RACSAM (2019) 113:1359–1378
https://doi.org/10.1007/s13398-018-0553-5

ORIGINAL PAPER

On the Darboux transform and the solutions of some integrable systems

D. Barrios Rolanía[1]

Received: 22 December 2017 / Accepted: 16 May 2018 / Published online: 23 May 2018

Abstract The relation between the concept of Darboux transform and the full Kostant Toda lattice is analyzed. The main result is Theorem 1, where the discrete Korteweg de Vries equation is used to obtain new solutions of the full Kostant Toda lattice. In addition, an iterative method to obtain the generalized Darboux factorization for a Hessenberg banded matrix is provided, which is the basis to obtain the new solutions.

Keywords Integrable systems · Darboux transforms · Hessenberg banded matrices

Mathematics Subject Classification 39A70 · 30E10 · 15A23

1 Introduction

In [2] some aspects of the relation between the $(p+2)$-banded matrices

$$J = \begin{pmatrix} a_{0,0} & 1 & & & & \\ a_{1,0} & a_{1,1} & 1 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ a_{p,0} & a_{p,1} & \cdots & a_{p,p} & 1 & \\ 0 & a_{p+1,1} & & \ddots & \ddots & \ddots \\ & 0 & \ddots & & & \\ & & \ddots & & & \end{pmatrix} \qquad (1)$$

and the integrable system

$$\dot{a}_{i,j} = \left(a_{i,i} - a_{j,j}\right) a_{i,j} + a_{i+1,j} - a_{i,j-1}\,, \qquad i, j = 0, 1, \ldots \qquad (2)$$

✉ D. Barrios Rolanía
dolores.barrios.rolania@upm.es

1 ETS Ingeniería Civil, Universidad Politécnica de Madrid, C/ Alfonso XII, 3 y 5, 28014 Madrid, Spain

were studied. In particular, a method for constructing solutions of this system was given in the case $p = 2$. This method is based on the extension of the concept of Darboux transform, which can be consulted in [4] for the classical tridiagonal case $p = 1$. Due to the matrix interpretation of this method, the concept of Darboux transform was extended in [2] for an arbitrary $p \in \mathbb{N}$ and a banded matrix J as in (1). However, just in [3] the existence of such a kind of generalized transforms was determined. As a consequence, now we are under the appropriate conditions to generalize the method for constructing solutions of (2), given in [2] in the case $p = 2$, to any $p \in \mathbb{N}$. This is precisely the goal of this paper.

For simplicity of the reading, we recall here some concepts introduced in [2] and used in [3] which will be employed in our work. The system (2) is usually called full Kostant Toda lattice. Here and in the sequel, the dot means differentiation with respect to $t \in \mathbb{R}$. However, in most of the cases we suppress the explicit t-dependence for brevity.

Definition 1 The infinite matrix J is called a solution of (2) if:

1. For each $j = 0, 1 \ldots$ the entries $a_{i,j} = a_{i,j}(t)$, $i = j, j+1, \ldots, j+p$, of J are continuous functions with complex values defined in an open interval $\mathscr{I}_j$ such that
$$\bigcap_{j=0}^{N} \mathscr{I}_j \neq \emptyset \quad \text{for every } N \in \mathbb{N}. \tag{3}$$
2. The entries $a_{i,j}$ of J verify (2).

An important tool for us is the called discrete Korteweg de Vries (KdV) equation,
$$\dot{\gamma}_n = \gamma_n \left(\sum_{i=1}^{p} \gamma_{n+i} - \sum_{i=1}^{p} \gamma_{n-i} \right), \quad n \in \mathbb{N}. \tag{4}$$
This system is an extension of the Volterra lattices studied in [10] and [6]. As in (2), the matrix theory is used to analyze the KdV equations. The matrix associated with this system is
$$\Gamma = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ 0 & \vdots & & & \\ \gamma_1 & 0 & & & \\ 0 & \gamma_2 & \ddots & & \\ & \ddots & \ddots & & \end{pmatrix},$$
with γ_1 in the p-th row. Also we assume that the entries $\gamma_j = \gamma_j(t)$, $j \in \mathbb{N}$, of Γ are continuous functions with complex values defined in the open intervals $\mathscr{O}_j$ such that
$$\bigcap_{j=0}^{N} \mathscr{O}_j \neq \emptyset, \text{ for every } N \in \mathbb{N}. \tag{5}$$

Definition 2 The matrix Γ is called a solution of (4) if the sequence $\{\gamma_n\}$ satisfies (4) and (5).

With respect to the extension of the concept of Darboux transform, the following definition was introduced in [2] and analyzed in [3]. As usual, here and in the sequel M_n denotes the leading principal submatrix of M with size $n \times n$.

Definition 3 Let $B = \left(b_{ij}\right)$, $i, j \in \mathbb{N}$, be a lower Hessenberg $(p+2)$-banded matrix,

$$B = \begin{pmatrix} b_{0,0} & 1 & & & & \\ b_{1,0} & b_{1,1} & 1 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ b_{p,0} & b_{p,1} & \cdots & b_{p,p} & 1 & \\ 0 & b_{p+1,1} & & \ddots & \ddots & \ddots \\ & 0 & \ddots & & & \\ & & \ddots & & & \end{pmatrix} \tag{6}$$

such that $\det(B_n) \neq 0$ for any $n \in \mathbb{N}$. Let L and U be lower and upper triangular matrices, respectively, such that the entries in the diagonal of L are $l_{ii} = 1$ and $B = LU$ is the (unique) LU factorization of B in these conditions. Assume $L = L^{(1)} L^{(2)} \ldots L^{(p)}$, where

$$U = \begin{pmatrix} \gamma_1 & 1 & & \\ & \gamma_{p+2} & 1 & \\ & & \gamma_{2p+3} & \ddots \\ & & & \ddots \end{pmatrix}, \; L^{(i)} = \begin{pmatrix} 1 & & & \\ \gamma_{i+1} & 1 & & \\ & \gamma_{p+i+2} & 1 & \\ & & \gamma_{2p+i+3} & \ddots \\ & & & \ddots \end{pmatrix}, \; i = 1, \ldots, p\,. \tag{7}$$

Then the matrix decomposition

$$L^{(1)} L^{(2)} \cdots L^{(p)} U \tag{8}$$

is called a Darboux factorization of B. Moreover, any circular permutation

$$L^{(i+1)} \cdots L^{(p)} U L^{(1)} \cdots L^{(i)}, \quad i = 1, 2, \ldots, p\,, \tag{9}$$

of (8) is called a Darboux transform of B. (We understand $U L^{(1)} \cdots L^{(p)}$ in (9) when $i = p$.)

Notice that the Hessenberg banded matrix (6) in Definition 3, in general, does not depend on $t \in \mathbb{R}$. However, our object is to obtain some solutions of (2) using the Darboux transforms of $J - CI$ under certain conditions for $C \in \mathbb{C}$. In other words, we need to work with banded matrices $J - CI$ whose entries are functions verifying (2)–(3). Our main result is the following.

Theorem 1 *Let J be a $(p+2)$-banded matrix as in (1). Assume that J is a solution of (2) verifying $a_{p+i,i} \neq 0$, $i = 0, 1, \ldots,$ and let $C \in \mathbb{C}$ be such that $\det(J_n - C I_n) \neq 0$ for any $n \in \mathbb{N}$. Then there exist p solutions $J^{(1)}, \ldots, J^{(p)}$ of (2) such that the following relations hold.*

$$J^{(i)} = CI + L^{(i+1)} \cdots L^{(p)} U L^{(1)} \cdots L^{(i)}, \quad i = 0, 1, 2, \ldots, p \tag{10}$$

(assuming $J^{(0)} = J$), where $L^{(i)}$, $i = 1, 2, \ldots, p$, and U are as in (7). Moreover the entries of U, $L^{(1)}, \cdots, L^{(p)}$ provide the sequence $\{\gamma_n\}$, which defines a solution Γ of (4).

Along the paper it is convenient to have another expression of (10), which is obtained when each entry of the matrix $J^{(i)}$ is given in terms of the entries of matrices of U, $L^{(1)}, \cdots, L^{(p)}$. This is, if we write

$$J^{(j)} = \begin{pmatrix} a_{0,0}^{(j)} & 1 & & & & & \\ a_{1,0}^{(j)} & a_{1,1}^{(j)} & 1 & & & & \\ \vdots & \vdots & \ddots & \ddots & & & \\ a_{p,0}^{(j)} & a_{p,1}^{(j)} & \cdots & a_{p,p}^{(j)} & 1 & & \\ 0 & a_{p+1,1}^{(j)} & & \ddots & \ddots & \ddots & \\ & 0 & \ddots & & & & \\ & & \ddots & & & & \end{pmatrix}, \quad j = 0, 1, \ldots, p, \tag{11}$$

then for each $j = 0, 1, \ldots, p$, from the products on the right hand side of (10) we get

$$a_{i,i}^{(j)} = C + \sum_{s=j+1}^{j+p+1} \gamma_{(i-1)p+i+s}, \tag{12}$$

$$a_{i+k,i}^{(j)} = \sum_{E_k^{(j)}} \gamma_{(i-1)p+i_1+i} \gamma_{ip+i_2+i} \cdots \gamma_{(k+i-1)p+i_{k+1}+i}, \\ i = 0, 1, \ldots, \quad k = 1, 2, \ldots, p. \tag{13}$$

(notice that $J^{(0)} = J$ and, consequently, $a_{s,r}^{(0)} = a_{s,r}$). The sum in (13) is extended to the set of indices $E_k^{(j)}$ defined as

$$E_k^{(j)} = \{(i_1, \ldots, i_{k+1}) : j + k + 1 \leq i_{k+1} \leq \cdots \leq i_1 \leq j + p + 1\}. \tag{14}$$

We are interested in the solutions of (2) and (4), which are connected by (12)–(13). Therefore, in the following we call *Bäcklund transformations* to relations (12)–(13) (and by extension (10)). In fact, in the literature of integrable systems this term is used to appoint relations between solutions of different systems (see [6]), being (12)–(13) extensions of this kind of relations.

The connection between Darboux transforms and banded matrices is a classical topic in the study of integrable systems and differential equations (see for instance [9], [11]). We underline that our concepts and results for the Darboux transforms not only extend those corresponding to tridiagonal matrices but also those established in the past for banded matrices (see [1]). In particular, notice the relevance of the Darboux factorization (8) for obtaining the new solutions $J^{(i)}$, $i = 1, \ldots, p$, given in (10). In this work we study this problem, given a constructive approach to arrive at (8) from J, which is another contribution of this paper.

In Sect. 2 some tools for our work are presented and the main auxiliary results are introduced. In Sect. 3 an iterative method for obtaining the Darboux factorization (8) is presented. Finally, the proof of Theorem 1 appears in Sect. 4.

2 Auxiliary results

An important tool in our approach is the sequence of polynomials $\{P_n(z)\} = \{P_n(t, z)\}$, $n \in \mathbb{N}$, associated with the matrix J. This family is defined by the following recurrence relation.

$$\left.\begin{array}{r}\sum_{i=n-p}^{n-1} a_{n,i} P_i(z) + (a_{n,n} - z) P_n(z) + P_{n+1}(z) = 0, \quad n = 0, 1, \dots \\ P_0(z) \equiv 1\,, \quad P_{-1}(z) = \cdots = P_{-p}(z) = 0\,.\end{array}\right\} \tag{15}$$

If J is a solution of (2) then each polynomial $P_n = P_n(t, z)$ is a continuous function on t. Furthermore, in Lemma 2 of [2] was proved

$$\dot{P}_n(z) = -\sum_{i=n-p}^{n-1} a_{n,i} P_{i(z)}\,, \quad n = 0, 1, \dots\,. \tag{16}$$

As a consequence of the above comments, for each $n = 0, 1, \dots$ we have that $\dot{P}_n(z)$ is also a continuous function on t in some open interval of $\mathbb{R}$. Moreover, the following expression for the derivative is straightly obtained from (15) and (16).

$$\dot{P}_n(z) = (a_{n,n} - z) P_n(z) + P_{n+1}(z)\,, \quad n = 0, 1, \dots\,. \tag{17}$$

On the other hand, it is well known that for each matrix J and $C \in \mathbb{C}$ in the conditions of Theorem 1 there exists the LU factorization of $J - CI$. This is, there exists a banded lower triangular matrix

$$L = \begin{pmatrix} 1 & & & & & \\ l_{1,1} & 1 & & & & \\ \vdots & \ddots & \ddots & & & \\ l_{p,1} & l_{p,2} & \dots & 1 & & \\ 0 & l_{p+1,2} & \ddots & & \ddots & \\ & 0 & \ddots & \ddots & \ddots & \end{pmatrix}, \tag{18}$$

and there exists an upper triangular matrix $U = U(t)$ as in (7) such that

$$J - CI = LU, \tag{19}$$

being

$$J_n - CI_n = L_n U_n \tag{20}$$

for each $n \in \mathbb{N}$ (see for instance [5]).

The following auxiliary result also will be used in the proof of Theorem 1.

Lemma 1 *In the above conditions, the entries* γ_{np+n+1} *,* $n = 0, 1, \dots,$ *of* U *verify (4).*

Proof It is obvious that the recurrence relation (15) can be rewritten as

$$(J_n - zI_n) \begin{pmatrix} P_0(z) \\ P_1(z) \\ \vdots \\ P_{n-1}(z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -P_n(z) \end{pmatrix}.$$

In particular, for $z = C$ from (20) we have

$$L_n U_n \begin{pmatrix} P_0(C) \\ P_1(C) \\ \vdots \\ P_{n-1}(C) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -P_n(C) \end{pmatrix}.$$

Then, using the fact that L is a triangular matrix whose diagonal entries are 1,

$$U_n \begin{pmatrix} P_0(C) \\ P_1(C) \\ \vdots \\ P_{n-1}(C) \end{pmatrix} = L_n^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -P_n(C) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -P_n(C) \end{pmatrix}.$$

This is, taking into account the structure of U [see (7)],

$$\gamma_{np+n+1} = -\frac{P_{n+1}(C)}{P_n(C)}, \quad n = 0, 1, \ldots \tag{21}$$

Taking derivatives in (21),

$$\dot{\gamma}_{np+n+1} = -\frac{P_n(C)}{P_{n-1}(C)} \left(\frac{\dot{P}_n(C)}{P_n(C)} - \frac{\dot{P}_{n-1}(C)}{P_{n-1}(C)} \right).$$

From this and (17),

$$\dot{\gamma}_{np+n+1} = \gamma_{np+n+1} \left(a_{n,n} - a_{n-1,n-1} - \gamma_{np+n+1} + \gamma_{(n-1)p+n} \right).$$

Then since (12) (with $j = 0$) we arrive at (4). □

The next result guarantees the existence of the Darboux factorization, which is used in the proof of Theorem 1.

Lemma 2 (Theorem 1 in [3]) *Let L be a lower triangular matrix as in (18) with complex entries, such that $l_{p+j,j+1} \neq 0$ for each $j = 0, 1, \ldots$ Then there exists a set of $p(p-1)/2$ complex numbers*

$$\left.\begin{array}{lllll} \gamma_2, & \gamma_{p+3} & \cdots & \cdots & \gamma_{(p-1)p}, \\ \gamma_3, & \gamma_{p+4} & \cdots & \gamma_{(p-2)p}, & \\ \vdots & \vdots & \iddots & & \\ \gamma_{p-1}, & \gamma_{2p}, & & & \\ \gamma_p & & & & \end{array}\right\} \tag{22}$$

as well as p triangular matrices $L^{(i)}$, $i = 1, \ldots, p$, as in (7) such that

$$L = L^{(1)} L^{(2)} \cdots L^{(p)}, \tag{23}$$

where $\gamma_{k(p+1)+i+1} \neq 0$ for $i = 1, 2, \ldots, p$ and $k = 0, 1, \ldots$ Moreover, the factorization (23) is unique for each fixed set of points (22).

Lemma 2 in [3] was obtained as a consequence of the following result. Here, the necessary conditions to obtain the set (22) are given in an explicit way.

Lemma 3 (Theorem 2 in [3]) *Let us consider a $(p+1)$-banded lower triangular matrix L as in (18) such that $l_{p+j,j+1} \neq 0$ for each $j = 0, 1, \ldots$ Assume $\alpha_1, \alpha_2, \ldots, \alpha_{p-1} \in \mathbb{C}$ such that*

$$\sum_{s=0}^{p-1} (-1)^s \alpha_{p-s} \alpha_{p-s+1} \cdots \alpha_{p-1} C_k^{(s)} \neq 0, \quad \textit{for all } k = 1, 2, \ldots, \tag{24}$$

where $C_1^{(s)} := l_{p-s-1,1}$ *and*

$$C_k^{(s)} := \begin{vmatrix} l_{p-s-1,1} & l_{p-s-1,2} & \cdots & \cdots & l_{p-s-1,k} \\ l_{p,1} & l_{p,2} & \ddots & \ddots & l_{p,k} \\ 0 & l_{p+1,2} & \ddots & \ddots & l_{p+1,k} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & l_{p+k-2,k-1} & l_{p+k-2,k} \end{vmatrix}, \quad k \geq 2, \tag{25}$$

for each $s = 0, 1, \ldots, p-1$. *(We understand* $\alpha_{p-s}\alpha_{p-s+1}\cdots\alpha_{p-1} = 1$ *for* $s = 0$ *and* $l_{i,j} = 0$ *for* $j > i+1$.*) Then there exist a bi-diagonal matrix*

$$D^{(1)} = \begin{pmatrix} 1 & & & & \\ \alpha_1 & 1 & & & \\ & \alpha_2 & 1 & & \\ & & \alpha_3 & \ddots & \\ & & & \ddots & \end{pmatrix} \tag{26}$$

and a p*-banded lower triangular matrix*

$$A = \begin{pmatrix} 1 & & & & & \\ \delta_{2,1} & 1 & & & & \\ \vdots & \ddots & \ddots & & & \\ \delta_{p,1} & \delta_{p,2} & \ldots & 1 & & \\ 0 & \delta_{p+1,2} & \ddots & & \ddots & \\ & 0 & \ddots & \ddots & \ddots & \end{pmatrix} \tag{27}$$

such that $\delta_{p+k-1,k} \neq 0$, $k = 1, 2, \ldots$, *and*

$$L = D^{(1)} A\,. \tag{28}$$

Moreover, if the p *entries* $\alpha_1, \alpha_2, \ldots, \alpha_{p-1} \in \mathbb{C}$ *of* $D^{(1)}$ *are fixed verifying (24), then (28) is the unique factorization of* L *in these conditions.*

Lemma 2 can be obtained as a corollary of Lemma 3. The key of this fact is the next lemma, which we use in the proof of our main result.

Lemma 4 *For* $s \in \{0, 1, \ldots, p-2\}$ *and* $N \in \mathbb{N}$, *let us consider the triangular matrix*

$$T^{(s)} = \begin{pmatrix} 1 & & & & & \\ m_{1,1}^{(s)} & 1 & & & & \\ \vdots & \ddots & \ddots & & & \\ m_{p-s,1}^{(s)} & m_{p-s,2}^{(s)} & \ldots & 1 & & \\ 0 & m_{p-s+1,2}^{(s)} & \ddots & & \ddots & \\ & 0 & \ddots & \ddots & \ddots & \end{pmatrix}, \tag{29}$$

where $m_{p-s+j,j+1}^{(s)} \neq 0$, $j = 0, 1, \ldots$ *Then there exist* $p-s-1$ *complex values*

$$\alpha_i^{(s)} \neq 0, \quad i = 1, 2, \ldots, p-s-1, \tag{30}$$

such that

$$\sum_{j=0}^{p-s-1} (-1)^j \alpha_{p-s-j}^{(s)} \alpha_{p-s-j+1}^{(s)} \cdots \alpha_{p-s-1}^{(s)} R_k^{(s,j)} \neq 0 \tag{31}$$

for each $k = 1, 2, \ldots, N$, *where* $R_1^{(s,r)} = m_{p-s-r-1,1}^{(s)}$ *and*

$$R_k^{(s,r)} := \begin{vmatrix} m_{p-s-r-1,1}^{(s)} & m_{p-s-r-1,2}^{(s)} & \cdots & \cdots & m_{p-s-r-1,k}^{(s)} \\ m_{p-s,1}^{(s)} & m_{p-s,2}^{(s)} & \cdots & \cdots & m_{p-s,k}^{(s)} \\ 0 & m_{p-s+1,2}^{(s)} & \ddots & \ddots & m_{p-s+1,k}^{(s)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & m_{p-s+k-2,k-1}^{(s)} & m_{p-s+k-2,k}^{(s)} \end{vmatrix}, \quad k \geq 2 . \tag{32}$$

Proof For the complex numbers $R_k^{(s,r)}$, $r = 0, \ldots, p-s-1$, given in (32) and $k = 1, 2, \ldots, N$, we consider the hyperplanes π_k in $\mathbb{R}^{p-s-1}$ given by the equation of the form

$$R_k^{(s,1)} x_1 + R_k^{(s,2)} x_2 + \cdots + R_k^{(s,p-s-1)} x_{p-s-1} = R_k^{(s,0)} . \tag{33}$$

Also we consider the hyperplanes Ψ_i of equation $x_i = 0$, $i = 1, \ldots, p-s-1$. We define $\pi := \bigcup_{k=1}^{N} \pi_k$ and $\Psi := \bigcup_{i=1}^{p-s-1} \Psi_i$. Then, if μ is the Lebesgue measure in $\mathbb{R}^{p-s-1}$, it is well known that $\mu(\pi \cup \Psi) = 0$ (see [12] for details). As a consequence, there exists a nonnumerable set of points $X \in \mathbb{R}^{p-s-1}$ such that $X = (x_1, \ldots, x_{p-s-1}) \notin \pi \cup \Psi$. We choose one of these points and we define iteratively

$$\alpha_{p-s-j}^{(s)} = \begin{cases} x_1 & , \quad \text{if } j = 1 , \\ \dfrac{(-1)^{j+1} x_j}{\alpha_{p-s-j+1}^{(s)} \alpha_{p-s-j+2}^{(s)} \cdots \alpha_{p-s-1}^{(s)}} & , \quad \text{if } j = 2, \ldots, p-s-1 . \end{cases} \tag{34}$$

Note that $\alpha_1^{(s)}, \ldots, \alpha_{p-s-1}^{(s)}$ are well defined because $X \notin \Psi$ and, consequently, $x_j \neq 0$ for each $j = 1, \ldots, p-s-1$. Therefore $\alpha_i^{(s)} \neq 0$, $i = 1, \ldots, p-s-1$.

From (33) and (34) we arrive at (31). □

3 Construction of the matrices $L^{(i)}$, $i = 1, \ldots, p$

Let L be a matrix as in (18) verifying $l_{p+j,j+1} \neq 0$, $j = 0, 1, \ldots$ Then from the set of data (22) it is possible to build $L^{(1)}, \ldots, L^{(p)}$ which are the factors in (23). With this purpose we will construct Table 1 where, on each row, the entries of these matrices will be given. From the Backlünd transformations (13) (for $j = 0$) it is easy to arrive at

$$\delta_k^{(i)} \gamma_{(k+i+1)p+i} = a_{k+i+1,i-1} - \sum_{\widetilde{E}_{k+2}^{(0)}} \gamma_{(i-2)p+i+i_1-1} \gamma_{(i-1)p+i+i_2-1} \cdots \gamma_{(k+i)p+i+i_{k+3}-1}$$
$$i \in \mathbb{N}, \quad k = -1, 0, \ldots, p-2 , \tag{35}$$

where

$$\delta_k^{(i)} = \gamma_{(i-1)p+i} \gamma_{ip+i} \cdots \gamma_{(k+i)p+i} \tag{36}$$

Table 1 Constructing $L^{(1)}, \ldots, L^{(p)}$

U	γ_1	γ_{p+2}	$\cdots$	$\gamma_{mp+(m+1)}$	$\cdots$	$\cdots$	$\cdots$	$\gamma_{(p+i-1)p+(i-1)} \cdots$
$L^{(1)}$	γ_2	γ_{p+3}	$\cdots$	$\gamma_{mp+(m+2)}$	$\cdots$	$\cdots$	$\cdots$	$\gamma_{(p+i-1)p+i} \cdots$
$L^{(2)}$	γ_3	γ_{p+4}	$\cdots$	$\gamma_{mp+(m+3)}$	$\cdots$	$\cdots$	$\gamma_{(p+i-2)p+i} \cdots$	
$\vdots$	$\vdots$	$\vdots$		$\vdots$			$\iddots$	
$L^{(p-m-1)}$	γ_{p-m}	γ_{2p-m+1}	$\cdots$	$\gamma_{(m+1)p}$	$\cdots$	$\gamma_{(m+i+1)p+i}$		
$\vdots$	$\vdots$	$\vdots$			$\iddots$			
$L^{(p-2)}$	γ_{p-1}	γ_{2p}	$\cdots$	$\gamma_{(i+2)p+i}$				
$L^{(p-1)}$	γ_p	$\cdots$	$\gamma_{(i+1)p+i}$					
$L^{(p)}$	$\cdots$	γ_{ip+i}						

and

$$\widetilde{E}^{(0)}_{k+2} = \{(i_1, \dots, i_{k+3}) : k+3 \le i_{k+3} \le \cdots \le i_1 \le p+1\,, i_{k+3} < p+1\}. \qquad (37)$$

We remark that the subsequence $\{\gamma_{mp+m+1}\}$, $m \in \mathbb{N}$, in (35)–(36) is known and given in (21). This subsequence of $\{\gamma_n\}$ constitutes the first row of Table 1 and defines the matrix U. In order to complete the rest of the γ's, the main idea is to obtain $\gamma_{(k+i+1)p+i}$ for $k = -1, 0, \dots, p-2$, iteratively for each fixed $i \in \mathbb{N}$. Firstly we take $i = 1$ in (35). This is,

$$\delta^{(1)}_k \gamma_{(k+2)p+1} = a_{k+2,0} - \sum_{\widetilde{E}^{(0)}_{k+2}} \gamma_{-p+i_1}\gamma_{i_2}\gamma_{p+i_3}\cdots\gamma_{(k+1)p+i_{k+3}}\,,$$
$$k = -1, 0, 1, \dots, p-2\,, \qquad (38)$$

and $\delta^{(1)}_k = \gamma_1\gamma_{p+1}\gamma_{2p+1}\cdots\gamma_{(k+1)p+1}$. We note that $\gamma_{-p+i_1} = 0$ when $i_1 \ne p+1$ in the sum of (38). Therefore this sum can be rewritten extended to $\widetilde{\widetilde{E}}_{k+2} := \{(i_2, \dots, i_{k+3}) : k+3 \le i_{k+3} \le \cdots \le i_2 \le p+1\,, i_{k+3} < p+1\}$. This is,

$$\delta^{(1)}_k \gamma_{(k+2)p+1} = a_{k+2,0} - \gamma_1 \sum_{\widetilde{\widetilde{E}}_{k+2}} \gamma_{i_2}\gamma_{p+i_3}\cdots\gamma_{(k+1)p+i_{k+3}}\,. \qquad (39)$$

For $k = -1$ we have $\delta^{(1)}_{-1} = \gamma_1$ and since (39)

$$\gamma_{p+1} = \frac{a_{1,0}}{\gamma_1} - \sum_{j=2}^{p} \gamma_j\,.$$

Now we can calculate $\delta^{(1)}_0 = \gamma_1\gamma_{p+1}$. Taking $k = 0$ in (39),

$$\gamma_{2p+1} = \frac{a_{2,0}}{\delta^{(1)}_0} - \frac{1}{\gamma_{p+1}} \sum_{\widetilde{\widetilde{E}}_2} \gamma_{i_2}\gamma_{p+i_3}\,.$$

In this way, in p steps given for $k = -1, 0, \dots, p-2$ we obtain the entries

$$\gamma_{p+1},\ \gamma_{2p+1},\ \dots,\ \gamma_{p^2+1},$$

corresponding to the matrices $L^{(p)}$, $L^{(p-1)}, \dots, L^{(1)}$ respectively (see (7)). These values of γ's constitute the secondary diagonal in Table 1 for $i = 1$. Note that in the step $k+2$ it is possible to obtain $\gamma_{(k+2)p+1}$ in (39) because the entries $\gamma_{i_2},\ \gamma_{p+i_3},\ \dots,\ \gamma_{(k+1)p+i_{k+3}}$ in the right hand side are in the upper triangular part of Table 1, over the secondary diagonal. In fact $\gamma_{sp+i_{s+2}}$ is in the column $s+1$, including the secondary diagonal, for $s = 0, 1, \dots, k$, because $i_{s+2} \le p+1$. The last factor, $\gamma_{(k+1)p+i_{k+3}}$, is in the column $k+2$ but $i_{k+3} \le p$ and this factor is not in the secondary diagonal, whose entry $\gamma_{(k+1)p+(p+1)}$ is being obtained in this step.

We will iterate the above procedure for constructing any parallel diagonal

$$\gamma_{ip+i},\ \gamma_{(i+1)p+i},\ \dots,\ \gamma_{(p+i-1)p+i}$$

in Table 1. Assume that we have constructed the parallel diagonals

$$\gamma_{sp+s},\ \gamma_{(s+1)p+s},\ \dots,\ \gamma_{(p+s-1)p+s}\,, \quad s = 1, 2, \dots, i-1\,. \qquad (40)$$

Then we take $k = -1$ in (36) and we have $\delta^{(i)}_{-1} = \gamma_{(i-1)p+i}$, which is an entry of U. Therefore γ_{ip+i} can be obtained from (35) when $k = -1$. In general, for each

$k \in \{-1, 0, \ldots, p-2\}$ we can obtain $\gamma_{(k+i+1)p+i}$ from (35), because for this value of k we have that $\delta_k^{(i)}$ is known in the previous step. As in the case $i = 1$, the factors $\gamma_{(i-2)p+i+i_1-1}$, $\gamma_{(i-1)p+i+i_2-1}, \ldots, \gamma_{(k+i)p+i+i_{k+3}-1}$ on the right hand side of (35) are the entries of matrices $L^{(j)}$, $j = 1, \ldots, p$, corresponding to the upper triangular part of Table 1, which are known from the previous steps. In this form, the parallel diagonal $\gamma_{(k+i+1)p+i}$, $k = -1, 0, \ldots, p-2$, is obtained, corresponding with the value $s = i$ in (40).

4 Proof of Theorem 1

If J and C satisfy the conditions of the statement in Theorem 1, then (19) holds. Moreover for each $N \in \mathbb{N}$ there exists some open interval $\mathscr{I}_N \neq \emptyset$, $\mathscr{I}_N \subset \mathbb{R}$, such that the entries $\gamma_{ip+i+1}(t)$ of U (see (7)) and the entries $l_{i,j}(t)$ of L, $i, j \leq N$, (see (18)) are continuous functions on $\mathscr{I}_N$. In fact, these entries can be expressed in terms of products and sums of the entries of $J - CI$, which are continuous functions (see [8] for instance). We assume a fixed $N \geq p-1$ in the sequel for convenience, and we let $t_0 \in \mathscr{I}_N$.

4.1 Factorization of L

In [3], Lemma 2 and the factorization (23) were proved for a fixed matrix L that could not depend on $t \in \mathbb{R}$. Also Lemma 3 was proved in [3] for this kind of fixed matrices. Here, we need to extend these results for our matrix L, which depends on t. In other words, we want to prove that the first entries of the factors $L^{(i)}$, $i = 1, \ldots, p$, in (23) are defined in some open interval $\mathscr{I} \subset \mathbb{R}$ such that $t_0 \in \mathscr{I}$. With this purpose we will apply Lemma 4 successively for $s = 0, 1, \ldots, p-2$.

First we take $s = 0$ and $T^{(0)} = L(t_0)$ in (29). Then there exist $\alpha_1^{(0)}, \ldots, \alpha_{p-1}^{(0)}$ verifying (31), this is

$$\sum_{j=0}^{p-1} (-1)^j \alpha_{p-j}^{(0)} \alpha_{p-j+1}^{(0)} \ldots \alpha_{p-1}^{(0)} R_k^{(0,j)}(t_0) \neq 0\,, \quad k = 1, \ldots, N. \tag{41}$$

For each $i \in \{1, 2, \ldots, p-1\}$ we consider the following initial value problem,

$$\left.\begin{array}{rcl} \dot{\gamma}_{(i-1)p+(i+1)}(t) & = & \gamma_{(i-1)p+(i+1)}(t)\left(D_i^{(0)}(t) - \gamma_{(i-1)p+(i+1)}(t)\right) \\ \gamma_{(i-1)p+(i+1)}(t_0) & = & \alpha_i^{(0)} \end{array}\right\} \tag{42}$$

being $D_i^{(0)} = a_{i,i} - a_{i-1,i-1} + \gamma_{(i-2)p+i}$. We analyze (42) iteratively. If $i = 1$ then $D_1^{(0)} = a_{1,1} - a_{0,0}$ and it is well known that (42) has a unique solution $\gamma_2(t)$ in some open interval $\mathscr{I}_1^{(0)}$ containing t_0 (see [7] for details). If $i = 2$ then $D_2^{(0)} = a_{2,2} - a_{1,1} + \gamma_2$ is defined in $\mathscr{I}_1^{(0)}$ and the solution γ_{p+3} is defined in some open interval $\mathscr{I}_2^{(0)}$ containing t_0, being $\mathscr{I}_2^{(0)} \subset \mathscr{I}_1^{(0)}$. Iterating this procedure, suppose that $\gamma_2, \gamma_{p+3}, \ldots, \gamma_{(j-1)p+(j+1)}$, $j < p-1$, are the solutions of (42) which are continuous functions defined in some open interval $\mathscr{I}_j^{(0)}$ containing t_0. If $i = j+1$ then $D_i^{(0)} = a_{j+1,j+1} - a_{j,j} + \gamma_{(j-1)p+(j+1)}$ is a continuous function defined in such interval $\mathscr{I}_j^{(0)}$. Therefore (42) has a solution $\gamma_{jp+(j+2)}$ in these conditions, which is defined in some interval $\mathscr{I}_{j+1}^{(0)} \subset \mathscr{I}_j^{(0)}$. Thus we have proved the existence of $p-1$ continuous functions

$$\gamma_{(i-1)p+(i+1)}\,, \quad i = 1, 2, \ldots, p-1\,, \tag{43}$$

being these functions the respective solutions of (42) in some open interval $\mathscr{I}^{(0)}$ containing t_0. We take $\mathscr{I}^{(0)} := \mathscr{I}^{(0)}_{p-1}$.

Note that in (43) we have obtained the first row of (22). These functions, defined in the interval $\mathscr{I}^{(0)}$, are the first $p-1$ entries in the subdiagonal of $L^{(1)}(t)$.

(41) can be written as

$$\sum_{j=0}^{p-1}(-1)^j \gamma_{(p-j-1)p+(p-j+1)}(t_0)\cdots\gamma_{(p-2)p+p}(t_0)R_k^{(0,j)}(t_0) \neq 0\,, \quad k = 1, \ldots, N. \tag{44}$$

Due to the continuity of functions (43) and $R_k^{(0,j)}$, it is possible to choose $\mathscr{I}^{(0)}$ sufficiently small such that (44) is satisfied for each $t \in \mathscr{I}^{(0)}$. This is,

$$\sum_{j=0}^{p-1}(-1)^j \gamma_{(p-j-1)p+(p-j+1)}(t)\cdots\gamma_{(p-2)p+p}(t)R_k^{(0,j)}(t) \neq 0\,, \quad k = 1, \ldots, N,$$

for $t \in \mathscr{I}^{(0)}$.

In these conditions we apply Lemma 3 and we see that $L(t)$ is factorized as in (28) for each $t \in \mathscr{I}^{(0)}$. This is,

$$L(t) = L^{(1)}(t)T^{(1)}(t)\,, \quad t \in \mathscr{I}^{(0)}\,, \tag{45}$$

where $T^{(1)}$ is given in (29) for $s = 1$ satisfying $m^{(1)}_{p-1+j,j+1}(t) \neq 0\,,\ t \in \mathscr{I}^{(0)}\,,\ j = 0, 1, \ldots,$ and $L^{(1)}$ has the structure given in (7). We underline that the factorization (45) should be understood in a formal sense, because we need to fix $N \in \mathbb{N}$ to have the first entries of these matrices defined in some nonempty open interval. In other words, it is possible that the entries for the infinite matrices $L^{(1)}(t)$ and $T^{(1)}(t)$ are defined in $t = t_0$ but they are not simultaneously defined in a nonempty open interval. This remark should be applied in the sequel to matrices depending on t, as in (46) and (47).

We seek to prove the existence of the factors $L^{(i)}(t)\,,\ i = 1, \ldots, p$, satisfying

$$L(t) = L^{(1)}(t)\cdots L^{(p)}(t) \tag{46}$$

whose first entries are continuous functions in some open interval containing t_0. We proceed by induction. Let $r \in \mathbb{N}$ be satisfying $0 \leq r < p-2$. For $s = 0, 1, \ldots, r$, we assume

$$T^{(s)}(t) = L^{(s+1)}(t)T^{(s+1)}(t)\,, \quad t \in \mathscr{I}^{(s)}\,, \tag{47}$$

where $\mathscr{I}^{(s)}$ is an open interval and

$$t_0 \in \mathscr{I}^{(r)} \subseteq \cdots \mathscr{I}^{(1)} \subseteq \mathscr{I}^{(0)}\,. \tag{48}$$

In (47) we assume $T^{(0)} = L$ and $T^{(s+1)}\,,\ s = 0, 1, \ldots, r$, as in (29) such that

$$m^{(s)}_{p-s+j,j+1} \neq 0\,, \quad j = 0, 1, \ldots \tag{49}$$

Also we assume the matrices $L^{(s+1)}(t)$, $s = 0, \dots, r$, with the structure given in (7). This is,

$$L^{(s+1)}(t) = \begin{pmatrix} 1 & & & & \\ \gamma_{s+2}(t) & 1 & & & \\ & \gamma_{p+s+3}(t) & 1 & & \\ & & \gamma_{2p+s+4}(t) & \ddots & \\ & & & \ddots & \end{pmatrix}, \tag{50}$$

being $\gamma_{s+2}(t), \dots, \gamma_{(p-s-1)p}(t)$ continuous functions in $\mathscr{I}^{(s)}$ such that

$$\dot{\gamma}_{(i-1)p+(i+s+1)}(t) = \gamma_{(i-1)p+(i+s+1)}(t)\left(D_i^{(s)}(t) - \gamma_{(i-1)p+(i+s+1)}(t)\right) \tag{51}$$

[see row $s+1$ in (22)], and

$$D_i^{(s)} = a_{i,i} - a_{i-1,i-1} - 2\sum_{j=1}^{s}\gamma_{(i-1)p+(i+j)} + \sum_{j=1}^{s}\gamma_{ip+(i+j+1)} + \sum_{j=0}^{s}\gamma_{(i-2)p+(i+j)}\,. \tag{52}$$

We have proved the factorization (47) in the above conditions for $s = 0$ [see (45)]. We want to prove that (47)–(52) are satisfied for $s = r+1$ in some interval $\mathscr{I}^{(r+1)}$ such that $t_0 \in \mathscr{I}^{(r+1)} \subseteq \mathscr{I}^{(r)}$. Since (49) and Lemma 4 (for $s = r+1$) we know that there exist $p-r-2$ complex numbers

$$\alpha_i^{(r+1)} \neq 0\,, \quad i = 1, 2, \dots, p-r-2\,,$$

satisfying

$$\sum_{j=0}^{p-r-2}(-1)^j\alpha_{p-r-j-1}^{(r+1)}\alpha_{p-r-j}^{(r+1)}\cdots\alpha_{p-r-2}^{(r+1)}R_k^{(r+1,j)} \neq 0\,, \quad k = 1, \dots, N\,. \tag{53}$$

Moreover we define

$$\alpha_i^{(s)} := \gamma_{(i-1)p+i+s+1}(t_0)\,, \quad s = 1, 2, \dots, r\,, \quad i = 1, 2, \dots, p-s-1\,. \tag{54}$$

We consider the following initial value problem for each $s = 1, 2, \dots, r+1$ and $i = 1, 2, \dots, p-s-1$,

$$\left.\begin{aligned} \dot{\gamma}_{(i-1)p+i+s+1}(t) &= \gamma_{(i-1)p+i+s+1}(t)\left(D_i^{(s)}(t) - \gamma_{(i-1)p+i+s+1}(t)\right) \\ \gamma_{(i-1)p+i+s+1}(t_0) &= \alpha_i^{(s)} \end{aligned}\right\} \tag{55}$$

where $D_i^{(s)}$ is given in (52). Since (42), (51) and (54), the continuous functions

$$\gamma_{s+2}(t),\ \dots, \gamma_{(p-s-1)p}(t)\,,\ t \in \mathscr{I}^{(s)}\,,\ s = 0, \dots, r\,,$$

are the solutions of (55) for $i = 1, 2, \dots, p-s-1$ respectively. We study this initial value problem when $s = r+1$ taking $i = 1, 2, \dots, p-r-2$. First, if $i = 1$ we have that

$$D_1^{(r+1)} = a_{1,1} - a_{0,0} - 2\sum_{j=1}^{r+1}\gamma_{j+1} + \sum_{j=1}^{r+1}\gamma_{p+j+2}$$

is a continuous function defined in $\mathscr{I}^{(r)}$. Then there exists a solution $\gamma_{r+3}(t)$ of (55) (with $i = 1,\ s = r + 1$), which is a continuous function in some open interval $\mathscr{I}_1^{(r+1)}$ containing t_0 such that $\mathscr{I}_1^{(r+1)} \subseteq \mathscr{I}^{(r)}$. Iterating the procedure, we suppose $\gamma_{(i-1)p+i+s+1}$ solutions of (55) when $s = r + 1$ and $i = 1, 2, \ldots, \widetilde{i}$, being $\widetilde{i} < p - r - 2$. Then the continuous functions

$$\gamma_{(i-1)p+i+r+2}(t)\,, \quad i = 1, 2, \ldots, \widetilde{i}\,,$$

are defined in some open interval $\mathscr{I}_{\widetilde{i}}^{(r+1)}$ containing t_0, being $\mathscr{I}_{\widetilde{i}}^{(r+1)} \subseteq \mathscr{I}_{\widetilde{i}-1}^{(r+1)} \subseteq \cdots \subseteq \mathscr{I}_1^{(r+1)}$. We recall that also the functions

$$\gamma_{(i-1)p+i+s+1}(t)\,, \quad i = 1, 2, \ldots, p - s - 1\,, \quad s = 0, 1, \ldots, r,$$

are determined in the previous steps. Therefore, $D_{\widetilde{i}+1}^{(r+1)}$ is a continuous function defined in some open interval $\mathscr{I}_{\widetilde{i}+1}^{(r+1)}$ such that $t_0 \in \mathscr{I}_{\widetilde{i}+1}^{(r+1)} \subseteq \mathscr{I}_{\widetilde{i}}^{(r+1)}$. Then it is possible to set that (55), for $i = \widetilde{i} + 1$ and $s = r + 1$, also has a solution in the above conditions. Thus for $s = r + 1$ the continuous functions

$$\gamma_{(i-1)p+i+r+2}(t)\,, \quad i = 1, 2, \ldots, p - r - 2\,, \tag{56}$$

are defined in some open interval, namely $\mathscr{I}^{(r+1)} := \mathscr{I}_{p-r-2}^{(r+1)}$, containing t_0. Note that $\mathscr{I}^{(r+1)} \subseteq \mathscr{I}^{(r)}$. Moreover, from (53) and the continuity of the functions (56), we can assume $\mathscr{I}^{(r+1)}$ sufficiently small such that

$$\sum_{j=0}^{p-r-2} (-1)^j \gamma_{(p-r-j-2)p+p-j}(t)\gamma_{(p-r-j-1)p+p-j+1}(t) \cdots \gamma_{(p-r-2)p}(t) R_k^{(r+1,j)}(t) \neq 0\,,$$
$$k = 1, \ldots, N\,, \quad t \in \mathscr{I}^{(r+1)}\,.$$

Therefore we apply Lemma 3 (substituting p by $p-r-1$) and we arrive at (47) for $s = r+1$. Consequently, (47) and the conditions given in (48)–(52) are satisfied for $s = 0, 1, \ldots, p-2$. In particular,

$$T^{(p-2)}(t) = L^{(p-1)}(t)T^{(p-1)}(t)\,, \quad t \in \mathscr{I}^{(p-2)}\,,$$

where $L^{(p)} := T^{(p-1)}$ is a by-diagonal matrix with the structure given in (7). Then we obtain the factorization (46) by applying successively (47) in $t \in \mathscr{I}^{(p-2)}$ for $s = 0, 1, \ldots, p - 2$. We recall that (46) has a formal sense. This is, for each fixed $N \in \mathbb{N}$ there exists some open interval $\mathscr{I}_N$ such that the entries $l_{i,j}(t)\,,\ i, j \leq N\,,$ of $L(t)$ are obtained in terms of the corresponding products and sums of entries of $L^{(1)}\,, \ldots\,, L^{(p)}$ for $t \in \mathscr{I}_N \cap \mathscr{I}^{(p-2)}$. However, it is possible to have $\bigcap_{N \in \mathbb{N}} \mathscr{I}_N = \{t_0\}$.

4.2 Solution of the KdV equation

Now we analyze the derivative $\dot{\gamma}_n$ of the entries of U and the recently defined matrices $L^{(i)}\,,\ i = 1, \ldots, p$, to prove that the matrix Γ defined by the sequence $\{\gamma_n\}$ is a solution of (4) (see Definition 2).

The functions

$$\gamma_{(i-1)p+i+s+1}(t)\,, \quad i = 1, 2, \ldots, p - s - 1\,, \quad s = 0, 1, \ldots, p - 2\,, \tag{57}$$

defined for $t \in \mathscr{I}^{(p-2)}$ were obtained as the solutions of the initial value problem (55). These functions correspond to the set (22). Moreover the entries $a_{i,j}$ of J can be obtained since (19) and (46) in terms of the entries of the factors U, $L^{(i)}$, $i = 1, \ldots, p$. This is (12) (taking $j = 0$) for the diagonal entries of J. From this and (52) we arrive immediately to (4) for the functions γ_n given in (57). On the other hand, (4) was proved for the entries of U in Lemma 1. Thus we only need to verify (4) for the rest of the entries of $L^{(i)}$, $i = 1, \ldots, p$, that are not given in (57).

The terms of the sequence $\{\gamma_n\}$ not given in (57) are

$$\gamma_{(k+i+1)p+i}\,, \quad i = 1, 2, \ldots, \quad k = -1, 0, \ldots, p-2\,. \tag{58}$$

The construction of these terms from the data (57) was analyzed in Sect. 3. The functions (57), whose derivatives verify (4), can be rewritten as in (58) taking $i = -p+2, -p+3, \ldots, -1, 0$ (with the convention $\gamma_n \equiv 0$ when $n \leq 0$). We recall that for each fixed $i \in \mathbb{N}$ a parallel secondary diagonal of Table 1 is obtained in (35) for $k = -1, 0, \ldots, p-2$. Next we look for an iterative expression like (35) for the derivatives of these functions. This is, we want to use this construction for obtaining iteratively and simultaneously each function $\gamma_{(k+i+1)p+i}$ and its derivative $\dot{\gamma}_{(k+i+1)p+i}$ in terms of the functions and the derivatives obtained in the previous steps. In this way we assume that all the functions γ_n on the right hand side of (35) and their derivatives are known.

We shall prove the next expression for the derivative of the function $\delta_k^{(i)}$ given in (36),

$$\dot{\delta}_k^{(i)} = \delta_k^{(i)} \left(\sum_{j=0}^{p} \gamma_{(k+i)p+i+j} - \sum_{j=0}^{p} \gamma_{(i-2)p+i+j} \right), \qquad i = 1, 2, \ldots, \quad k = -1, 0, \ldots, p-2\,. \tag{59}$$

We underline that for each i, k the factors γ_n of $\delta_k^{(i)}$ given in (36) and its derivatives $\dot{\gamma}_n$ verify (4) from the previous steps. From this fact,

$$\begin{aligned} \frac{\dot{\delta}_k^{(i)}}{\delta_k^{(i)}} &= \sum_{r=-1}^{k} \frac{\dot{\gamma}_{(r+i)p+i}}{\gamma_{(r+i)p+i}} = \sum_{r=-1}^{k} \left(\sum_{j=1}^{p} \gamma_{(r+i)p+i+j} - \sum_{j=1}^{p} \gamma_{(r+i-1)p+i+j-1} \right) \\ &= \sum_{j=1}^{p} \gamma_{(k+i)p+i+j} + \sum_{r=0}^{k} \left(\gamma_{(r+i-1)p+i+p} - \gamma_{(r+i-1)p+i} \right) - \sum_{j=1}^{p} \gamma_{(i-2)p+i+j-1}\,, \end{aligned}$$

which leads to (59).

Now, for $i \in \mathbb{N}$ and $k \in \{-1, 0, \ldots, p-2\}$ fixed, we study the derivatives of the terms on the right hand side of (35). With this purpose for each $(i_1, i_2, \ldots, i_{k+3}) \in \widetilde{E}_{k+2}^{(0)}$ we define

$$\Delta_k^{(i)} := \gamma_{(i-2)p+i+i_1-1} \gamma_{(i-1)p+i+i_2-1} \cdots \gamma_{(i+k)p+i+i_{k+3}-1}\,, \tag{60}$$

(see (37)), where we assume that the derivatives of the functions γ_n verify (4). From this we have

$$\begin{aligned} \frac{\dot{\Delta}_k^{(i)}}{\Delta_k^{(i)}} &= \sum_{j=1}^{p} \gamma_{(k+i)p+i+i_{k+3}+j-1} - \sum_{j=1}^{p} \gamma_{(i-3)p+i+i_1+j-2} \\ &\quad + \sum_{r=1}^{k+2} \left(\sum_{j=i+i_r}^{i+i_r+p-1} \gamma_{(r+i-3)p+j} - \sum_{j=i+i_{r+1}-1}^{i+i_{r+1}+p-2} \gamma_{(r+i-3)p+j} \right). \end{aligned} \tag{61}$$

We are interested in (61) when $(i_1, i_2, \ldots, i_{k+3}) \in \widetilde{E}_{k+2}^{(0)}$ and $k \in \{-1, 0, \ldots, p-2\}$, as in (35).

Firstly, if $k \geq 0$ we have $i_r - i_{r+1} \leq p - k - 2 \leq p - 2$ for $r = 1, 2, \ldots, k+2$. Then

$$i + i_{r+1} - 1 < i + i_r \leq i + i_{r+1} + p - 2 < i + i_r + p - 1 \tag{62}$$

for $(i_1, i_2, \ldots, i_{k+3}) \in E_{k+2}^{(0)}$ given in (14) and, in particular, for $(i_1, i_2, \ldots, i_{k+3}) \in \widetilde{E}_{k+2}^{(0)}$. Then we simplify (61) because

$$\begin{aligned} &\sum_{j=i+i_r}^{i+i_r+p-1} \gamma_{(r+i-3)p+j} - \sum_{j=i+i_{r+1}-1}^{i+i_{r+1}+p-2} \gamma_{(r+i-3)p+j} \\ &= \sum_{j=i+i_{r+1}+p-1}^{i+i_r+p-1} \gamma_{(r+i-3)p+j} - \sum_{j=i+i_{r+1}-1}^{i+i_r-1} \gamma_{(r+i-3)p+j} \,. \end{aligned} \tag{63}$$

Secondly, if $k = -1$ then in the above conditions we have $(i_1, i_2) \in \widetilde{E}_1^{(0)}$ and

$$2 \leq i_2 \leq i_1 \leq p + 1 \,. \tag{64}$$

Thus $i_1 - i_2 \leq p - 1$. If

$$i_1 - i_2 \leq p - 2 \tag{65}$$

then (62) also holds (with $r = 1$) and consequently we arrive at (63). Moreover, (65) holds when either $i_2 \neq 2$ or $i_1 \neq p+1$ in (64). In this case, when $i_2 = 2$ and $i_1 = p+1$, we arrive straight at (63).

Therefore (63) holds for $r = 1, 2, \ldots, k+2$, $k = -1, 0, \ldots, p-2$ and $(i_1, i_2, \ldots, i_{k+3}) \in E_{k+2}^{(0)}$. From this and (61) we have

$$\begin{aligned} \frac{\dot{\Delta}_k^{(i)}}{\Delta_k^{(i)}} &= \sum_{j=1}^{p} \gamma_{(k+i)p+i+i_{k+3}+j-1} - \sum_{j=1}^{p} \gamma_{(i-3)p+i+i_1+j-2} \\ &+ \sum_{r=1}^{k+2} \left(\sum_{j=i+i_{r+1}+p-1}^{i+i_r+p-1} \gamma_{(r+i-3)p+j} - \sum_{j=i+i_{r+1}-1}^{i+i_r-1} \gamma_{(r+i-3)p+j} \right) . \end{aligned} \tag{66}$$

Moreover for each $j = 0, 1, \ldots, p$ we can show

$$\begin{aligned} &\sum_{\widetilde{E}_{k+2}^{(j)}} \Delta_k^{(i)} \left(\sum_{s=i+i_r+p-1}^{i+i_{r-1}+p-1} \gamma_{(r+i-4)p+s} - \sum_{s=i+i_{r+1}+p-1}^{i+i_r+p-1} \gamma_{(r+i-4)p+s} \right) = 0 \,, \\ &k = -1, 0, \ldots, p-2 \,, \; r = 2, \ldots, k+2 \,. \end{aligned} \tag{67}$$

Indeed, in the first term of (67) for each $s = i + \tilde{j} + p - 1$, $i_r \leq \tilde{j} \leq i_{r-1}$ we have

$$\Delta_k^{(i)} \gamma_{(r+i-4)p+s} = \gamma_{(i-3)p+i+\tilde{i}_0-1} \gamma_{(i-2)p+i+\tilde{i}_1-1} \cdots \gamma_{(i+k)p+i+\tilde{i}_{k+3}-1} \tag{68}$$

where

$$\tilde{i}_q = \begin{cases} i_{q+1} + p \;, & q = 0, 1, \ldots, r-2 \\ \tilde{j} + p \;, & q = r-1 \\ i_q \;, & q = r, r+1, \ldots, k+3 \,. \end{cases} \tag{69}$$

In the second term of (67), for each $s = i + \tilde{j} + p - 1$ and $i_{r+1} \leq \tilde{j} \leq i_r$ we have (68) for

$$\tilde{i}_q = \begin{cases} i_{q+1} + p & , \quad q = 0, 1, \ldots, r-1 \\ \tilde{j} & , \quad q = r \\ i_q & , \quad q = r+1, r+2, \ldots, k+3\,. \end{cases} \tag{70}$$

In both cases, (69) and (70), it is verified $(i_1, i_2, \ldots, i_{k+3}) \in \widetilde{E}^{(j)}_{k+2}$ and $j + k + 2 \leq \tilde{i}_{k+3} \leq \cdots \leq \tilde{i}_0 \leq j + 2p + 1$, being $\tilde{i}_{r-1} - \tilde{i}_r \geq p$ and taking $(\tilde{i}_0, \tilde{i}_1, \ldots, \tilde{i}_{k+3})$ all the values in these conditions. Thus both sums coincide and (67) is verified.

Taking into account (67) and making some computations in (66) we obtain

$$\sum_{\widetilde{E}^{(0)}_{k+2}} \dot{\Delta}^{(i)}_k = \sum_{\widetilde{E}^{(0)}_{k+2}} \Delta^{(i)}_k \left(\sum_{j=1}^{p} \gamma_{(k+i)p+i+i_{k+3}+j-1} - \sum_{j=1}^{p} \gamma_{(i-3)p+i+i_1+j-2} \right.$$
$$\left. - \sum_{j=i_2}^{i_1} \gamma_{(i-2)p+i+j-1} + \sum_{j=i_{k+3}}^{i_{k+2}} \gamma_{(k+i)p+i+j-1} \right). \tag{71}$$

Due to $\widetilde{E}^{(0)}_{k+2} = E^{(0)}_{k+2} \backslash \{(\overbrace{p+1, \cdots, p+1}^{(k+3)})\}$, since (71) we have

$$\sum_{\widetilde{E}^{(0)}_{k+2}} \dot{\Delta}^{(i)}_k = \sum_{E^{(0)}_{k+2}} \Delta^{(i)}_k \left(\sum_{j=1}^{p} \gamma_{(k+i)p+i+i_{k+3}+j-1} - \sum_{j=1}^{p} \gamma_{(i-3)p+i+i_1+j-2} \right.$$
$$\left. - \sum_{j=i_2}^{i_1} \gamma_{(i-2)p+i+j-1} + \sum_{j=i_{k+3}}^{i_{k+2}} \gamma_{(k+i)p+i+j-1} \right)$$
$$-\delta^{(i)}_k \gamma_{(k+i+1)p+i} \left(\sum_{j=0}^{p} \gamma_{(k+i+1)p+i+j} - \sum_{j=1}^{p+1} \gamma_{(i-2)p+i+j-1} \right). \tag{72}$$

On the other hand we know $(i_1, i_2, \ldots, i_{k+4}) \in E^{(0)}_{k+3}$ if and only if $k + 3 \leq i_{k+4} - 1 \leq \cdots \leq i_2 - 1 \leq i_1 - 1 \leq p$. This is, $(i_2 - 1, i_3 - 1, \ldots, i_{k+4} - 1) \in E^{(0)}_{k+2}$ and $i_2 \leq i_1 \leq p+1$. Then

$$\sum_{E^{(0)}_{k+3}} \Delta^{(i-1)}_{k+1} = \sum_{E^{(0)}_{k+2}} \Delta^{(i)}_k \sum_{j=i_1}^{p} \gamma_{(i-3)p+i+j-1}\,. \tag{73}$$

Moreover, $(i_1, i_2, \ldots, i_{k+4}) \in E^{(0)}_{k+3}$ if and only if $k + 3 \leq i_{k+3} \leq \cdots \leq i_1 \leq p + 1$ and $k + 4 \leq i_{k+4} \leq i_{k+3}$. This is, $(i_1, \ldots, i_{k+3}) \in E^{(0)}_{k+2}$ and $k + 4 \leq i_{k+4} \leq i_{k+3}$. Then

$$\sum_{E^{(0)}_{k+3}} \Delta^{(i)}_{k+1} = \sum_{E^{(0)}_{k+2}} \Delta^{(i)}_k \sum_{j=k+4}^{i_{k+3}} \gamma_{(k+i+1)p+i+j-1}\,. \tag{74}$$

Since (73) and (74), using (13) we have $a_{k+i+2,i-1} - a_{k+i+1,i-2} =$

$$\sum_{E^{(0)}_{k+2}} \Delta^{(i)}_k \left(\sum_{j=k+4}^{i_{k+3}} \gamma_{(k+i+1)p+i+j-1} - \sum_{j=i_1}^{p} \gamma_{(i-3)p+i+j-1} \right). \tag{75}$$

Using again the Bäcklund transformations (12)–(13),

$$\left(a_{k+i+1,k+i+1} - a_{i-1,i-1}\right) a_{k+i+1,k+i+1} \tag{76}$$

$$= \sum_{E_{k+2}^{(0)}} \Delta_k^{(i)} \left(\sum_{j=1}^{p+1} \gamma_{(k+i)p+k+i+j+1} - \sum_{j=1}^{p+1} \gamma_{(i-2)p+i+j-1} \right) . \tag{77}$$

From (75), (76) and (2) we obtain $\dot{a}_{k+i+1,i-1} =$

$$\sum_{E_{k+2}^{(0)}} \Delta_k^{(i)} \left(\sum_{j=k+3}^{p+i_{k+3}} \gamma_{(k+i)p+i+j-1} - \sum_{j=i_1}^{2p+1} \gamma_{(i-3)p+i+j-1} \right) . \tag{78}$$

Similarly to (67) it is easy to verify

$$\sum_{E_{k+2}^{(q)}} \Delta_k^{(i)} \left(\sum_{s=i_{k+3}}^{i_{k+2}} \gamma_{(k+i)p+i+s-1} - \sum_{s=q+k+3}^{i_{k+3}} \gamma_{(k+i)p+i+s-1} \right) = 0 \tag{79}$$

and

$$\sum_{E_{k+2}^{(q)}} \Delta_k^{(i)} \left(\sum_{s=i_1}^{q+p+1} \gamma_{(i-2)p+i+s-1} - \sum_{s=i_2}^{i_1} \gamma_{(i-2)p+i+s-1} \right) = 0 \tag{80}$$

for each $q = 0, 1, \ldots, p$. Taking derivatives in (35) and using (59), (72), (78) and (80) (with $q = 0$) we arrive at

$$\dot{\gamma}_{(k+i+1)p+i} = \gamma_{(k+i+1)p+i} \left(\sum_{j=0}^{p} \gamma_{(k+i+1)p+i+j} - \sum_{j=0}^{p} \gamma_{(k+i+1)p+i-j} \right) ,$$

which is (4) for $n = (k+i+1)p+i$ with k, i in the mentioned conditions.

4.3 New solutions of the Kostant Toda lattice

Our next target is to prove that the matrices given in (10) are the solutions of (2). With this purpose we show that the entries $a_{q,r}^{(j)}$ of each $J^{(j)}$ verify (2). Because the sequence $\{\gamma_n\}$ verifies (4), if we take derivatives in (12) and we make some extra computations we arrive at $\dot{a}_{i,i}^{(j)} =$

$$\sum_{j+1 \le i_2 \le i_1 \le j+p} \gamma_{(i-1)p+i+i_1+1} \gamma_{ip+i+i_2+1} - \sum_{j+1 \le i_2 \le i_1 \le j+p} \gamma_{(i-2)p+i+i_1} \gamma_{(i-1)p+i+i_2}$$

which is $a_{i+1,i}^{(j)} - a_{i,i-1}^{(j)}$. Moreover, for $k = 1, 2, \ldots, p$ and using the notation of (60), from (4) we see

$$\dot{a}^{(j)}_{i+k,i} = \sum_{E^{(j)}_k} \Delta^{(i+1)}_{k-2} \left(\sum_{r=1}^{p} \gamma_{(i+k-1)p+i+i_{k+1}+r} + \sum_{s=0}^{k-1} \sum_{r=1}^{p} \big(\gamma_{(i+s-1)p+i+i_{s+1}+r} - \gamma_{(i+s-1)p+i+i_{s+2}+r-1}\big) - \sum_{r=1}^{p} \gamma_{(i-1)p+i+i_1-r} \right) . \tag{81}$$

For $(i_1, \ldots, i_{k+1}) \in E^{(j)}_k$ we have $j+k+1 \le i_{s+2} \le i_{s+1} \le j+p+1$ for $s = 0, \ldots, k-1$. Then $i_{s+2} \le i_{s+1} \le i_{s+2} + p - 1 \le i_{s+1} + p$ and

$$\sum_{r=1}^{p} \big(\gamma_{(i+s-1)p+i+i_{s+1}+r} - \gamma_{(i+s-1)p+i+i_{s+2}+r-1}\big) = \sum_{r=i_{s+2}}^{i_{s+1}} \gamma_{(i+s)p+i+r} - \sum_{r=i_{s+2}}^{i_{s+1}} \gamma_{(i+s-1)p+i+r} . \tag{82}$$

Taking into account (82) in the right hand side of (81), since (68) we have

$$\dot{a}^{(j)}_{i+k,i} = \sum_{E^{(j)}_k} \Delta^{(i+1)}_{k-2} \left(\sum_{r=1}^{p} \gamma_{(i+k-1)p+i+i_{k+1}+r} + + \sum_{r=i_{k+1}}^{i_k} \gamma_{(i+k-1)p+i+r} - \sum_{r=i_2}^{i_1} \gamma_{(i-1)p+i+r} - \sum_{r=1}^{p} \gamma_{(i-1)p+i+i_1+r} \right) .$$

Then using (79)-(80) we arrive at

$$\dot{a}^{(j)}_{i+k,i} = \sum_{E^{(j)}_k} \Delta^{(i+1)}_{k-2} \left(\sum_{r=j+k+1}^{i_{k+1}+p} \gamma_{(i+k-1)p+i+r} - \sum_{r=i_1-p}^{j+p+1} \gamma_{(i-1)p+i+r} \right) . \tag{83}$$

On the other hand, from (12)–(13) we have

$$(a^{(j)}_{i+k,i+k} - a^{(j)}_{i,i}) a^{(j)}_{i+k,i} = \sum_{E^{(j)}_k} \Delta^{(i+1)}_{k-2} \left(\sum_{s=j+1}^{j+p+1} \gamma_{(i+k-1)p+i+k+s} - \sum_{s=j+1}^{j+p+1} \gamma_{(i-1)p+i+s} \right) , \tag{84}$$

$$a^{(j)}_{i+k+1,i} = \sum_{E^{(j)}_{k+1}} \gamma_{(i+k)p+i+i_{k+2}} \Delta^{(i+1)}_{k-2} . \tag{85}$$

Moreover $(i_1, \ldots, i_{k+2}) \in E^{(j)}_{k+1}$ if and only if $(\widetilde{i}_1, \ldots, \widetilde{i}_{k+1}) \in E^{(j)}_k$ and $\widetilde{i}_1 + 1 \le i_1 \le j + p + 1$, being $\widetilde{i}_r = i_{r+1} - 1$, $r = 1, \ldots, k+1$. Then

$$a^{(j)}_{i+k,i-1} = \sum_{E^{(j)}_{k+1}} \Delta^{(i)}_{k-1} = \sum_{E^{(j)}_k} \Delta^{(i+1)}_{k-2} \sum_{s=i_1+1}^{j+p+1} \gamma_{(i-2)p+i+s-1} . \tag{86}$$

Since (84), (85) and (86) we have $(a^{(j)}_{i+k,i+k} - a^{(j)}_{i,i})a^{(j)}_{i+k,i} + a^{(j)}_{i+k+1,i} - a^{(j)}_{i+k,i-1} =$

$$= \sum_{E_k^{(j)}} \Delta^{(i+1)}_{k-2} \left(\sum_{r=j+k+1}^{p+i_{k+1}} \gamma_{(i+k-1)p+i+r} - \sum_{r=i_1-p}^{j+p+1} \gamma_{(i-1)p+i+r} \right). \tag{87}$$

Finally, comparing (83) and (87) we arrive at (2) for the entries of the matrices $J^{(j)}$, $j = 1, \ldots, p$.

With this, Theorem 1 is proved. ◻

Acknowledgements This work was partially supported by Dirección General de Investigación Científica y Técnica, Ministerio de Economía y Competitividad, under grant MTM2014-54053-P.

References

1. Adler, M., van Moerbeke, P.: Darboux transforms on band matrices, weights, and associated polynomials. Int. Math. Res. Not. **18**, 935–984 (2000)
2. Barrios Rolanía, D., Branquinho, A., Foulquié Moreno, A.: On the full Kostant-Toda system and the discrete Korteweg-de Vries equations. J. Math. Anal. Appl. **401**, 811–820 (2013)
3. Barrios Rolanía, D., Manrique, D.: On the existence of Darboux transformations for banded matrices. Appl. Math. Comput. **253**, 116–125 (2015)
4. Bueno, M.I., Marcellán, F.: Darboux transformation and perturbation of linear functionals. Linear Algebra Appl. **384**, 215–242 (2004)
5. Gantmacher, F.R.: The Theory of Matrices, vol. I. AMS Chelsea Pub, Providence (2000)
6. Gesztesy, F., Holden, H., Simon, B., Zhao, Z.: On the Toda and Kac-van Moerbeke systems. Trans. Am. Math. Soc. **339**(2), 849–868 (1993)
7. Hale, J.K.: Ordinary Differential Equations. Dover Publications Inc., New York (2009)
8. Isaacson, E., Keller, H.B.: Analysis of Numerical Methods. Dover Pub. Inc, New York (1994)
9. Matveev, V.B., Salle, M.A.: Darboux transformations and solitons. Springer series in Non-linear Dynamics, Springer, Berlin (1991)
10. Peherstorfer, F.: On Toda lattices and orthogonal polynomials. J. Comput. Appl. Math. **133**, 519–534 (2001)
11. Samsonov, B.F., Pecheritsin, A.A.: Chains of Darboux transformations for the matrix Schrödinger. J. Phys. A Math. Gen. **33**(1), 239–250 (2004)
12. Taylor, A.E.: General Theory of Functions ans Integration. Dover Pub. Inc, New York (1985)