|f'(b)|\right]\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|dt \\
&\le\frac{b-a}{2\Lambda(1)}\left[|f'(a)|+|f'(b)|\right]K(t)
\end{aligned}$$

where $K(t)$ is defined in (3.6).This completes the proof. □

Remark 3 Under assumption of Theorem 4 with $\varphi(t) = t$, then Theorem 4 reduce to Corollary 1 in [16].

Corollary 2 *Under assumption of Theorem 4 with $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we have the following inequalities*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{2^{1-\frac{\alpha}{k}}(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\left[I^{\alpha}_{a^+,k}f\left(\frac{a+b}{2}\right) - I^{\alpha}_{b^-,k}f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{b-a}{2}A(\alpha,k)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

where

$$A(\alpha,k) = \left(\frac{2}{3}\right)^{\frac{k}{\alpha}+1}\left(1-\frac{k}{k+\alpha}\right)+\frac{k}{2(\alpha+k)}-\frac{1}{3}.$$

Proof In Theorem 4, if we take $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we write

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{2^{1-\frac{\alpha}{k}}(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\left[I^{\alpha}_{a^+,k}f\left(\frac{a+b}{2}\right) - I^{\alpha}_{b^-,k}f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{b-a}{2}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_0^1\left|\frac{t^{\frac{\alpha}{k}}}{2}-\frac{1}{3}\right|dt$$

and by simply compuations we get

$$\int_0^1\left|\frac{t^{\frac{\alpha}{k}}}{2}-\frac{1}{3}\right|dt = \int_0^{\left(\frac{2}{3}\right)^{\frac{k}{\alpha}}}\left(\frac{1}{3}-\frac{t^{\frac{\alpha}{k}}}{2}\right)dt+\int_{\left(\frac{2}{3}\right)^{\frac{k}{\alpha}}}^1\left(\frac{t^{\frac{\alpha}{k}}}{2}-\frac{1}{3}\right)dt = \left(\frac{2}{3}\right)^{\frac{k}{\alpha}+1}\left(1-\frac{k}{k+\alpha}\right)+\frac{k}{2(\alpha+k)}-\frac{1}{3}$$

which is completes the proof. □

Remark 4 If we take $\alpha = k = 1$ in Corollary 2, then Corollary 2 reduces to Corollary 1 in [16].

Corollary 3 *Under assumption of Corollary 2 with $k = 1$, then we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{2^{1-\alpha}(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left[I^{\alpha}_{a^+}f\left(\frac{a+b}{2}\right) - I^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{b-a}{2}B(\alpha)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

where

$$B(\alpha)=\left(\frac{2}{3}\right)^{\frac{1}{\alpha}+1}\left(\frac{\alpha}{1+\alpha}\right)+\frac{1}{2(\alpha+1)}-\frac{1}{3}.$$

Theorem 5 *Let $f: I=[a,b]\subset R\to R$ be an absolutely continuous mapping on I° such that $f'\in L_1([a,b])$, where $a,b\in I^\circ$ with $a<b$. If the mapping $|f'|^q$, $q>1$, is convex on $[a,b]$, then we have the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi f\left(\frac{a+b}{2}\right)+{}_{b^-}I_\varphi f\left(\frac{a+b}{2}\right)\right]\right|\\&\le\frac{b-a}{2\Lambda(1)}\left(\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof From Lemma 3 and by Hölder's inequality, we get

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{2\Lambda(1)}\left[{}_{a^+}I_\varphi f\left(\frac{a+b}{2}\right)+{}_{b^-}I_\varphi f\left(\frac{a+b}{2}\right)\right]\right|\\&\le\frac{b-a}{2\Lambda(1)}\int_0^1\left[\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|\right.\\&\left.+\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt\\&\le\left(\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|^p\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\\&+\left(\int_0^1\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|^p\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\\&\le\left(\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|^p\right)^{\frac{1}{p}}\left[\int_0^1\left(\frac{1+t}{2}|f'(b)|^q+\frac{1-t}{2}|f'(a)|^q\right)\right]^{\frac{1}{q}}dt\\&+\left(\int_0^1\left|\frac{\Lambda(1)}{3}-\frac{\Lambda(t)}{2}\right|^p\right)^{\frac{1}{p}}\left[\int_0^1\left(\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q\right)\right]^{\frac{1}{q}}dt\\&\le\left(\int_0^1\left|\frac{\Lambda(t)}{2}-\frac{\Lambda(1)}{3}\right|^p\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

□

Remark 5 Under assumption of Theorem 5 with $\varphi(t)=t$, then Theorem 5 reduce to Theorem 4 in [16].

Remark 6 Under assumption of Theorem 5 with $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, then Theorem 5 reduce to Corollary 2.10 in [2].

Corollary 4 *Under assumption of Theorem 5 with* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *we have the following inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{2^{1-\frac{\alpha}{k}}(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\left[I^{\alpha}_{a^+,k}f\left(\frac{a+b}{2}\right)-I^{\alpha}_{b^-,k}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\le \frac{b-a}{2}\left(\int_0^1\left|\frac{t^{\frac{\alpha}{k}}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right).$$

References

1. Alomari, M., Darus, M., Dragomir, S.S.: New inequalities of Simpsonís type for s-convex functions with applications. RGMIA Res Rep Coll 12(4); (2009) (Article 9)
2. Chen, J., Huang, X.: Some new inequalities of Simpson's type for s-convex functions via fractional integrals. Filomat **31**(15), 4989–4997 (2017)
3. Dragomir, S.S., Agarwal, R.P., Cerone, P.: On Simpsonís inequality and applications. J. Inequal. Appl. **5**, 533–579 (2000)
4. Dragomir, S.S.: On Simpson's quadrature formula for differentiable mappings whose derivatives belong to l_p spaces and applications. J. KSIAM **2**, 57–65 (1998)
5. Dragomir, S.S.: On Simpson's quadrature formula for Lipschitzian mappings and applications Soochow. J. Math. **25**, 175–180 (1999)
6. Du, T., Li, Y., Yang, Z.: A generalization of Simpson's inequality via differentiable mapping using extended (s, m) -convex functions. Appl. Math Comput. **293**, 358–369 (2017)
7. Hussain, S., Qaisar, S.: More results on Simpson's type inequality through convexity for twice differentiable continuous mappings, vol. 5. Springer, Berlin, pp 77 (2016)
8. Liu, B.Z.: An inequality of Simpson type. Proc. R. Soc. A **461**, 2155–2158 (2005)
9. Mubeen, S., Habibullah, G.M.: k-Fractional integrals and application. Int. J. Contemp. Math. Sci. **7**(2), 89–94 (2012)
10. Pecaric, J., Proschan, F., Tong, Y.L.: Convex functions, partial ordering and statistical applications. Academic Press, New York (1991)
11. Pecaric, J., Varosanec, S.: A note on Simpson's inequality for functions of bounded variation. Tamkang J. Math. 31(3), 239–242 (2000)
12. Qaisar, S., He, C.J., Hussain, S.: A generalizations of Simpson's type inequality for differentiable functions using (α, m)-convex functions and applications. J. Inequal. Appl. 13 (2013) (Article 158)
13. Kavurmaci, H., Akdemir, A.O., Set, E., Sarikaya, M.Z.: Simpson's type inequalities for m-and (α, m) -geometrically convex functions. Konuralp J. Math. **2**(1), 90–101 (2014)
14. Ozdemir, M.E., Akdemir, A.O., Kavurmacı, H.: On the Simpson's inequality for convex functions on the co-ordinates. Turk. J. Anal. Number Theory **2**(5), 165–169 (2014)
15. Sarıkaya, M. Z., Ertuğral, F.: On the generalized Hermite-Hadamard inequalities 2017 (submitted)
16. Sarikaya, M.Z., Set, E., Ozdemir, M.E.: On new inequalities of Simpson's type for s-convex functions. Comput. Math. Appl. **60**, 2191–2199 (2010)
17. Sarikaya, M.Z., Set, E., Özdemir, M.E.: On new inequalities of Simpson's type for convex functions. RGMIA Res. Rep. Coll. 13(2) (2010) (Article 2)
18. Sarikaya, M.Z., Set, E., Ozdemir, M.E.: On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex. J. Appl. Math. Stat. Inf. 9(1) (2013)

19. Sarıkaya, M.Z., Tunç, T., Budak, H.: Simpson's type inequality for F-convex function. Facta Universitatis Ser. Math. Inform. (in press)
20. Set, E., Ozdemir, M.E., Sarikaya, M.Z.: On new inequalities of Simpson's type for quasi-convex functions with applications. Tamkang J. Math. **43**(3), 357–364 (2012)
21. Set, E., Sarikaya, M. Z., Uygun, N.: On new inequalities of Simpson's type for generalized quasi-convex functions. Adv. Inequal. Appl. 3, 1–11 (2017)
22. Tseng, K.L., Yang, G.S., Dragomir, S.S.: On weighted Simpson type inequalities and applications. J. Math. Inequal. **1**(1), 13–22 (2007)
23. Ujevic, N.: Double integral inequalities of Simpson type and applications. J. Appl. Math. Comput. **14**(1–2), 213–223 (2004)
24. Yang, Z.Q., Li, Y.J., Du, T.: A generalization of Simpson type inequality via differentiable functions using (s, m)-convex functions. Ital. J. Pure Appl. Math. **35**, 327–338 (2015)