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ORIGINAL PAPER

Remarks on vector fields with simply connected trajectories and their associated derivations

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Abstract
Let X be a polynomial vector field on $\mathbb{C}^2$ with at most isolated zeros and whose trajectories are all simply connected. Let us suppose that there is a polynomial $P \in \mathbb{C}[x, y]$ such that (i) $dP(X) = 1$ or (ii) $dP(X) = a \cdot P$, with $a \in \mathbb{C}^*$. In (Bustinduy and Giraldo, in Adv Math 285:1339–1357, 2015; Bustinduy and Giraldo, in J Differ Equ 264:3933–3939, 2018) the authors determined X and P, up to an algebraic change of coordinates, when $P \in \mathbb{C}[x, y]$ is primitive. In this note, we extend these results for an arbitrary P. Finally, as an application, we show that if a polynomial vector field X on $\mathbb{C}^2$ with at most isolated zeros has all its trajectories simply connected and there exist $P \in \mathbb{C}[x, y]$ and $n \in \mathbb{N}^+$ such that $X^n(P) = 0$ and $X^{n-1}(P) \neq 0$ or $X^{n+1}(P) = a \cdot X^n(P)$ with $a \in \mathbb{C}^*$, X is complete and present some questions on the study of derivations whose image is a Mathieu subspace.

Keywords Foliation transverse to a fibration · Foliation P-complete · Simply connected trajectories · Eigenfunctions of derivations

Mathematics Subject Classification Primary 32M25; Secondary 32L30 · 32S65

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To Felipe Cano, on his 60th birthday.

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1 Introduction

1.1 Vector fields and trajectories

A holomorphic vector field X on $\mathbb{C}^2$ is a section of the tangent bundle of $\mathbb{C}^2$. Set coordinates x, y in $\mathbb{C}^2$, hence

$$X = P_1(x, y)\frac{\partial}{\partial x} + P_2(x, y)\frac{\partial}{\partial y},$$

with P_1, P_2 holomorphic functions. A point which is a common zero of P_1 and P_2 is called a singular point of X. Take $z = (x, y)$ in $\mathbb{C}^2$ and the differential equation $\varphi_z'(t) = X(\varphi_z(t))$ with $\varphi_z(0) = z$. The local solution φ_z can be extended by analytic continuation along paths from $t = 0$ in $\mathbb{C}$ to a maximal connected Riemann surface $\pi_z : \Omega_z \to \mathbb{C}$, which is a Riemann domain over $\mathbb{C}$. The solution of X through z is $\varphi_z : \Omega_z \to \mathbb{C}^2$. The (complex) trajectory C_z of X through z is the Riemann surface $\varphi_z(\Omega_z)$ immersed in $\mathbb{C}^2$.

If $\Omega_z = \mathbb{C}$, as domain in $\mathbb{C}$ (then, π_z is an analytic isomorphism), X is said to be complete on C_z. In this case, C_z is uniformized by $\mathbb{C}$, and then analytically isomorphic to (= of type) $\mathbb{C}$ or $\mathbb{C}^*$ (maximum principle).

We say that X is complete if it is complete on C_z for any z. In this case, the flow $\varphi : \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}^2$ of X, $(t, z) \mapsto \varphi(t, z) = \varphi_z(t)$, defines a holomorphic action of $(\mathbb{C}, +)$ on $\mathbb{C}^2$ by analytic automorphisms and $X = \frac{\partial}{\partial t}\varphi(t, z)_{|t=0}$.

There are different types of flows:

- φ is *algebraic*, if φ is a polynomial map.
- φ is *quasi-algebraic*, if φ_t, for any $t \in \mathbb{C}$, is a polynomial automorphism.
- φ is *proper*, if the topological closure $\overline{C_z}$ of C_z in $\mathbb{C}^2$, for any z, is an analytic curve.

1.2 Vector fields and simply-connected trajectories

Let X be a polynomial vector field (then $P_1, P_2 \in \mathbb{C}[x, y]$) with simply-connected trajectories. We addressed in [2,3] the problem of deciding if X is complete or not, under the assumption that there existed a nonconstant primitive polynomial $P \in \mathbb{C}[x, y]$ such that

(i) $dP(X) = 1$, or
(ii) $dP(X) = a \cdot P$ with $a \neq 0$.

Concretely, in both cases we proved that $P = x$ and

$$X = [ax + d]\frac{\partial}{\partial x} + [b(x)y + c(x)]\frac{\partial}{\partial y} \tag{1}$$

after a polynomial automorphism. In fact, it is obtained that $a = 0$ and $d = 1$ in case (i) [2], and $a \neq 0$, $d = 0$, $b(0) = 0$ and $c(0) \neq 0$ in case (ii) [3]. In particular, X is complete. There are several motivations to study these problems.

1. If X is a complete polynomial vector field on $\mathbb{C}^2$ with at most isolated zeros and simply connected trajectories (of type $\mathbb{C}$), by classification of complete polynomial vector fields [1] X is as (1) after a polynomial change of coordinates. Moreover, after performing another polynomial change of coordinates, we can assume that $d = 0$, if $a \neq 0$; and $d = 1$, if $a = 0$. Note that these vector fields satisfy one of the following two properties with respect to $P = x$: either $dP(X) = ax$, if $a \neq 0$; or $dP(X) = 1$, if $a = 0$.

On the other hand, if X has no zeros and flow φ, one also knows, according to [13], [16, Théorème 2] and [16, Théorème 4] (see also [2, Introduction]):

– Algebraic $\varphi \Rightarrow$ Quasi-algebraic $\varphi \Rightarrow$ Proper φ and trajectories of type $\mathbb{C}$.
– In these three situations for φ, after a holomorphic automorphism, there is a polynomial $P = x$ such that $dP(X) = 1$.

Then, it is natural to study if a reciprocal of Brunella's result is valid:
If a polynomial vector field X on $\mathbb{C}^2$ with at most isolated zeros and simply-connected trajectories satisfies for a primitive polynomial P that $dP(X) = aP$, with $a \neq 0$, or $dP(X) = 1$, is X complete?

The affirmative answer to this question is given in [2] and [3], and it implies that such an X has no trajectories of type $\mathbb{D}$, and they are all of type $\mathbb{C}$.

Note that in case (i), the trajectories are always proper in $\mathbb{C}^2$ and X is the constant horizontal vector field after a holomorphic change of coordinates. However, in case (ii), the trajectories are not necessarily proper.

2. Let X be a polynomial vector field on $\mathbb{C}^2$, and the $\mathbb{C}$-derivation D_X of $\mathbb{C}[x, y]$ associated to X:

$$\begin{aligned} D_X : \mathbb{C}[x, y] &\to \mathbb{C}[x, y] \\ f &\mapsto X(f). \end{aligned}$$

A *slice* s of D_X is a polynomial $s \in \mathbb{C}[x, y]$ such that $D_X(s) = 1$. Questions about slices and derivations are related to Cancellation Problem in affine spaces [11, Chapter 10]. Moreover, the Jacobian Conjecture can be formulated as a problem in terms of derivations with a slice [11, Chapter 3]. Furthermore, this famous conjecture has been also formulated by Van de Essen, Wright and Zhao [17] in terms of derivations: it holds if the image of every derivation of $\mathbb{C}[x, y]$ with zero divergence and having a slice is a Mathieu subspace (we will recall this notion in the last section).

If D_X is surjective, $1 \in \mathrm{Im}(D_X)$ and D_X has a slice. Surjective derivations in $\mathbb{C}[x, y]$ are studied in [5], and they are studied too in affine domains B over $\mathbb{C}$ with small dimension in [12]. An important property of a surjective derivation D_X of $\mathbb{C}[x, y]$ is that X has simply-connected trajectories [5, Proposition 1.6].

Motivated by these facts, we studied in [2] polynomial vector fields X on $\mathbb{C}^2$ with simply-connected trajectories such that D_X have a slice, and determined X, modulo a polynomial automorphism. Moreover, we applied this result to the study of surjective derivations. In particular we obtained in [2, Theorem 2] an affirmative answer to a conjecture stated by Cerveau in [6]: If *D_X is surjective, then, up to a polynomial change of coordinates,*

$$X = \frac{\partial}{\partial x} + by\frac{\partial}{\partial y} \tag{2}$$

with $b \in \mathbb{C}$.

1.3 Main result

In what follows, we will assume that X has at most isolated zeros. We extend the above results to the case of a non-primitive polynomial P with the following theorem:

Theorem 1 *Let X be a polynomial vector field in $\mathbb{C}^2$. If there is $P \in \mathbb{C}[x, y]$ such that (i) $dP(X) = 1$ or (ii) $dP(X) = a \cdot P$, with $a \in \mathbb{C}^*$; and the trajectories of X are simply connected, up to a polynomial change of coordinates:*

(1.1) *In case* (i), $P = x$, *and*

$$X = \frac{\partial}{\partial x} + [b(x)y + c(x)]\frac{\partial}{\partial y},$$

with $b, c \in \mathbb{C}[x]$, *and*

(1.2) *In case* (ii), $P = x^n$, *with* $n \in \mathbb{N}^+$, *and*

$$X = dx\frac{\partial}{\partial x} + [b(x)y + c(x)]\frac{\partial}{\partial y},$$

with $d = a/n$, $b, c \in \mathbb{C}[x]$, $b(0) = 0$ *and* $c(0) \neq 0$.

In particular, X is complete and has all its trajectories of type $\mathbb{C}$.

2 Proof of Theorem 1

Note that in case (i): $dP(X) = 1$, P is always primitive. Theorem 1, after [2] and [3], follows by this proposition:

Proposition 1 *Let X be a polynomial vector field in $\mathbb{C}^2$. If there is a non-primitive $P \in \mathbb{C}[x, y]$ such that $dP(X) = a \cdot P$, with $a \in \mathbb{C}^*$; and the trajectories of X are simply-connected, up to a polynomial change of coordinates, P and X are as in* (1.2) *with* $n > 1$.

Proof By Stein's factorization Theorem, we consider a primitive polynomial P_0 such that $P = h(P_0)$ with h a polynomial in $\mathbb{C}[z]$ of degree $n \geq 2$.

Lemma 1 *The polynomial $h \in \mathbb{C}[z]$ has only one root*

Proof Assume that $h(z)$ has k roots $\alpha_i \in \mathbb{C}$, for i from 1 to k, respectively of multiplicity $m_i \in \mathbb{N}^+$. Then

$$h(z) = \lambda \prod_{i=1}^{k} (z - \alpha_i)^{m_i},$$

with $\lambda \in \mathbb{C}^*$, and with n equal to $\sum_{i=1}^{k} m_i$. According to $dP(X) = aP$, it follows that $h'(P_0)dP_0(X) = ah(P_0)$. Then $h(P_0)/h'(P_0) \in \mathbb{C}[x, y]$, and thus $h(z)/h'(z) \in \mathbb{C}[z]$. Because

$$\frac{h(z)}{h'(z)} = \frac{\prod_{i=1}^{k}(z - \alpha_i)}{\sum_{i=1}^{k} m_i(\prod_{j\neq i}(z - \alpha_j))}$$

it is clear that if $k \geq 2$, the polynomial $\sum_{i=1}^{k} m_i(\prod_{j\neq i}(z - \alpha_j))$ has other roots different from α_i and we obtain a contradiction because $h(z)/h'(z)$ is not a polynomial. □

After Lemma 1, Proposition 1 follows easily from [3] .

Assume that $h(z) = (z - \alpha_1)^n$, with $n \geq 2$ ($\lambda = 1$). Condition $dP(X) = aP$ can be written as

$$n(P_0 - \alpha_1)^{n-1} dP_0(X) = a(P_0 - \alpha_1)^n.$$

Hence $dP_0(X) = a/n(P_0 - \alpha_1)$. As $dP_0 = d(P_0 - \alpha_1)$, if $Q = P_0 - \alpha_1$, one obtains that Q is a primitive polynomial such that $dQ(X) = a/nQ$. According to [3], we can assume that $Q = x$ and

$$X = (a/n)x\frac{\partial}{\partial x} + [b(x)y + c(x)]\frac{\partial}{\partial y},$$

where b, $c \in \mathbb{C}[x]$ with $b(0) = 0$ and $c(0) \neq 0$ after a polynomial automorphism. Then $P = x^n$, and Proposition 1 is proved.

□

3 An application and some questions on the image of derivations

First, we give an application of Theorem 1.

Theorem 2 *Let X be a polynomial vector field in $\mathbb{C}^2$ whose trajectories are all simply connected. If there is a nonconstant $P \in \mathbb{C}[x, y]$ and $n \in \mathbb{N}^+$ satisfying:*

a) $X^n(P) = 0$ and $X^{n-1}(P) \neq 0$, or
b) $X^{n+1}(P) = a \cdot X^n(P)$ for $a \in \mathbb{C}^*$

Then, X is complete.

Proof In case a), suppose first that $n = 1$; then $X(P) = 0$ and as P is not a constant polynomial, according to [15], after a polynomial change of coordinates, $X = \partial/\partial x$ which is complete. If $n \geq 2$, take $\bar{P} := X^{n-1}(P)$. If $\bar{P}$ is not constant, then $X(\bar{P}) = 0$ and as before [15] implies that after a polynomial change of coordinates $X = \partial/\partial x$, that is complete. Otherwise, if $\bar{P} = \lambda \in \mathbb{C}^*$, it is enough to apply Theorem 1 to $\tilde{X} := (1/\lambda)X$ and P if $n = 2$, and to $\tilde{X}$ and $\tilde{P} := X^{n-2}(P)$ if $n > 2$, because $\tilde{X}(P) = 1$ and $\tilde{X}(\tilde{P}) = 1$ respectively, to conclude that $\tilde{X}$, and then X, are complete.

In case b), we note that neither $X^n(P)$ nor $X^{n+1}(P)$ equals a nonzero constant. Denote $\hat{P} := X^n(P)$. Since $X(\hat{P}) = a \cdot \hat{P}$, Theorem 1 implies that X is complete. □

Consider the natural domain Ω, containing $\{0\} \times \mathbb{C}^2$, in $\mathbb{C} \times \mathbb{C}^2$, where the local flow $\varphi : \Omega \to \mathbb{C}^2$ of X is defined as $\varphi(t, x, y) = \varphi_z(t)$ (see Sect. 1).

Take $P \in \mathbb{C}[x, y]$. Then $P(\varphi(t, x, y))$ can be expressed according to the *Lie series* as:

$$P(\varphi(t, x, y)) = P(x, y) + X^2(P)(x, y)\frac{t^2}{2!} + X^3(P)(x, y)\frac{t^3}{3!} + \cdots$$

Theorem 2 implies the following:

Corollary 1 *Let X be a polynomial vector field in $\mathbb{C}^2$ whose trajectories are all simply connected. Consider the local flow $\varphi : \Omega \to \mathbb{C}^2$ of X. If there is a nonconstant $P \in \mathbb{C}[x, y]$ and $n \in \mathbb{N}^+$ satisfying:*

(a)

$$P(\varphi(t, x, y)) = P + X(P)t + X^2(P)\frac{t^2}{2!} + \cdots + X^{n-1}(P)\frac{t^{n-1}}{(n-1)!},$$

with $X^{n-1}(P) \neq 0$*, or*

(b)

$$P(\varphi(t,x,y)) = P + X(P)t + X^2(P)\frac{t^2}{2!} + \cdots + X^{n-1}(P)\frac{t^{n-1}}{(n-1)!} + \\ + a^{-n}X^n(P)\left[e^{at} - (1 + at + \frac{(at)^2}{2!} + \cdots + \frac{(at)^{n-1}}{(n-1)!})\right],$$

with $a \in \mathbb{C}^*$.

Then, φ *can be extended to* $\mathbb{C} \times \mathbb{C}^2$ *and* X *is complete.*

Remark 1 The Lie series is related to r-inflection points of the vector field with respect to curves of degree r, where $r = \deg(P)$. See [7,9]; see [10] for an extension of this idea for codimension one foliations.

Finally, we want to point out some ideas related with *Mathieu subspaces*, recently introduced by Zhao in [18], and the study of derivations of $\mathbb{C}[x, y]$. Let us first recall the following

Definition 1 Let R be a commutative k-algebra and M a k-subspace of R. Then M is a Mathieu subspace of R if the following condition holds: if $a \in R$ is such that $a^m \in M$ for all $m \geq 1$, then for any $b \in R$ there exists and $N \in \mathbb{N}$ such that $ba^m \in M$ for all $m \geq N$.

In our situation, $R = \mathbb{C}[x, y]$. It is clear that the image of a derivation $\mathrm{Im}(D_X)$ is a $\mathbb{C}$-subspace of $\mathbb{C}[x, y]$. However, $\mathrm{Im}(D_X)$ is not necessarily a Mathieu subspace. Indeed, Zhao proved in [18, Lemma 4,5] that if M is a Mathieu subspace of R and $1 \in M$, then $M = R$. The following example, taken from [17, Example 2.4],

$$X = \frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y}$$

shows that $\mathrm{Im}(D_X)$ is not a Mathieu subspace, as $1 \in \mathrm{Im}\,(D_X)$ but D_X is not surjective ($y \notin \mathrm{Im}\,(D_X)$).

Recall that D_X is locally finite if for any $f \in \mathbb{C}[x, y]$, the $\mathbb{C}$-vector space spanned by $\{X^n f \mid n \geq 0\}$ has finite dimension. If D_X is locally finite, $\mathrm{Im}(D_X)$ is a Mathieu subspace [17, Theorem 3.1]. In particular, if D_X is locally finite and has a slice, X is surjective, and then of the form (2) after a polynomial automorphism [17, Proposition 3.2].

It would be interesting to *determine polynomial vector fields* X *with all its trajectories simply-connected and such that* $\mathrm{Im}(D_X)$ *is a Mathieu subspace of* $\mathbb{C}[x, y]$*, up to a polynomial automorphism.*

A polynomial vector field X in $\mathbb{C}^2$ determines a locally finite derivation D_X if and only if its flow $\varphi : \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}^2$ is quasi-algebraic [8, Theorem 3.1]. In particular, X is complete.

As we mentioned before, in [17, Theorem 4.3] it is proved that the Jacobian conjecture in $\mathbb{C}^2$ holds if and only if for every derivation D of $\mathbb{C}[x, y]$ with zero divergence (where if $D = p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y}$, $\mathrm{Div}\,(D) = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}$) and having a slice, it holds that $\mathrm{Im}(D)$ is a Mathieu subspace.

Recall that the jacobian conjecture in $\mathbb{C}^2$ affirms that a polynomial map $F := (F_1, F_2) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ with $\det J_F = 1$ is an automorphism. We call a pair of polynomials $F_1, F_2 \in \mathbb{C}[x, y]$ with $\det J_{(F_1,F_2)} = 1$ a *Jacobian pair*. In a joint article with Muciño [4], the authors proved that the invertibility of the map given by the jacobian pair is equivalent to the fact that one of the vector fields

$$\frac{\partial}{\partial F_2} := \frac{\partial F_1}{\partial y}\frac{\partial}{\partial x} - \frac{\partial F_1}{\partial x}\frac{\partial}{\partial y} \quad \text{or} \quad \frac{\partial}{\partial F_1} := \frac{\partial F_2}{\partial y}\frac{\partial}{\partial x} - \frac{\partial F_2}{\partial x}\frac{\partial}{\partial y}$$

is complete. Hence, the condition that the image of a derivation D with zero divergence and having a slice is a Mathieu subspace is equivalent to the fact that the polynomial vector field inducing D is complete.

Thus, we note that for derivations there is a close relation between having as image a Mathieu subspace and being induced by a complete polynomial vector field. *We do not know examples of a derivation D_X determined by a non complete vector field X for which* $\mathrm{Im}(D_X)$ *is a Mathieu subspace of* $\mathbb{C}[x, y]$.

Example 1 Let us consider

$$X = \frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}.$$

X is complete with flow $\varphi(t, x_0, y_0) = (t + x_0, y_0 e^{\frac{t^2}{2}+x_0 t})$. Then D_X is not locally finite. Its trajectories are simply-connected. Moreover, D_X has x as slice.

$\mathrm{Im}(D_X)$ is not a Mathieu subspace. Otherwise, as $1 \in \mathrm{Im}(D_X)$, D_X should be surjective as observed above. But this is not possible because $y \notin \mathrm{Im}\,(D_X)$, as a simple calculation shows: writing $Q = a_0(x) + a_1(x)y + \cdots + a_n(n)y^n$, with $a_i(x) \in \mathbb{C}[x]$, $D_X(Q) = y$ implies

$$y = a_0'(x) + a_1'(x)y + \cdots + a_n(x)'y^n + xy[a_1(x) + 2a_2(x)y + \cdots + na_n(x)y^{n-1}]$$

hence it should hold that

$$a_1'(x) + xa_1(x) = 1,$$

which is impossible.

Example 2 [14, Theorem 2.6] Let us consider

$$X = bx^a y^{b-1}\frac{\partial}{\partial x} - ax^{a-1}y^b\frac{\partial}{\partial y}.$$

with $a, b \geq 1$. Then, $\mathrm{Im}(D_X)$ is a Mathieu subspace if and only if $a = b$.

Note that, when $a = b \geq 2$, X is a vector field with non isolated singularities, trajectories of type $\mathbb{C}^*$, and whose image is a Mathieu subspace.

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