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Property (t) and perturbations

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Abstract In this paper we study the properties of property (t), which is introduced by Rashid. We investigate the property (t) in connection with Weyl type theorems, and establish sufficient and necessary conditions for which property (t) holds. Especially, we obtain the equivalence of a-Weyl's theorem and property (t) without the condition that T is a-polaroid, which improves a corresponding result of Rashid (Mediterr J Math 11:1–16, 2014). We also study the stability of property (t) under perturbations by nilpotent operators, by finite rank operators, by quasi-nilpotent operators and by Riesz operators commuting with T.

Keywords Property (t) · Perturbation · Weyl type theorem

Mathematics Subject Classification Primary 47B20; Secondary 47A10

1 Introduction and basic results

Throughout this paper, we denote X an infinite dimensional complex Banach space and $L(X)$ the algebra of all bounded linear operators on X. For $T \in L(X)$, we denote the null space, the range, the spectrum, the point spectrum, the approximate point spectrum, the surjective spectrum, the isolated points of spectrum and the isolated points of approximate point spectrum by $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, $\sigma_s(T)$, iso$\sigma(T)$ and iso$\sigma_a(T)$, respectively. If $R(T)$ is closed and $\alpha(T) = \dim N(T) < \infty$ (resp. $\beta(T) = \dim X/R(T) < \infty$), then T is called an upper (resp. a lower) semi-Fredholm operator. In the sequel $\Phi_+(X)$ (resp. $\Phi_-(X)$)

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is written for the set of all upper (resp. lower) semi-Fredholm operators. The class of all semi-Fredholm operators is defined by $\Phi_{\pm}(X) = \Phi_{+}(X) \cup \Phi_{-}(X)$, and the index of T is given by $i(T) = \alpha(T) - \beta(T)$. Denote $\Phi(X) = \Phi_{+}(X) \cap \Phi_{-}(X)$ the set of all Fredholm operators. Define $W_{+}(X) = \{T \in \Phi_{+}(X) : i(T) \leq 0\}$, $W_{-}(X) = \{T \in \Phi_{-}(X) : i(T) \geq 0\}$. The set of all Weyl operators is defined by $W(X) = W_{+}(X) \cap W_{-}(X) = \{T \in \Phi(X) : i(T) = 0\}$. The classes of operators defined above generate the following spectrums: the Weyl spectrum of T is defined by $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin W(X)\}$ and the upper semi-Weyl spectrum of T is defined by $\sigma_{uw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin W_{+}(X)\}$. Following Coburn [13], Weyl's theorem is said to hold for T if $\sigma(T) \backslash \sigma_w(T) = \pi_{00}(T)$, where $\pi_{00}(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda I) < \infty\}$. According to Rakočević [18], a-Weyl's theorem is said to hold for T if $\sigma_a(T) \backslash \sigma_{uw}(T) = \pi_{00}^a(T)$, where $\pi_{00}^a(T) = \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda I) < \infty\}$. It's known that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse doesn't hold in general.

Recall that the ascent $p(T)$ of an operator T is defined by $p(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$. Similarly, the descent $q(T)$ of an operator T is defined by $q(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$. The class of all upper semi-Browder operators is defined by $B_{+}(X) = \{T \in \Phi_{+}(X) : p(T) < \infty\}$ and the class of all Browder operators is defined by $B(X) = \{T \in \Phi(X) : p(T) = q(T) < \infty\}$. The Browder spectrum of T is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\}$ and the upper semi-Browder spectrum is defined by $\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin B_{+}(X)\}$. For $T \in L(X)$, set $p_{00}(T) = \sigma(T) \backslash \sigma_b(T)$ and $p_{00}^a(T) = \sigma_a(T) \backslash \sigma_{ub}(T)$. In [15], Browder's theorem is said to hold for T if $\sigma(T) \backslash \sigma_w(T) = p_{00}(T)$; a-Browder's theorem is said to hold for T if $\sigma_a(T) \backslash \sigma_{uw}(T) = p_{00}^a(T)$. Note that a-Browder's theorem for T entails Browder's theorem for T,

Recall [5,6,11,12] that

1. Property (w) is said to hold for T if $\sigma_a(T) \backslash \sigma_{uw}(T) = \pi_{00}(T)$.
2. Property (b) is said to hold for T if $\sigma_a(T) \backslash \sigma_{uw}(T) = p_{00}(T)$.
3. Property (aw) is said to hold for T if $\sigma(T) \backslash \sigma_w(T) = \pi_{00}^a(T)$.
4. Property (ab) is said to hold for T if $\sigma(T) \backslash \sigma_w(T) = p_{00}^a(T)$.
5. Property (R) is said to hold for T if $p_{00}^a(T) = \pi_{00}(T)$.

The single valued extension property plays an important role in local spectral theory, see the recent monograph of Laursen and Neumann [16] and Aiena [1]. In this article we shall consider the following local version of this property.

Let $T \in L(X)$. The operator T is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbrev. SVEP at λ_0), the only analytic function $f : D \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in D$ is the function $f \equiv 0$. An operator T is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

From the identity theorem for analytic function it easily follows that $T \in L(X)$, as well as its dual T^*, has SVEP at every point of the boundary of the spectrum $\sigma(T) = \sigma(T^*)$, so both T and T^* have SVEP at every isolated point of the spectrum.

According to [4, Theorem 1.2], if $T \in L(X)$ and suppose that $\lambda_0 I - T \in \Phi_{\pm}(X)$. Then the following statements are equivalent:

1. T has SVEP at λ_0; 2. $p(T - \lambda_0 I) < \infty$; 3. $\sigma_a(T)$ doesn't cluster at λ_0.

Dually, if $\lambda_0 I - T \in \Phi_{\pm}(X)$, then the following statements are equivalent:

4. T^* has SVEP at λ_0; 5. $q(T - \lambda_0 I) < \infty$; 6. $\sigma_s(T)$ doesn't cluster at λ_0.

An important subspace in local spectral theory is given by the glocal spectral subspace $\chi_T(F)$ associated with a closed subspace $F \subseteq \mathbb{C}$. This is defined, for an arbitrary operator $T \in L(X)$ and a closed subspace F of $\mathbb{C}$, as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C}\backslash F \to X$ which satisfies the identity $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathbb{C}\backslash F$.

The basic role of SVEP arises in local spectral theory since all decomposable operators enjoy this property. Recall that $T \in L(X)$ has the decomposition property (δ) if $X = \chi_T(\overline{U}) + \chi_T(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$. Decomposable operators may be defined in several ways for instance as the union of the property (β) and property (δ), see [16, Theorem 2.5.19] for relevant definitions. Note that property (β) implies that T has SVEP, while the property (δ) implies SVEP for T^*, see [16, Theorem 2.5.19].

A bounded operator T is said to be a-polaroid if every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T. T is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. T is said to be finite-isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of finite multiplicity.

In Sect. 2, we study the property (t) in connection with Weyl type theorems. We prove that an operator T possessing property (t) possesses a-Weyl's theorem, but the converse is not true in general as shown by Example 2.4. And we obtain the equivalence of a-Weyl's theorem and property (t) without the condition that T is a-polaroid, which improves a corresponding result of [20, Theorem 3.3]. We also show the relations of property (t) with other Weyl type theorems. In Sect. 3, we prove that if $T \in L(X)$ and E is a nilpotent operator commuting with T, then T possesses property (t) if and only if $T + E$ possesses property (t). Finally we obtain the stability of property (t) under perturbations by finite rank operators and by quasi-nilpotent operators commuting with T. In the last part, as a conclusion, we give a diagram summarizing the different relations between Weyl type theorems, extending a similar diagram given [10].

2 Property (t)

Definition 2.1 [20] An operator T is said to satisfy property (t) if $\sigma(T)\backslash\sigma_{uw}(T) = \pi_{00}(T)$.

Theorem 2.2 *T satisfies property (t) if and only if $\sigma(T)\backslash\sigma_{uw}(T) = \pi_{00}^a(T)$.*

Proof Suppose T satisfies property (t), then $\sigma(T)\backslash\sigma_{uw}(T) = \pi_{00}(T)$, it follows from Theorem 2.6 of [20] that $\sigma(T) = \sigma_a(T)$, then $\pi_{00}(T) = \pi_{00}^a(T)$, we have $\sigma(T)\backslash\sigma_{uw}(T) = \pi_{00}^a(T)$. Conversely, we first prove $\sigma(T) = \sigma_a(T)$, since $\sigma_a(T) \subseteq \sigma(T)$ holds for every operator T, we need only to prove $\sigma(T) \subseteq \sigma_a(T)$. Let $\lambda \in \sigma(T)$. If $\lambda \notin \sigma_{uw}(T)$, since $\sigma(T)\backslash\sigma_{uw}(T) = \pi_{00}^a(T)$, then $\lambda \in \pi_{00}^a(T) \subseteq \sigma_a(T)$. If $\lambda \in \sigma_{uw}(T)$, it is easy to prove $\lambda \in \sigma_a(T)$, i.e., $\sigma(T) = \sigma_a(T)$, and so $\pi_{00}(T) = \pi_{00}^a(T)$, we have T satisfies property (t). □

Theorem 2.3 *Suppose that T satisfies property (t). Then T satisfies a-Weyl's theorem.*

Proof Suppose that T satisfies property (t). It follows from Theorem 2.6 of [20] and Theorem 2.2 that $\sigma(T) = \sigma_a(T)$ and $\sigma(T)\backslash\sigma_{uw}(T) = \pi_{00}^a(T)$, then $\sigma_a(T)\backslash\sigma_{uw}(T) = \pi_{00}^a(T)$, i.e., T satisfies a-Weyl's theorem. □

The following example shows that a-Weyl's theorem is weaker than property (t).

Example 2.4 Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the unilateral right shift operator defined by $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ for all $x = (x_1, x_2, \ldots) \in l^2(\mathbb{N})$. Then $\sigma(T) = D$, $\sigma_a(T) = \sigma_{uw}(T) = \partial D$ and $\pi_{00}(T) = \pi_{00}^a(T) = \phi$, where D denotes the closed unit disc and ∂D denotes the unit circle. It follows that $\sigma_a(T)\backslash\sigma_{uw}(T) = \pi_{00}^a(T) = \phi$, then T satisfies a-Weyl's theorem. While T doesn't satisfy property (t), since $\sigma(T)\backslash\sigma_{uw}(T) \neq \phi = \pi_{00}(T)$.

The following theorem improves the result of [20, Theorem 3.3], here we omit the condition that T is a-polaroid.

Theorem 2.5 *T satisfies property (t) if and only if the following two conditions hold:*

1. *T satisfies a-Weyl's theorem;*
2. *$\sigma(T) = \sigma_a(T)$.*

Proof If T satisfies property (t), it follows from Theorem 2.3 and [20] that T satisfies a-Weyl's theorem and $\sigma(T) = \sigma_a(T)$. Conversely, if T satisfies a-Weyl's theorem and $\sigma(T) = \sigma_a(T)$, we have $\sigma(T)\backslash\sigma_{uw}(T) = \sigma_a(T)\backslash\sigma_{uw}(T) = \pi_{00}^a(T) = \pi_{00}(T)$, i.e., T satisfies property (t). □

Corollary 2.6 *Suppose that T is decomposable. Then T satisfies property (t) if and only if T satisfies a-Weyl's theorem.*

Proof If T is decomposable, then T^* has SVEP, we have $\sigma(T) = \sigma_a(T)$. The equivalence then follows from Theorem 2.5.

The following example shows that property (t) for an operator is not transmitted to the dual T^*. □

Example 2.7 Let $L : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the unilateral left shift operator defined by $L(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ for all $x = (x_1, x_2, \ldots) \in l^2(\mathbb{N})$. Then $\sigma(T) = \sigma(T^*) = \sigma_{uw}(T) = D$, $\sigma_{uw}(T^*) = \partial D$ and $\pi_{00}(T^*) = \pi_{00}(T) = \phi$. It follows that $\sigma(T)\backslash\sigma_{uw}(T) = \pi_{00}(T) = \phi$, then T satisfies property (t). On the other hand, since $\sigma(T^*)\backslash\sigma_{uw}(T^*) \neq \phi = \pi_{00}(T^*)$, then T^* does not satisfy property (t).

Corollary 2.8 *Suppose that T satisfies property (t). Then T satisfies Weyl's theorem, a-Browder's theorem and Browder's theorem.*

The above Example 2.4 also shows that a-Browder's theorem and Browder's theorem are strictly weaker than property (t). However, we have:

Theorem 2.9 *T satisfies property (t) if and only if the following three conditions hold:*

1. *T satisfies a-Browder's theorem;*
2. *$\sigma_a(T) = \sigma(T)$;*
3. *$p_{00}^a(T) = \pi_{00}(T)$.*

Proof If T satisfies property (t), it follows from Theorem 2.6 and Proposition 2.7 of [20] that $\sigma_a(T) = \sigma(T)$, T satisfies a-Browder's theorem and $p_{00}^a(T) = \pi_{00}(T)$. On the other hand, if T satisfies a-Browder's theorem, $\sigma_a(T) = \sigma(T)$ and $p_{00}^a(T) = \pi_{00}(T)$, then $\sigma(T)\backslash\sigma_{uw}(T) = \sigma_a(T)\backslash\sigma_{uw}(T) = p_{00}^a(T) = \pi_{00}(T)$, i.e., T satisfies property (t). □

Corollary 2.10 *Suppose that T satisfies property (t). Then T satisfies property (R).*

The following example shows that property (R) does not entail property (t).

Example 2.11 Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the unilateral right shift operator defined by $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ for all $x = (x_1, x_2, \ldots) \in l^2(\mathbb{N})$. It is easy to verify $\pi_{00}(T) = p_{00}^a(T) = \phi$, i.e., T satisfies property (R). While T does not satisfy property (t).

Theorem 2.12 *T satisfies property (t) if and only if the following three conditions hold:*

1. *T satisfies Browder's theorem;*
2. $\sigma_{uw}(T) = \sigma_w(T)$*;*
3. $p_{00}(T) = \pi_{00}(T)$.

Proof If T satisfies property (t), it follows from Corollary 2.8, Theorem 2.10 of [20] and [2] that T satisfies Browder's theorem, $\sigma_{uw}(T) = \sigma_w(T)$ and $p_{00}(T) = \pi_{00}(T)$. Conversely, if T satisfies Browder's theorem, $\sigma_{uw}(T) = \sigma_w(T)$ and $p_{00}(T) = \pi_{00}(T)$, then $\sigma(T)\backslash\sigma_{uw}(T) = \sigma(T)\backslash\sigma_w(T) = p_{00}(T) = \pi_{00}(T)$, i.e., T satisfies property (t). □

Theorem 2.13 *Suppose that T satisfies property* (t)*. Then T has property* (aw)*.*

Proof Suppose that T satisfies property (t). Then $\sigma(T)\backslash\sigma_{uw}(T) = \pi_{00}(T)$. It follows from the proof of Theorem 2.2 and Theorem 2.12 that $\pi_{00}(T) = \pi_{00}^a(T)$ and $\sigma_{uw}(T) = \sigma_w(T)$. Then we have $\sigma(T)\backslash\sigma_w(T) = \pi_{00}^a(T)$. □

Corollary 2.14 *Suppose that T satisfies property* (t)*. Then T satisfies property* (ab)*.*

The following example shows that property (aw) and property (ab) are weaker than property (t).

Example 2.15 Let T be defined as in Example 2.11. Then $\sigma(T) = \sigma_w(T) = D$ and $\pi_{00}^a(T) = \phi$. It follows that $\sigma(T)\backslash\sigma_w(T) = \pi_{00}^a(T) = \phi$, then T satisfies property (aw), hence T satisfies property (ab). While T doesn't satisfy property (t).

Theorem 2.16 *T satisfies property* (t) *if and only if the following three conditions hold:*

1. *T satisfies property* (aw)*;*
2. $\sigma_{uw}(T) = \sigma_w(T)$*;*
3. $\pi_{00}(T) = \pi_{00}^a(T)$.

Proof If T satisfies property (t), it follows from the proof of Theorem 2.2, Theorem 2.12 and Theorem 2.13 that $\pi_{00}(T) = \pi_{00}^a(T)$, $\sigma_{uw}(T) = \sigma_w(T)$ and T satisfies property (aw). Conversely, if T satisfies property (aw), $\sigma_{uw}(T) = \sigma_w(T)$ and $\pi_{00}(T) = \pi_{00}^a(T)$, then $\sigma(T)\backslash\sigma_{uw}(T) = \sigma(T)\backslash\sigma_w(T) = \pi_{00}^a(T) = \pi_{00}(T)$, i.e., T satisfies property (t). □

Theorem 2.17 *T satisfies property* (t) *if and only if the following three conditions hold:*

1. *T satisfies property* (ab)*;*
2. $\sigma_{uw}(T) = \sigma_w(T)$*;*
3. $\pi_{00}(T) = p_{00}^a(T)$.

Proof If T satisfies property (t), it follows from Theorem 2.9, Theorem 2.12 and Corollary 2.14 that $\pi_{00}(T) = p_{00}^a(T)$, $\sigma_{uw}(T) = \sigma_w(T)$ and T satisfies property (ab). Conversely, if T satisfies property (ab), $\sigma_{uw}(T) = \sigma_w(T)$ and $\pi_{00}(T) = p_{00}^a(T)$, then $\sigma(T)\backslash\sigma_{uw}(T) = \sigma(T)\backslash\sigma_w(T) = p_{00}^a(T) = \pi_{00}(T)$, i.e., T satisfies property (t). □

Theorem 2.18 *T satisfies property* (t) *if and only if the following three conditions hold:*

1. *T satisfies property* (b)*;*
2. $\sigma_a(T) = \sigma(T)$*;*
3. $\pi_{00}(T) = p_{00}(T)$.

Proof If T satisfies property (t), it follows from [20] and Theorem 2.12 that T satisfies property (b), $\sigma_a(T) = \sigma(T)$ and $\pi_{00}(T) = p_{00}(T)$. Conversely, if T satisfies property (b), $\sigma_a(T) = \sigma(T)$ and $\pi_{00}(T) = p_{00}(T)$, then $\sigma(T)\backslash\sigma_{uw}(T) = \sigma_a(T)\backslash\sigma_{uw}(T) = p_{00}(T) = \pi_{00}(T)$, i.e., T satisfies property (t). □

3 Property (t) under perturbations

An operator T is called Riesz if its essential spectrum $\sigma_e(T) = \{0\}$. Compact operators, also quasi-nilpotent operators, are Riesz operators.

Lemma 3.1 [19,22] *Suppose that $T \in L(X)$ and that $R \in L(X)$ is a Riesz operator commuting with T. Then*

1. $\sigma_{uw}(T + R) = \sigma_{uw}(T)$.
2. $\sigma_w(T + R) = \sigma_w(T)$.
3. $\sigma_{ub}(T + R) = \sigma_{ub}(T)$.
4. $\sigma_b(T + R) = \sigma_b(T)$.

The next result shows that property (t) for T is transmitted to $T + E$ in the case where E is a nilpotent operator which commutes with T.

Theorem 3.2 *Suppose $T \in L(X)$ and let $E \in L(X)$ be a nilpotent operator which commutes with T. Then T satisfies property (t) if and only if $T + E$ satisfies property (t).*

Proof Since E is a nilpotent operator which commutes with T, it follows from [3] and Lemma 3.1 that $\pi_{00}(T + E) = \pi_{00}(T)$ and $\sigma_{uw}(T + E) = \sigma_{uw}(T)$. Suppose that T has property (t). Then $\sigma(T + E)\backslash\sigma_{uw}(T + E) = \sigma(T)\backslash\sigma_{uw}(T) = \pi_{00}(T) = \pi_{00}(T + E)$, therefore $T + E$ has property (t). The converse follows by symmetry. □

This example shows that the commutativity hypothesis of Theorem 3.2 is essential.

Example 3.3 Let $Q : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$Q(x_1, x_2, \ldots) = \left(0, 0, \frac{x_1}{2}, \frac{x_2}{2^2}, \frac{x_3}{2^3}, \ldots\right) \text{ for all } x = (x_1, x_2, \ldots) \in l^2(\mathbb{N}),$$

and

$$E(x_1, x_2, \ldots) = \left(0, 0, -\frac{x_1}{2}, 0, 0, \ldots\right) \text{ for all } x = (x_1, x_2, \ldots) \in l^2(\mathbb{N}).$$

Clearly E is a nilpotent operator, $\sigma(Q) = \sigma_{uw}(Q) = \{0\}$ and $\pi_{00}(Q) = \phi$. It follows that $\sigma(Q)\backslash\sigma_{uw}(Q) = \pi_{00}(Q) = \phi$, i.e., Q satisfies property (t). While $\sigma(Q + E) = \sigma_{uw}(Q + E) = \{0\}$ and $\pi_{00}(Q+E) = \{0\}$, it follows that $\sigma(Q+E)\backslash\sigma_{uw}(Q+E) = \phi \neq \pi_{00}(Q+E)$, i.e., $Q + E$ does not satisfy property (t).

The previous theorem does not extend to commuting finite rank operators as shown by the following example.

Example 3.4 Let $S : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be an injective quasi-nilpotent operator, and let $U : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be defined by $U(x_1, x_2, \ldots) := (-\frac{1}{2}x_1, 0, 0, \ldots)$ for all $x = (x_1, x_2, \ldots) \in l^2(\mathbb{N})$. Define

$$T = \begin{pmatrix} \frac{1}{2}I & 0 \\ 0 & S \end{pmatrix} \text{ and } F = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\sigma(T) = \sigma_{uw}(T) = \{0, \frac{1}{2}\}$ and $\pi_{00}(T) = \phi$, it follows that $\sigma(T)\backslash\sigma_{uw}(T) = \pi_{00}(T) = \phi$, i.e., T satisfies property (t).

On the other hand, since $\sigma(T+F) = \sigma_{uw}(T+F) = \{0, \frac{1}{2}\}$ and $\pi_{00}(T+F) = \{0\}$, then $\sigma(T+F)\backslash\sigma_{uw}(T+F) = \phi \neq \pi_{00}(T+F)$, i.e., $T+F$ does not satisfy property (t). It is easy to verify that F is a finite rank operator commuting with T.

However, we have:

Theorem 3.5 *Suppose that $T \in L(X)$ is isoloid, F is an operator that commutes with T and for which there exists a positive integer n such that F^n is finite rank. If T satisfies property (t), then $T+F$ satisfies property (t).*

Proof Suppose that T satisfies property (t). It follows from Theorem 2.12 that T satisfies Browder's theorem and $\sigma_w(T) = \sigma_{uw}(T)$, and hence $T+F$ satisfies Browder's theorem and $\sigma_w(T+F) = \sigma_{uw}(T+F)$ by Lemma 3.1. By Theorem 2.12, in order to show that $T+F$ satisfies property (t), we need only to show $p_{00}(T+F) = \pi_{00}(T+F)$. Since $p_{00}(T+F) \subseteq \pi_{00}(T+F)$ holds for every operator, it is sufficient to prove $\pi_{00}(T+F) \subseteq p_{00}(T+F)$. Let $\lambda \in \pi_{00}(T+F)$. If $T-\lambda$ is invertible, then $T+F-\lambda$ is Fredholm, and hence $\lambda \in p_{00}(T+F)$. If $\lambda \in \sigma(T)$, it follows from [17, Lemma 2.3] that $\lambda \in \text{iso}\sigma(T)$. Since T is isoloid, we have $0 < \alpha(\lambda - T)$, as F is a finite rank operator commuting with T, $(T+F-\lambda)^n|_{N(T-\lambda)} = F^n|_{N(T-\lambda)}$ has finite-dimension range and kernel, it is easy to obtain that $\alpha(\lambda - T) < \infty$, i.e., $\lambda \in \pi_{00}(T)$. We have $\lambda \in p_{00}(T)$ by Theorem 2.12, then $\lambda - T$ is Browder. It follows from Lemma 3.1 that $\lambda - (T+F)$ is also Browder, hence $\lambda \in \sigma(T+F)\backslash\sigma_b(T+F) = p_{00}(T+F)$, i.e., $T+F$ satisfies property (t). □

The following example shows that Theorem 3.5 fails if we do not assume that T is isoloid.

Example 3.6 Let T be defined as in Example 3.4. Then $\sigma(T) = \sigma_{uw}(T) = \{0, \frac{1}{2}\}$ and $\pi_{00}(T) = \phi$, it follows that $\sigma(T)\backslash\sigma_{uw}(T) = \pi_{00}(T) = \phi$, i.e., T satisfies property (t).

On the other hand, since $\sigma(T+F) = \sigma_{uw}(T+F) = \{0, \frac{1}{2}\}$ and $\pi_{00}(T+F) = \{0\}$, then $\sigma(T+F)\backslash\sigma_{uw}(T+F) = \phi \neq \pi_{00}(T+F)$, i.e., $T+F$ does not satisfy property (t). It is easy to verify that F^n is a finite rank operator commuting with T.

Corollary 3.7 *Suppose that $T \in L(X)$ is isoloid, F is a finite rank operator that commutes with T. If T satisfies property (t), then $T+F$ satisfies property (t).*

Theorem 3.8 *Suppose that $T \in L(X)$ and iso$\sigma_a(T) = \phi$. If T satisfies property (t) and F is a finite rank operator commuting with T, then T+F satisfies property* (t).

Proof Suppose that T satisfies property (t). It follows from Theorem 2.12 that T satisfies Browder's theorem and $\sigma_w(T) = \sigma_{uw}(T)$, and hence $T+F$ satisfies Browder's theorem and $\sigma_w(T+F) = \sigma_{uw}(T+F)$ by Lemma 3.1. By Theorem 2.12, in order to show that $T+F$ satisfies property (t), we need only to show $p_{00}(T+F) = \pi_{00}(T+F)$. Since $p_{00}(T+F) \subseteq \pi_{00}(T+F)$ holds for every operator, it is sufficient to prove $\pi_{00}(T+F) \subseteq p_{00}(T+F)$. Since $\text{iso}\sigma_a(T) = \phi$ and F is a finite rank operator commuting with T, by the proof of [3, Theorem 2.8], $\sigma_a(T) = \sigma_a(T+F)$, then $\text{iso}\sigma_a(T+F) = \phi$. Since $\text{iso}\sigma(T+F) \subseteq \text{iso}\sigma_a(T+F)$, $\text{iso}\sigma(T+F) = \phi$. It follows that $\pi_{00}(T+F) = \phi$, hence $\pi_{00}(T+F) \subseteq p_{00}(T+F)$, i.e., $T+F$ satisfies property (t). □

Theorem 3.2 does not extend to commuting quasi-nilpotent operators as shown by the following example.

Example 3.9 Let $Q : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $Q(x_1, x_2, \ldots) = (\frac{x_2}{2^2}, \frac{x_3}{2^3}, \frac{x_4}{2^4}, \ldots)$ for all $x = (x_1, x_2, \ldots) \in l^2(\mathbb{N})$ and $T = 0$. Clearly T satisfies property (t). While Q is quasi-nilpotent and $TQ = QT$, so $\sigma(Q) = \sigma_{uw}(Q) = \{0\}$ and $\pi_{00}(Q) = \{0\}$, it follows that $\sigma(Q)\backslash\sigma_{uw}(Q) = \phi \neq \pi_{00}(Q)$ i.e., $T + Q = Q$ does not satisfy property (t).

Theorem 3.10 *Suppose that $T \in L(X)$ satisfies $\sigma_p(T) \cap iso\sigma(T) \subseteq \pi_{00}(T)$, Q is a quasi-nilpotent operator that commutes with T. If T satisfies property (t), then $T + Q$ satisfies property (t).*

Proof Since $\sigma(T + Q) = \sigma(T)$ and $\sigma_{uw}(T + Q) = \sigma_{uw}(T)$, it is sufficient to show $\pi_{00}(T + Q) = \pi_{00}(T)$. Let $\lambda \in \pi_{00}(T) = \sigma(T)\backslash\sigma_{uw}(T)$. Then $\lambda \in \text{iso}\sigma(T) = \text{iso}\sigma(T + Q)$ and $T - \lambda$ is upper semi-Fredholm. Therefore $T + Q - \lambda$ is upper semi-Fredholm, and hence $\lambda \in p_{00}(T + Q) \subseteq \pi_{00}(T + Q)$. Conversely, suppose $\lambda \in \pi_{00}(T + Q)$. Since Q is a quasi-nilpotent operator that commutes with T, we obtain that the restriction of $T - \lambda$ to the finite-dimension subspace $N(T + Q - \lambda)$ is not invertible, and hence $N(T - \lambda)$ is non-trivial. Therefore, $\sigma_p(T) \cap \text{iso}\sigma(T) \subseteq \pi_{00}(T)$, thus $T + Q$ satisfies property (t). □

We shall show that property (t) is preserved under injective quasi-nilpotent perturbations. We need first some preliminary results.

Lemma 3.11 [3] *Let $T \in L(X)$ be such that $\alpha(T) < \infty$. Suppose that there exists an injective quasi-nilpotent operator $Q \in L(X)$ such that $TQ = QT$. Then $\alpha(T) = 0$.*

Theorem 3.12 *Suppose that $T \in L(X)$ and $Q \in L(X)$ is an injective quasi-nilpotent operator commuting with T. If T satisfies property (t), then also $T + Q$ satisfies property (t).*

Proof Since T satisfies property (t), it follows from Lemma 3.1 that

$$\sigma(T + Q)\backslash\sigma_{uw}(T + Q) = \sigma(T)\backslash\sigma_{uw}(T) = \pi_{00}(T).$$

To show property (t) for $T + Q$, we need only to prove that $\pi_{00}(T) = \pi_{00}(T + Q) = \phi$. Assume that $\pi_{00}(T) \neq \phi$. Let $\lambda \in \pi_{00}(T)$. Then $\alpha(T - \lambda) < \infty$, it follows from Lemma 3.11 that $\alpha(\lambda - T) = 0$, a contradiction, thus $\pi_{00}(T) = \phi$. We can prove $\pi_{00}(T + Q) = \phi$ by the same way.

In Theorem 3.12, the condition quasi-nilpotent can't be replaced by the condition compact. □

Example 3.13 Let $U : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $U(x_1, x_2, \ldots) = (0, \frac{x_2}{2^2}, \frac{x_3}{2^3}, \ldots)$ for all $x = (x_1, x_2, \ldots) \in l^2(\mathbb{N})$ and $V(x_1, x_2, \ldots) = (x_1, -\frac{x_2}{2^2}, -\frac{x_3}{2^3}, \ldots)$ for all $x = (x_1, x_2, \ldots) \in l^2(\mathbb{N})$. Define $T = U \oplus I$ and $K = V \oplus Q$, where Q is an injective compact quasi-nilpotent operator. Clearly $\sigma(T) = \left\{\frac{1}{2^n} : n = 2, 3, \ldots\right\} \cup \{0, 1\}$, $\sigma_{uw}(T) = \{0, 1\}$ and $\pi_{00}(T) = \left\{\frac{1}{2^n} : n = 2, 3, \ldots\right\}$, it follows that $\sigma(T)\backslash\sigma_{uw}(T) = \{\frac{1}{2^n} : n = 2, 3, \ldots\} = \pi_{00}(T)$, thus property (t) holds for T.

Note that K is an injective compact operator, $KT = TK$ and $\sigma(T + K) = \sigma_{uw}(T + K) = \{0, 1\}$ and $\pi_{00}(T + K) = \{1\}$, it follows that $\sigma(T + K)\backslash\sigma_{uw}(T + K) = \phi \neq \pi_{00}(T + K)$, then $T + K$ does not satisfy property (t).

Theorem 3.14 *Suppose that $T \in L(X)$ is a finite-isoloid operator, R is a Riesz operator that commutes with T. If T satisfies property (t), then $T + R$ satisfies property (t).*

Proof Suppose that T satisfies property (t). It follows from Theorem 2.12 that T satisfies Browder's theorem and $\sigma_w(T) = \sigma_{uw}(T)$, and hence $T + R$ satisfies Browder's theorem and $\sigma_w(T + R) = \sigma_{uw}(T + R)$ by Lemma 3.1. By Theorem 2.12, in order to show that $T + R$ satisfies property (t), we need only to show $p_{00}(T + R) = \pi_{00}(T + R)$. Since $p_{00}(T + R) \subseteq \pi_{00}(T + R)$ holds for every operator, it is sufficient to prove $\pi_{00}(T + R) \subseteq p_{00}(T + R)$. Let $\lambda \in \pi_{00}(T + R)$. If $T - \lambda$ is invertible, then $T + R - \lambda$ is Fredholm, and hence $\lambda \in p_{00}(T + R)$. If $\lambda \in \sigma(T)$, it follows from [17, Lemma 2.3] that $\lambda \in \text{iso}\sigma(T)$. Since T is finite-isoloid, we have $0 < \alpha(\lambda - T) < \infty$, i.e., $\lambda \in \pi_{00}(T)$. We have $\lambda \in p_{00}(T)$ by Theorem 2.12, then $\lambda - T$ is Browder. It follows from Lemma 3.1 that $\lambda - (T + R)$ is also Browder, hence $\lambda \in \sigma(T + R)\backslash\sigma_b(T + R) = p_{00}(T + R)$, i.e., $T + R$ satisfies property (t). □

Corollary 3.15 *Suppose that $T \in L(X)$ is a finite-isoloid operator, K is a compact operator that commutes with T. If T satisfies property (t), then $T + K$ satisfies property (t).*

4 Conclusion

In the last part, we give a summary of the known Weyl type theorems as in [10], including the properties introduced in [5,7,10,11,20,21], and in this paper. We use the abbreviations gW; W; (gw); (w); (gt); (t); aW; (aw); (R) and (aR) to signify that an operator $T \in L(X)$ obeys generalized Weyl's theorem, Weyl's theorem, property (gw), property (w), property (gt), property (t), a-Weyl's theorem, property (aw), property (R) and property (aR). Similarly, the abbreviations gB; B; gaB and aB have analogous meaning with respect to Browder's theorem .

The following Fig. 1 summarizes the meaning of various theorems and properties.

In the following Fig. 2, which extends the similar diagram presented in [10], arrows signify implications between various Weyl type theorems, Browder type theorems, property (gw), property (gaw), property (gR), property (gaR), property (gS), property (w), property (aw), property (R), property (aR) and property (S). The numbers near the arrows are references

gW	$\sigma(T)\backslash\sigma_{BW}(T) = E(T)$	(gt)	$\sigma(T)\backslash\sigma_{SBF_+^-}(T) = E(T)$
W	$\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T)$	(t)	$\sigma(T)\backslash\sigma_{uw}(T) = \pi_{00}(T)$
(gw)	$\sigma_a(T)\backslash\sigma_{SBF_+^-}(T) = E(T)$	aW	$\sigma_a(T)\backslash\sigma_{uw}(T) = \pi_{00}^a(T)$
(w)	$\sigma_a(T)\backslash\sigma_{uw}(T) = \pi_{00}(T)$	(aw)	$\sigma(T)\backslash\sigma_w(T) = \pi_{00}^a(T)$
gB	$\sigma(T)\backslash\sigma_{BW}(T) = \Pi(T)$	(b)	$\sigma_a(T)\backslash\sigma_{uw}(T) = p_{00}(T)$
B	$\sigma(T)\backslash\sigma_w(T) = p_{00}(T)$	(ab)	$\sigma(T)\backslash\sigma_w(T) = p_{00}^a(T)$
gaB	$\sigma_a(T)\backslash\sigma_{SBF_+^-}(T) = \Pi_a(T)$	(R)	$p_{00}^a(T) = \pi_{00}(T)$
aB	$\sigma_a(T)\backslash\sigma_{uw}(T) = p_{00}^a(T)$	(aR)	$p_{00}(T) = \pi_{00}^a(T)$

Fig. 1 Various Weyl type theorems

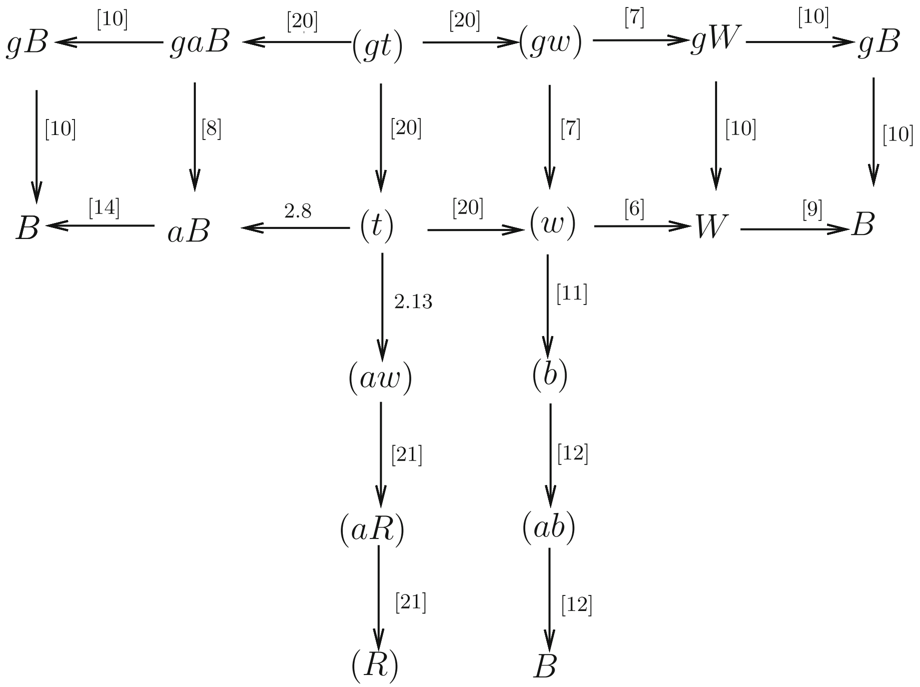

Fig. 2 The relationships between various Weyl type theorems

to the results in the present paper (numbers without brackets) or to the bibliography therein (the numbers in square brackets).

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