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ORIGINAL PAPER

Degenerate Bernstein polynomials

Taekyun Kim[1] · **Dae San Kim**[2]

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Abstract
Here we consider the degenerate Bernstein polynomials as a degenerate version of Bernstein polynomials, which are motivated by Simsek's recent work 'Generating functions for unification of the multidimensional Bernstein polynomials and their applications' (Simsek in Filomat 30(7):1683–1689, 2016, Math Methods Appl Sci 1–12, 2018) and Carlitz's degenerate Bernoulli polynomials. We derived their generating function, symmetric identities, recurrence relations, and some connections with generalized falling factorial polynomials, higher-order degenerate Bernoulli polynomials and degenerate Stirling numbers of the second kind.

Keywords Bernoulli polynomials · Generating functions · Degenerate Bernstein polynomials · Stirling numbers

Mathematics Subject Classification 11B83

1 Introduction

For $\lambda \in \mathbb{R}$, the degenerate Bernoulli polynomials of order k are defined by Carlitz as

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{k}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}\frac{t^{n}}{n!},\quad \text{(see [4,5])}. \tag{1.1}$$

Note that $\lim_{\lambda\to 0}\beta_{n,\lambda}^{(k)}(x)=B_{n}^{(k)}(x)$ are the ordinary Bernoulli polynomials of order k given by

$$\left(\frac{t}{e^{t}-1}\right)^{k}e^{xt}=\sum_{n=0}^{\infty}B_{n}^{(k)}(x)\frac{t^{n}}{n!},\quad \text{(see [1,13,14])}.$$

✉ Taekyun Kim
tkkim@kw.ac.kr

Dae San Kim
dskim@sogang.ac.kr

1 Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

2 Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea

It is known that the Stirling numbers of the second kind are defined as

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad \text{(see [2,4,8,10])}, \tag{1.2}$$

where $(x)_l = x(x-1)\cdots(x-l+1)$, $(l \geq 1)$, $(x)_0 = 1$.

For $\lambda \in \mathbb{R}$, the $(x)_{n,\lambda}$ is defined as

$$(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n \geq 1) \tag{1.3}$$

In [8–10], $\binom{x}{n}_\lambda$ is defined as

$$\binom{x}{n}_\lambda = \frac{(x)_{n,\lambda}}{n!} = \frac{x(x-\lambda)\cdots(x-(n-1)\lambda)}{n!},\ (n \geq 1),\ \binom{x}{0}_\lambda = 1. \tag{1.4}$$

Thus, by (1.4), we get

$$(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \binom{x}{n}_\lambda t^n,\ (|\lambda t| < 1),\ \text{(see [7])}. \tag{1.5}$$

From (1.5), we note that

$$\sum_{m=0}^{n} \binom{y}{m}_\lambda \binom{x}{n-m}_\lambda = \binom{x+y}{n}_\lambda,\ (n \geq 0). \tag{1.6}$$

The degenerate Stirling numbers of the second kind are defined by

$$\frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!},\ (k \geq 0), \quad \text{(see [7,8])}. \tag{1.7}$$

By (1.7), we easily get

$$\lim_{\lambda \to 0} S_{2,\lambda}(n,k) = S_2(n,k),\ (n \geq k \geq 0), \quad \text{(see [8,10])}.$$

In this paper, we use the following notation.

$$(x \oplus_\lambda y)^n = \sum_{k=0}^{n} \binom{n}{k} (x)_{k,\lambda}(y)_{n-k,\lambda},\ (n \geq 0). \tag{1.8}$$

The Bernstein polynomials of degree n is defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ (n \geq k \geq 0), \quad \text{(see [6,11,19])}. \tag{1.9}$$

Let $C[0,1]$ be the space of continuous functions on $[0,1]$. The Bernstein operator of order n for f is given by

$$\mathbb{B}_n(f|x) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) B_{k,n}(x), \tag{1.10}$$

where $n \in \mathbb{N} \cup \{0\}$ and $f \in C[0,1]$, (see [3,6,15]).

A Bernoulli trial involves performing a random experiment and noting whether a particular event A occurs. The outcome of Bernoulli trial is said to be "success" if A occurs and a "failure" otherwise. The probability $P_n(k)$ of k successes in n independent Bernoulli trials is given by the binomial probability law:

$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}, \text{ for } k = 0, 1, 2, \ldots,$$

From the definition of Bernstein polynomials we note that Bernstein basis is probability mass of binomial distribution with parameter $(n, x = p)$.

Here we would like to mention that in [18] the author studies the so-called Bernstein type polynomials, which are different from our degenerate Bernstein polynomials, and derives many interesting results on those polynomials.

Let us assume that the probability of success in an experiment is p. We wondered if we can say the probability of success in the ninth trial is still p after failing eight times in a ten trial experiment. Because there's a psychological burden to be successful.

It seems plausible that the probability is less than p. This speculation motivated the study of the degenerate Bernstein polynomials associated with the probability distribution.

In this paper, we consider the degenerate Bernstein polynomials as a degenerate version of Bernstein polynomials. We derive their generating function, symmetric identities, recurrence relations, and some connections with generalized falling factorial polynomials, higher-order degenerate Bernoulli polynomials and degenerate Stirling numbers of the second kind.

2 Degenerate Bernstein polynomials

For $\lambda \in \mathbb{R}$ and $k, n \in \mathbb{N} \cup \{0\}$, with $k \le n$, we define the degenerate Bernstein polynomials of degree n which are given by

$$B_{k,n}(x|\lambda) = \binom{n}{k} (x)_{k,\lambda} (1-x)_{n-k,\lambda}, \quad (x \in [0, 1]). \tag{2.1}$$

Note that $\lim_{\lambda \to 0} B_{k,n}(x|\lambda) = B_{k,n}(x)$, $(0 \le k \le n)$. From (2.1), we derive the generating function of $B_{k,n}(x|\lambda)$, which are given by

$$\begin{aligned}
\sum_{n=k}^{\infty} B_{k,n}(x|\lambda) \frac{t^n}{n!} &= \sum_{n=k}^{\infty} \binom{n}{k} (x)_{k,\lambda} (1-x)_{n-k,\lambda} \frac{t^n}{n!} \\
&= \frac{(x)_{k,\lambda}}{k!} \sum_{n=k}^{\infty} \frac{1}{(n-k)!} (1-x)_{n-k,\lambda} t^n \\
&= \frac{(x)_{k,\lambda}}{k!} \sum_{n=0}^{\infty} \frac{(1-x)_{n,\lambda}}{n!} t^{n+k} \\
&= \frac{(x)_{k,\lambda}}{k!} t^k \sum_{n=0}^{\infty} \binom{1-x}{n}_{\lambda} t^n \\
&= \frac{(x)_{k,\lambda}}{k!} t^k (1+\lambda t)^{\frac{1-x}{\lambda}}.
\end{aligned} \tag{2.2}$$

Therefore, by (2.2), we obtain the following theorem.

Theorem 2.1 *For $x \in [0,1]$ and $k = 0, 1, 2, \ldots$, we have*

$$\frac{1}{k!}(x)_{k,\lambda}t^k(1+\lambda t)^{\frac{1-x}{\lambda}} = \sum_{n=k}^{\infty} B_{k,n}(x|\lambda)\frac{t^n}{n!}.$$

From (2.1), we note that

$$B_{k,n}(x|\lambda) = \binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda} = \binom{n}{n-k}(x)_{k,\lambda}(1-x)_{n-k,\lambda}. \tag{2.3}$$

By replacing x by $1-x$, we get

$$B_{k,n}(1-x|\lambda) = \binom{n}{n-k}(1-x)_{k,\lambda}(x)_{n-k,\lambda} = B_{n-k,n}(x|\lambda), \tag{2.4}$$

where $n, k \in \mathbb{N} \cup \{0\}$, with $0 \le k \le n$.

Therefore, by (2.4), we obtain the following theorem.

Theorem 2.2 (Symmetric identities) *For $n, k \in \mathbb{N} \cup \{0\}$, with $k \le n$, and $x \in [0,1]$, we have*

$$B_{n-k,n}(x|\lambda) = B_{k,n}(1-x|\lambda).$$

Now, we observe that

$$\begin{aligned}
&\frac{n-k}{n}B_{k,n}(x|\lambda) + \frac{k+1}{n}B_{k+1,n}(x|\lambda) \\
&= \frac{n-k}{n}\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda} + \frac{k+1}{n}\binom{n}{k+1}(x)_{k+1,\lambda}(1-x)_{n-k-1,\lambda} \\
&= \frac{(n-1)!}{k!(n-k-1)!}(x)_{k,\lambda}(1-x)_{n-k,\lambda} + \frac{(n-1)!}{k!(n-k-1)!}(x)_{k+1,\lambda}(1-x)_{n-k-1,\lambda} \\
&= (1-x-(n-k-1)\lambda)B_{k,n-1}(x|\lambda) + (x-k\lambda)B_{k,n-1}(x|\lambda) \\
&= (1+\lambda(1-n))B_{k,n-1}(x|\lambda).
\end{aligned} \tag{2.5}$$

Therefore, by (2.5), we obtain the following theorem.

Theorem 2.3 *For $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, with $k \le n-1$, and $x \in [0,1]$, we have*

$$(n-k)B_{k,n}(x|\lambda) + (k+1)B_{k+1,n}(x|\lambda) = (1+\lambda(1-n))B_{k,n-1}(x|\lambda). \tag{2.6}$$

From (2.1), we have

$$\begin{aligned}
&\left(\frac{n-k+1}{k}\right)\left(\frac{n-(k-1)\lambda}{1-x-(n-k)\lambda}\right)B_{k-1,n}(x|\lambda) \\
&= \left(\frac{n-k+1}{k}\right)\left(\frac{n-(k-1)\lambda}{1-x-(n-k)\lambda}\right)\binom{n}{k-1}(x)_{k-1,\lambda}(1-x)_{n-k+1,\lambda} \\
&= \frac{n!}{k!(n-k)!}(x)_{k-1,\lambda}(1-x)_{n-k,\lambda} = B_{k,n}(x|\lambda).
\end{aligned} \tag{2.7}$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.4 *For $n, k \in \mathbb{N}$, with $k \le n$, we have*

$$\left(\frac{n-k+1}{k}\right)\left(\frac{n-(k-1)\lambda}{1-x-(n-k)\lambda}\right)B_{k-1,n}(x|\lambda) = B_{k,n}(x|\lambda).$$

For $0 \leq k \leq n$, we get

$$\begin{aligned}
&(1-x-(n-k-1)\lambda)B_{k,n-1}(x|\lambda)+(x-(k-1)\lambda)B_{k-1,n-1}(x|\lambda)\\
&=(1-x-(n-k-1)\lambda)\binom{n-1}{k}(x)_{k,\lambda}(1-x)_{n-1-k,\lambda}\\
&\quad+(x-(k-1)\lambda)\binom{n-1}{k-1}(x)_{k-1,\lambda}(1-x)_{n-k,\lambda}\\
&=\binom{n-1}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda}+\binom{n-1}{k-1}(x)_{k,\lambda}(1-x)_{n-k,\lambda}\\
&=\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)(x)_{k,\lambda}(1-x)_{n-k,\lambda}=\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda}.
\end{aligned} \tag{2.8}$$

Therefore, by (2.8), we obtain the following theorem.

Theorem 2.5 (Recurrence formula). *For $k, n \in \mathbb{N}$, with $k \leq n-1$, $x \in [0, 1]$, we have*

$$(1-x-(n-k-1)\lambda)B_{k,n-1}(x|\lambda)+(x-(k-1)\lambda)B_{k-1,n-1}(x|\lambda)=B_{k,n}(x|\lambda).$$

Remark 1 For $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{k}{n}B_{k,n}(x|\lambda)=\sum_{k=0}^{n}\frac{k}{n}\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda}\\
&=\sum_{k=1}^{n}\binom{n-1}{k-1}(x)_{k,\lambda}(1-x)_{n-k,\lambda}=\sum_{k=0}^{n-1}\binom{n-1}{k}(x)_{k+1,\lambda}(1-x)_{n-1-k,\lambda}\\
&=(x-k\lambda)\sum_{k=0}^{n-1}\binom{n-1}{k}(x)_{k,\lambda}(1-x)_{n-1-k,\lambda}=(x-k\lambda)(x\oplus_{\lambda}(1-x))^{n-1}.
\end{aligned}$$

Now, we observe that

$$\begin{aligned}
\sum_{k=2}^{n}\frac{\binom{k}{2}}{\binom{n}{2}}B_{k,n}(x|\lambda)&=\sum_{k=2}^{n}\frac{k(k-1)}{n(n-1)}\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda}\\
&=\sum_{k=2}^{n}\frac{k(k-1)}{n(n-1)}\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda}\\
&=\sum_{k=2}^{n}\binom{n-2}{k-2}(x)_{k,\lambda}(1-x)_{n-k,\lambda}\\
&=\sum_{k=0}^{n-2}\binom{n-2}{k}(x)_{k+2,\lambda}(1-x)_{n-2-k,\lambda}\\
&=(x-k\lambda)(x-(k+1)\lambda)\sum_{k=0}^{n-2}\binom{n-2}{k}(x)_{k,\lambda}(1-x)_{n-2-k,\lambda}.
\end{aligned} \tag{2.9}$$

Similarly, we have

$$\begin{aligned}
\sum_{k=i}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}B_{k,n}(x|\lambda)&=(x-k\lambda)_{i,\lambda}\sum_{k=0}^{n-i}\binom{n-i}{k}(x)_{k,\lambda}(1-x)_{n-i-k,\lambda}\\
&=(x-k\lambda)_{i,\lambda}(x\oplus_{\lambda}(1-x))^{n-i}.
\end{aligned} \tag{2.10}$$

From (2.10), we note that

$$(x-k\lambda)_{i,\lambda}=\frac{1}{(x\oplus_{\lambda}(1-x))^{n-i}}\sum_{k=i}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}B_{k,n}(x|\lambda), \tag{2.11}$$

where $n, i \in \mathbb{N}$, with $i \leq n$, and $x \in [0, 1]$.

Therefore, by (2.11), we obtain the following theorem.

Theorem 2.6 *For $n, i \in \mathbb{N}$, with $i \leq n$, and $x \in [0, 1]$, we have*

$$(x-k\lambda)_{i,\lambda}=\frac{1}{(x\oplus_{\lambda}(1-x))^{n-i}}\sum_{k=i}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}B_{k,n}(x|\lambda).$$

From Theorem 2.1, we note that

$$\begin{aligned}\frac{t^k}{k!}(x)_{k,\lambda}(1+\lambda t)^{\frac{1-x}{\lambda}}&=\frac{(x)_{k,\lambda}}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^k\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^k(1+\lambda t)^{\frac{1-x}{\lambda}}\\&=(x)_{k,\lambda}\left(\sum_{m=k}^{\infty}S_{2,\lambda}(m,k)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\beta_{l,\lambda}^{(k)}(1-x)\frac{t^l}{l!}\right)\\&=(x)_{k,\lambda}\sum_{n=k}^{\infty}\left(\sum_{m=k}^{n}\binom{n}{m}S_{2,\lambda}(m,k)\beta_{n-m,\lambda}^{(k)}(1-x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.12}$$

On the other hand,

$$\frac{(x)_{k,\lambda}}{k!}t^k(1+\lambda t)^{\frac{1-x}{\lambda}}=\sum_{n=k}^{\infty}B_{k,n}(x|\lambda)\frac{t^n}{n!}. \tag{2.13}$$

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 2.7 *For $n, k \in \mathbb{N}\cup\{0\}$ with $n \geq k$, we have*

$$B_{k,n}(x|\lambda)=(x)_{k,\lambda}\sum_{m=k}^{n}\binom{n}{m}S_{2,\lambda}(m,k)\beta_{n-m,\lambda}^{(k)}(1-x).$$

Let Δ be the shift difference operator with $\Delta f(x) = f(x+1) - f(x)$. Then we easily get

$$\Delta^n f(0)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}f(k),\quad (n\in\mathbb{N}\cup\{0\}). \tag{2.14}$$

Let us take $f(x) = (x)_{m,\lambda}$, $(m \geq 0)$. Then, by (2.14), we get

$$\Delta^n(0)_{m,\lambda}=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(k)_{m,\lambda}. \tag{2.15}$$

For more details on (2.14) and (2.15), we let the reader refer to Chapter 7 of the book [12].

From (1.7), we note that

$$\begin{aligned}\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}&=\frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^k=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(1+\lambda t)^{\frac{l}{\lambda}}\\&=\sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l)_{n,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.16}$$

Thus, by comparing the coefficients on both sides of (2.16), we have

$$\frac{1}{k!}\Delta^k(0)_{n,\lambda} = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l)_{n,\lambda} = \begin{cases} S_{2,\lambda}(n,k) & \text{if } n \geq k, \\ 0 & \text{if } n < k. \end{cases} \tag{2.17}$$

By (2.17), we get

$$\frac{1}{k!}\Delta^k(0)_{n,\lambda} = S_{2,\lambda}(n,k), \text{ if } n \geq k. \tag{2.18}$$

From Theorem 7 and (2.18), we obtain the following corollary.

Corollary 2.8 *For $n, k \in \mathbb{N} \cup \{0\}$ with $n \geq k$, we have*

$$B_{k,n}(x|\lambda) = (x)_{k,\lambda}\sum_{m=k}^{n}\binom{n}{m}\beta^{(k)}_{n-m,\lambda}(1-x)\frac{1}{k!}\Delta^k(0)_{m,\lambda}.$$

Now, we observe that

$$\begin{aligned}
(1+\lambda t)^{\frac{x}{\lambda}} &= \left((1+\lambda t)^{\frac{1}{\lambda}} - 1 + 1\right)^x = \sum_{k=0}^{\infty}\binom{x}{k}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^k \\
&= \sum_{k=0}^{\infty}(x)_k\frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^k \\
&= \sum_{k=0}^{\infty}(x)_k\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(x)_k S_{2,\lambda}(n,k)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.19}$$

On the other hand,

$$(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\binom{\frac{x}{\lambda}}{n}\lambda^n t^n = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad (|\lambda t| < 1). \tag{2.20}$$

Therefore, by (2.19) and (2.20), we obtain the following theorem.

Theorem 2.9 *For $n \geq 0$, we have*

$$(x)_{n,\lambda} = \sum_{k=0}^{n}(x)_k S_{2,\lambda}(n,k).$$

By Theorem 2.9, we easily get

$$(x-k\lambda)_{i,\lambda} = \sum_{l=0}^{i}(x-k\lambda)_l S_{2,\lambda}(i,l). \tag{2.21}$$

From Theorem 2.6, we have the following theorem.

Theorem 2.10 *For $n, i \in \mathbb{N}$, with $i \leq n$, and $x \in [0,1]$, we have*

$$\sum_{l=0}^{i}(x-k\lambda)_l S_{2,\lambda}(i,l) = \frac{1}{(x \oplus_\lambda (1-x))^{n-i}}\sum_{k=i}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}B_{k,n}(x|\lambda).$$

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