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ORIGINAL PAPER

Classes of harmonic functions associated with Ruscheweyh derivatives

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Abstract In the paper we introduce the classes of functions defined by generalized Ruscheweyh derivatives and we show that they can be presented as dual sets. Moreover, by using extreme points theory, we obtain estimations of classical convex functionals on the defined classes of functions. Some applications of the main results are also considered.

Keywords Harmonic functions · Subordination · Starlike functions · Ruscheweyh operator · Dual sets · Correlated coefficients

Mathematics Subject Classification Primary 30C55; Secondary 30C45

1 Introduction

A real-valued function u is said to be harmonic in a domain $D \subset \mathbb{C}$ if it has continuous second order partial derivatives in D, which satisfy the Laplace equation

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We say that a complex-valued continuous function $f : D \to \mathbb{C}$ is harmonic in D if both functions $u := \operatorname{Re} f$ and $v := \operatorname{Im} f$ are real-valued harmonic functions in D. We note that every complex-valued function f harmonic in D with $0 \in D$, can be uniquely represented as

$$f = h + \overline{g}, \tag{1}$$

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where h, g are analytic functions in D with $g(0) = 0$. Then we call h the analytic part and g the co-analytic part of f (see [4]). It is easy to verify, that the Jacobian of f is given by

$$J_f(z) = \left|h'(z)\right|^2 - \left|g'(z)\right|^2 \quad (z \in D).$$

The mapping f is locally univalent if $J_f(z) \neq 0$ in D. A result of Lewy [15] shows that the converse is true for harmonic mappings. Therefore, f is locally univalent and sense-preserving if and only if

$$\left|h'(z)\right| > \left|g'(z)\right| \quad (z \in D). \tag{2}$$

Let $\mathcal{H}$ denote the class of harmonic functions in the unit disc $\mathbb{U}$. Any function $f \in \mathcal{H}$ can be written in the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n} \quad (z \in \mathbb{U}). \tag{3}$$

Let $\mathbb{N}_l := \{l, l+1, \ldots\}$, $\mathbb{N} := \mathbb{N}_1$, $k \in \mathbb{N}_2$, and let $\mathcal{H}(k)$ denote the class of function with missing coefficients i.e. the functions $f \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{n=k}^{\infty} \left(a_n z^n + \overline{b_n z^n}\right) \quad (z \in \mathbb{U}), \tag{4}$$

which are univalent and sense-preserving in $\mathbb{U}$.

We say that a function $f \in \mathcal{H}(2)$ is harmonic starlike in $\mathbb{U}(r)$ if

$$\frac{\partial}{\partial t}\left(\arg f\left(re^{it}\right)\right) > 0 \ \ (0 \leq t \leq 2\pi)$$

i.e. f maps the circle $\partial\mathbb{U}(r)$ onto a closed curve that is starlike with respect to the origin. It is easy to verify, that the above condition is equivalent to the following

$$\operatorname{Re} \frac{D_{\mathcal{H}} f(z)}{f(z)} > 0 \quad (|z| = r),$$

where

$$D_{\mathcal{H}} f(z) := zh'(z) - \overline{zg'(z)} \quad (z \in \mathbb{U}).$$

Ruscheweyh [20] introduced an operator $\mathcal{D}^m : \mathcal{A} \to \mathcal{A}$, defined by

$$\mathcal{D}^m f(z) = \frac{z\left(z^{m-1} f(z)\right)^{(m)}}{m!} \quad (m \in \mathbb{N}_0,\ z \in \mathbb{U}). \tag{5}$$

The Ruscheweyh derivative $\mathcal{D}^m$ was extended in [17] (see also [6,8,10,23]) on the class of harmonic functions. Let $D^m_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ denote the linear operator defined for a function $f = h + \overline{g} \in \mathcal{H}$ by

$$D^m_{\mathcal{H}} f := \mathcal{D}^m h + (-1)^m \overline{\mathcal{D}^m g}.$$

We say that a function $f \in \mathcal{H}$ is *subordinate* to a function $F \in \mathcal{H}$, and write $f(z) \prec F(z)$ (or simply $f \prec F$) if there exists a complex-valued function ω which maps $\mathbb{U}$ into oneself with $\omega(0) = 0$, such that $f(z) = F(\omega(z)) \quad (z \in \mathbb{U})$.

Let A, B be complex parameters, $A \neq B$. We denote by $\mathcal{S}^m_{\mathcal{H}}(k; A, B)$ the class of functions $f \in \mathcal{H}(k)$ such that

$$\frac{D_{\mathcal{H}}\left(D^m_{\mathcal{H}} f\right)(z)}{D^m_{\mathcal{H}} f(z)} \prec \frac{1 + Az}{1 + Bz}. \tag{6}$$

Also, by $\mathcal{R}_{\mathcal{H}}^{m}(k;A,B)$ we denote the class of functions $f\in\mathcal{H}(k)$ such that

$$\frac{D_{\mathcal{H}}^{m}f(z)}{z}\prec\frac{1+Az}{1+Bz}.$$

The classes $\mathcal{S}_{\mathcal{H}}^{m}(k;A,B)$ and $\mathcal{R}_{\mathcal{H}}^{m}(k;A,B)$ with restrictions $-B\leq A<B\leq 1$, $k=2$, were investigated in [8]. In particular, the class

$$\mathcal{S}_{\mathcal{H}}^{m}(\alpha):=\mathcal{S}_{\mathcal{H}}^{m}(2\alpha-1,1)\quad(0\leq\alpha<1)$$

is related to the class of Sălăgean-type harmonic functions studied by Yalçin [22]. The classes

$$\mathcal{R}_{\mathcal{H}}(k;A,B):=\mathcal{R}_{\mathcal{H}}^{1}(k;A,B),\ \mathcal{S}_{\mathcal{H}}(k;A,B):=\mathcal{S}_{\mathcal{H}}^{0}(k;A,B),\ \mathcal{K}_{\mathcal{H}}(k;A,B):=\mathcal{S}_{\mathcal{H}}^{1}(k;A,B)$$

are defined in [6] (see also [7]).

The object of the present paper is to show that the defined classes of functions can be presented as dual sets. Also, by using extreme points theory, we obtain estimations of classical convex functionals on the defined classes of functions with correlated coefficients. Some applications of the main results are also considered.

2 Dual sets

For functions $f_1,f_2\in\mathcal{H}$ of the form of the form

$$f_l(z)=\sum_{k=0}^{\infty}\left(a_{l,k}z^k+\overline{b_{l,k}z^k}\right)\quad(z\in\mathbb{U},l\in\mathbb{N})\tag{7}$$

we define *the Hadamard product or convolution* of f_1 and f_2 by

$$(f_1*f_2)(z)=\sum_{k=0}^{\infty}\left(a_{1,k}a_{2,k}z^k+\overline{b_{1,k}b_{2,k}z^k}\right)\quad(z\in\mathbb{U}).$$

Let $\mathcal{V}\subset\mathcal{H}$, $\mathbb{U}_0:=\mathbb{U}\setminus\{0\}$. Motivated by Ruscheweyh [19] we define the dual set of $\mathcal{V}$ by

$$\mathcal{V}^*:=\left\{f\in\mathcal{H}(k):\bigwedge_{g\in\mathcal{V}}(f*g)(z)\neq 0\quad(z\in\mathbb{U}_0)\right\}.$$

Theorem 1

$$\mathcal{S}_{\mathcal{H}}^{m}(k;A,B)=\left\{D_{\mathcal{H}}^{m}\left(\psi_\xi\right):\ |\xi|=1\right\}^*,$$

where

$$\psi_\xi(z):=z\frac{1+B\xi-(1+A\xi)(1-z)}{(1-z)^2}$$
$$-\overline{z}\frac{1+B\xi+(1+A\xi)(1-\overline{z})}{(1-\overline{z})^2}\quad(z\in\mathbb{U}).$$

Proof Let $f\in\mathcal{H}(k)$ be of the form (1). Then $f\in\mathcal{S}_{\mathcal{H}}^{m}(k;A,B)$ if and only if it satisfies (6) or equivalently

$$\frac{D_{\mathcal{H}}\left(D_{\mathcal{H}}^{m}f\right)(z)}{D_{\mathcal{H}}^{m}f(z)}\neq\frac{1+A\xi}{1+B\xi}\quad(z\in\mathbb{U}_0,\ |\xi|=1).$$

Since

$$D_{\mathcal{H}}\left(D_{\mathcal{H}}^{m}h\right)(z) = D_{\mathcal{H}}^{m}h(z) * z/(1-z)^2,\ D_{\mathcal{H}}^{m}h(z) = D_{\mathcal{H}}^{m}h(z) * \frac{z}{1-z},$$

the above inequality yields

$$\begin{aligned}
&(1+B\xi)\, D_{\mathcal{H}}\left(D_{\mathcal{H}}^{m}f\right)(z) - (1+A\xi)\, D_{\mathcal{H}}^{m}f(z)\\
&\quad = (1+B\xi)\, D_{\mathcal{H}}\left(D_{\mathcal{H}}^{m}h\right)(z) - (1+A\xi)\, D_{\mathcal{H}}^{m}h(z)\\
&\qquad - (-1)^m \left[(1+B\xi)\,\overline{D_{\mathcal{H}}\left(D_{\mathcal{H}}^{m}g\right)(z)} + (1+A\xi)\,\overline{D_{\mathcal{H}}^{m}g(z)}\right]\\
&\quad = D_{\mathcal{H}}^{m}h(z) * \left(\frac{(1+B\xi)\,z}{(1-z)^2} - \frac{(1+A\xi)\,z}{1-z}\right)\\
&\qquad - (-1)^m\,\overline{D_{\mathcal{H}}^{m}g(z)} * \left(\frac{(1+B\xi)\,\overline{z}}{(1-\overline{z})^2} + \frac{(1+A\xi)\,\overline{z}}{1-\overline{z}}\right)\\
&\quad = f(z) * D_{\mathcal{H}}^{m}\psi_{\xi}(z) \neq 0 \quad (z \in \mathbb{U}_0,\ |\xi| = 1).
\end{aligned}$$

Thus, $f \in \mathcal{S}_{\mathcal{H}}^{m}(k; A, B)$ if and only if $f(z) * D_{\mathcal{H}}^{m}\psi_{\xi}(z) \neq 0$ for $z \in \mathbb{U}_0,\ |\xi| = 1$ *i.e.* $\mathcal{S}_{\mathcal{H}}^{m}(k; A, B) = \left\{D_{\mathcal{H}}^{m}\left(\psi_{\xi}\right) : |\xi| = 1\right\}^*$. □

Similarly as Theorem 1 we prove the following theorem.

Theorem 2

$$\mathcal{R}_{\mathcal{H}}^{m}(k; A, B) = \left\{\delta_{\xi} : |\xi| = 1\right\}^*,$$

where

$$\delta_{\xi}(z) := z\frac{1+B\xi - (1+A\xi)(1-z)^{m+1}}{(1-z)^{m+1}} + (-1)^m\,\overline{z}\frac{1+B\xi}{(1-\overline{z})^{m+1}} \quad (z \in \mathbb{U}).$$

In particular, by Theorems 1 and 2 we obtain the following results.

Theorem 3

$$\mathcal{S}_{\mathcal{H}}^{m}(\alpha) = \left\{D_{\mathcal{H}}^{m}\left(\psi_{\xi}\right) : |\xi| = 1\right\}^*,$$

where

$$\begin{aligned}
\psi_{\xi}(z) := {} & z\frac{1+\xi - (1-\xi+2\alpha\xi)(1-z)}{(1-z)^2}\\
& -\overline{z}\frac{1+\xi + (1-\xi+2\alpha\xi)(1-\overline{z})}{(1-\overline{z})^2} \quad (z \in \mathbb{U}).
\end{aligned}$$

Theorem 4

$$\mathcal{S}_{\mathcal{H}}(k; A, B) = \left\{\psi_{\xi} : |\xi| = 1\right\}^*,$$

where

$$\psi_{\xi}(z) := z\frac{(B-A)\,\xi + (1+A\xi)\,z}{(1-z)^2} - \overline{z}\frac{2 + (A+B)\,\xi - (1+A\xi)\,\overline{z}}{(1-\overline{z})^2} \quad (z \in \mathbb{U}).$$

Theorem 5

$$\mathcal{K}_{\mathcal{H}}(k; A, B) = \left\{\psi_{\xi} : |\xi| = 1\right\}^*,$$

where

$$\psi_{\xi}(z) := z\frac{(B-A)\,\xi + (2+A\xi+B\xi)\,z}{(1-z)^3} + \overline{z}\frac{2 + (A+B)\,\xi + (B-A)\,\xi\overline{z}}{(1-\overline{z})^3} \quad (z \in \mathbb{U}).$$

Theorem 6

$$\mathcal{R}_{\mathcal{H}}(k; A, B) = \left\{\delta_\xi : |\xi| = 1\right\}^*,$$

where

$$\delta_\xi(z) := z\frac{(1 + B\xi) - (1 + A\xi)(1 - z)^2}{(1 - z)^2} - \frac{(1 + B\xi)\overline{z}}{(1 - \overline{z})^2} \quad (z \in \mathbb{U}).$$

3 Correlated coefficients

Let us consider the function $\varphi \in \mathcal{H}$ of the form

$$\varphi = u + \overline{v}, \quad u(z) = \sum_{n=0}^{\infty} u_n z^n, \quad v(z) = \sum_{n=1}^{\infty} v_n z^n \quad (z \in \mathbb{U}). \tag{8}$$

We say that a function $f \in \mathcal{H}$ of the form (4) has coefficients correlated with the function φ, if

$$u_n a_n = -|u_n||a_n|, \quad v_n b_n = |v_n||b_n| \quad (n \in \mathbb{N}_k). \tag{9}$$

In particular, if there exists a real number η such that

$$\varphi(z) = \frac{z}{1 - e^{i\eta}z} + \frac{\overline{z}}{1 - e^{i\eta}\overline{z}} = \sum_{n=1}^{\infty} e^{i(n-1)\eta}\left(z^n + \overline{z}^n\right) \quad (z \in \mathbb{U}),$$

then we obtain functions with varying coefficients defined by Jahangiri and Silverman [11] (see also [7]). Moreover, if we take

$$\varphi(z) = 2\Re\frac{z}{1 - z} = \sum_{n=1}^{\infty}\left(z^n + \overline{z}^n\right) \quad (z \in \mathbb{U}),$$

then we obtain functions with negative coefficients introduced by Silverman [21]. These functions were intensively investigated by many authors (for example, see [5–9, 11, 13, 25]).

Let $\mathcal{T}^m(k, \eta)$ denote the class of functions $f \in \mathcal{H}(k)$ with coefficients correlated with respect to the function

$$\varphi(z) = \frac{z}{(1 - e^{i\eta}z)^{m+1}} + \frac{(-1)^m\overline{z}}{(1 - e^{i\eta}\overline{z})^{m+1}} \quad (z \in \mathbb{U}). \tag{10}$$

Moreover, let us define

$$\mathcal{S}^m_{\mathcal{T}}(k, \eta; A, B) := \mathcal{T}^m(k, \eta) \cap \mathcal{S}^m_{\mathcal{H}}(k; A, B), \ \mathcal{R}^m_{\mathcal{T}}(k, \eta; A, B) := \mathcal{T}^m(k, \eta) \cap \mathcal{R}^m_{\mathcal{H}}(k; A, B),$$

where η; A, B are real parameters with $B > \max\{0, A\}$.

Let $f \in \mathcal{H}(k)$ be of the form (4). Thus, by (5) we have

$$D^m_{\mathcal{H}} f(z) = z + \sum_{n=k}^{\infty} \lambda_n a_n z^n + (-1)^m \sum_{n=k}^{\infty} \lambda_n \overline{b_n}\,\overline{z}^n \quad (z \in \mathbb{U}),$$

where

$$\lambda_n := \frac{(m + 1) \cdot \cdots \cdot (m + n - 1)}{(n - 1)!} \quad (n \in \mathbb{N}_k). \tag{11}$$

Theorem 7 *If a function* $f \in \mathcal{H}(k)$ *of the form* (4) *satisfies the condition*

$$\sum_{n=k}^{\infty} (\alpha_n |a_n| + \beta_n |b_n|) \leq B - A, \tag{12}$$

where

$$\alpha_n = \lambda_n \{n(1+B) - (1+A)\}, \quad \beta_n = \lambda_n \{n(1+B) + (1+A)\}, \tag{13}$$

then $f \in \mathcal{S}_{\mathcal{H}}^m(k; A, B)$.

Proof It is clear that the theorem is true for the function $f(z) \equiv z$. Let $f \in \mathcal{H}(k)$ be a function of the form (4) and let there exist $n \in \mathbb{N}_k$ such that $a_n \neq 0$ or $b_n \neq 0$. Since $\lambda_n \geq \lambda_k \geq 1$,we have

$$\frac{\alpha_n}{B-A} \geq n, \quad \frac{\beta_n}{B-A} \geq n, \quad n \in \mathbb{N}_k, \tag{14}$$

Thus, by (12) we get

$$\sum_{n=k}^{\infty} (n|a_n| + n|b_n|) \leq 1 \tag{15}$$

and

$$\begin{aligned} \left|h'(z)\right| - \left|g'(z)\right| &\geq 1 - \sum_{n=k}^{\infty} n|a_n||z|^n - \sum_{n=k}^{\infty} n|b_n||z|^n \geq 1 - |z| \sum_{n=k}^{\infty} (n|a_n| + n|b_n|) \\ &\geq 1 - \frac{|z|}{B-A} \sum_{n=k}^{\infty} (\alpha_n |a_n| + \beta_n |b_n|) \geq 1 - |z| > 0 \quad (z \in \mathbb{U}). \end{aligned}$$

Therefore, by (2) the function f is locally univalent and sense-preserving in $\mathbb{U}$. Moreover, if $z_1, z_2 \in \mathbb{U}$, $z_1 \neq z_2$, then.

$$\left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| = \left|\sum_{l=1}^{n} z_1^{l-1} z_2^{n-l}\right| \leq \sum_{l=1}^{n} |z_1|^{l-1} |z_2|^{n-l} < n \quad (n \in \mathbb{N}_k).$$

Hence, by (15) we have

$$\begin{aligned} |f(z_1) - f(z_2)| &\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\ &\geq \left|z_1 - z_2 - \sum_{n=k}^{\infty} a_n \left(z_1^n - z_2^n\right)\right| - \left|\sum_{n=k}^{\infty} \overline{b_n \left(z_1^n - z_2^n\right)}\right| \\ &\geq |z_1 - z_2| - \sum_{n=k}^{\infty} |a_n| \left|z_1^n - z_2^n\right| - \sum_{n=k}^{\infty} |b_n| \left|z_1^n - z_2^n\right| \\ &= |z_1 - z_2| \left(1 - \sum_{n=k}^{\infty} |a_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| - \sum_{n=k}^{\infty} |b_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right|\right) \\ &> |z_1 - z_2| \left(1 - \sum_{n=k}^{\infty} n|a_n| - \sum_{n=k}^{\infty} n|b_n|\right) \geq 0. \end{aligned}$$

This leads to the univalence of f *i.e.* $f \in \mathcal{S}_{\mathcal{H}}$. Therefore, $f \in \mathcal{S}^m(k; A, B)$ if and only if there exists a complex-valued function ω, $\omega(0) = 0$, $|\omega(z)| < 1$ $(z \in \mathbb{U})$ such that

$$\frac{D_{\mathcal{H}}\left(D_{\mathcal{H}}^m f\right)(z)}{D_{\mathcal{H}}^m f(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (z \in \mathbb{U}),$$

or equivalently

$$\left| \frac{D_{\mathcal{H}}\left(D^m_{\mathcal{H}} f\right)(z) - D^m_{\mathcal{H}} f(z)}{B D_{\mathcal{H}}\left(D^m_{\mathcal{H}} f\right)(z) - A D^m_{\mathcal{H}} f(z)} \right| < 1 \quad (z \in \mathbb{U}). \tag{16}$$

Thus, it is suffice to prove that

$$\left| D_{\mathcal{H}}\left(D^m_{\mathcal{H}} f\right)(z) - D^m_{\mathcal{H}} f(z) \right| - \left| B D_{\mathcal{H}}\left(D^m_{\mathcal{H}} f\right)(z) - A D^m_{\mathcal{H}} f(z) \right| < 0 \;\; (z \in \mathbb{U}_0).$$

Indeed, letting $|z| = r \;\; (0 < r < 1)$ we have

$$\begin{aligned}
&\left| D_{\mathcal{H}}\left(D^m_{\mathcal{H}} f\right)(z) - D^m_{\mathcal{H}} f(z) \right| - \left| B D_{\mathcal{H}}\left(D^m_{\mathcal{H}} f\right)(z) - A D^m_{\mathcal{H}} f(z) \right| \\
&\quad = \left| \sum_{n=k}^{\infty} (n-1)\lambda_n a_n z^n - (-1)^n \sum_{n=k}^{\infty} (n+1)\lambda_n \overline{b_n} \overline{z}^n \right| \\
&\qquad - \left| (B-A) z + \sum_{n=k}^{\infty} (Bn - A)\lambda_n a_n z^n + (-1)^n \sum_{n=k}^{\infty} (Bn + A)\lambda_n \overline{b_n} \overline{z}^n \right| \\
&\quad \le \sum_{n=k}^{\infty} (n-1)\lambda_n |a_n| r^n + \sum_{n=k}^{\infty} (n+1)\lambda_n |b_n| r^n - (B-A) r \\
&\qquad + \sum_{n=k}^{\infty} (Bn - A)\lambda_n |a_n| r^n + \sum_{n=k}^{\infty} (Bn + A)\lambda_n |b_n| r^n \\
&\quad \le r \left\{ \sum_{n=k}^{\infty} (\alpha_n |a_n| + \beta_n |b_n|) r^{n-1} - (B-A) \right\} < 0.
\end{aligned}$$

whence $f \in \mathcal{S}^m_{\mathcal{H}}(k; A, B)$. □

The next theorem, shows that the condition (12) is also the sufficient condition for a function $f \in \mathcal{H}$ of correlated coefficients to be in the class $\mathcal{S}^m_{\mathcal{T}}(k, \eta; A, B)$.

Theorem 8 *Let $f \in \mathcal{T}^m(k, \eta)$ be a function of the form (4). Then $f \in \mathcal{S}^m_{\mathcal{T}}(k, \eta; A, B)$ if and only if the condition (12) holds true.*

Proof In view of Theorem 7 we need only show that each function $f \in \mathcal{S}^m_{\mathcal{T}}(k, \eta; A, B)$ satisfies the coefficient inequality (12). If $f \in \mathcal{S}^m_{\mathcal{T}}(k, \eta; A, B)$, then it is of the form (4) with (9) and it satisfies (16) or equivalently

$$\left| \frac{\sum_{n=k}^{\infty} (n-1)\lambda_n a_n z^n - (-1)^n \sum_{n=k}^{\infty} (n+1)\lambda_n \overline{b_n} \overline{z}^n}{(B-A) z + \sum_{n=k}^{\infty} (Bn - A)\lambda_n a_n z^n + (-1)^n \sum_{n=k}^{\infty} (Bn + A)\lambda_n \overline{b_n} \overline{z}^n} \right| < 1 \quad (z \in \mathbb{U}).$$

Therefore, putting $z = re^{i\eta}$ $(0 \le r < 1)$ by (10) and (9) we obtain

$$\frac{\sum_{n=k}^{\infty} (n-1)\lambda_n |a_n| + (n+1)\lambda_n |b_n| r^{n-1}}{(B-A) - \sum_{n=k}^{\infty} \{(Bn - A)\lambda_n |a_n| + (Bn + A)\lambda_n |b_n|\} r^{n-1}} < 1. \tag{17}$$

It is clear that the denominator of the left hand side cannot vanish for $r \in \langle 0, 1)$. Moreover, it is positive for $r = 0$, and in consequence for $r \in \langle 0, 1)$. Thus, by (17) we have

$$\sum_{n=k}^{\infty} (\alpha_n |a_n| + \beta_n |b_n|) r^{n-1} < B - A \quad (0 \le r < 1). \tag{18}$$

The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=k}^{\infty}(\alpha_n |a_n| + \beta_n |b_n|)$ is nondecreasing sequence. Moreover, by (18) it is bounded by $B - A$. Hence, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=k}^{\infty} (\alpha_n |a_n| + \beta_n |b_n|) = \lim_{n\to\infty} S_n \leq B - A,$$

which yields the assertion (12). □

The following result may be proved in much the same way as Theorem 8.

Theorem 9 *Let $f \in \mathcal{H}$ be a function of the form (4). Then $f \in \mathcal{R}_{\mathcal{T}}^{m}(k, \eta; A, B)$ if and only if*

$$\sum_{n=k}^{\infty} \lambda_n (|a_n| + |b_n|) \leq \frac{B - A}{1 + B}.$$

By Theorems 8 and 9 we have the following corollary.

Corollary 1 *Let*

$$\phi(z) = z + \sum_{n=k}^{\infty} \left(\frac{1}{n-a} z^n + \frac{1}{n+a} \overline{z}^n \right) \quad \left(z \in \mathbb{U},\ a = \frac{1+A}{1+B} \right), \tag{19}$$

$$\omega(z) = z + \sum_{n=k}^{\infty} \left((n-a) z^n + (n+a) \overline{z}^n \right) \quad \left(z \in \mathbb{U},\ a = \frac{1+A}{1+B} \right).$$

Then

$$f \in \mathcal{R}_{\mathcal{T}}^{m}(k, \eta; A, B) \Leftrightarrow f * \phi \in \mathcal{S}_{\mathcal{T}}^{m}(k, \eta; A, B),$$
$$f \in \mathcal{S}_{\mathcal{T}}^{m}(k, \eta; A, B) \Leftrightarrow f * \omega \in \mathcal{R}_{\mathcal{T}}^{m}(k, \eta; A, B).$$

In particular,

$$\mathcal{R}_{\mathcal{T}}^{1}(k, \eta; -1, B) = \mathcal{S}_{\mathcal{T}}^{0}(k, \eta; -1, B).$$

4 Topological properties

We consider the usual topology on $\mathcal{H}$ defined by a metric in which a sequence $\{f_n\}$ in $\mathcal{H}$ converges to f if and only if it converges to f uniformly on each compact subset of $\mathbb{U}$. It follows from the theorems of Weierstrass and Montel that this topological space is complete.

Let $\mathcal{F}$ be a subclass of the class $\mathcal{H}$. A functions $f \in \mathcal{F}$ is called *an extreme point of* $\mathcal{F}$ if the condition

$$f = \gamma f_1 + (1-\gamma) f_2 \quad (f_1, f_2 \in \mathcal{F},\ 0 < \gamma < 1)$$

implies $f_1 = f_2 = f$. We shall use the notation $E\mathcal{F}$ to denote the set of all extreme points of $\mathcal{F}$. It is clear that $E\mathcal{F} \subset \mathcal{F}$.

We say that $\mathcal{F}$ is *locally uniformly bounded* if for each r, $0 < r < 1$, there is a real constant $M = M(r)$ so that

$$|f(z)| \leq M \quad (f \in \mathcal{F},\ |z| \leq r).$$

We say that a class $\mathcal{F}$ is *convex* if

$$\gamma f + (1-\gamma) g \in \mathcal{F} \qquad (f, g \in \mathcal{F},\ 0 \leq \gamma \leq 1).$$

Moreover, we define *the closed convex* hull of $\mathcal{F}$ as the intersection of all closed convex subsets of $\mathcal{H}$ that contain $\mathcal{F}$. We denote the closed convex hull of $\mathcal{F}$ by $\overline{co}\mathcal{F}$.

A real-valued functional $\mathcal{J}:\mathcal{H}\to\mathbb{R}$ is called *convex* on a convex class $\mathcal{F}\subset\mathcal{H}$ if

$$\mathcal{J}\left(\gamma f+(1-\gamma)\,g\right)\leq\gamma\mathcal{J}\left(f\right)+(1-\gamma)\,\mathcal{J}\left(g\right)\quad\left(f,g\in\mathcal{F},\ 0\leq\gamma\leq 1\right).$$

The Krein–Milman theorem (see [14]) is fundamental in the theory of extreme points. In particular, it implies the following lemma.

Lemma 1 [6, pp.45] *Let $\mathcal{F}$ be a non-empty compact convex subclass of the class $\mathcal{H}$ and $\mathcal{J}:\mathcal{H}\to\mathbb{R}$ be a real-valued, continuous and convex functional on $\mathcal{F}$. Then*

$$\max\left\{\mathcal{J}(f):f\in\mathcal{F}\right\}=\max\left\{\mathcal{J}(f):f\in E\mathcal{F}\right\}.$$

Since $\mathcal{H}$ is a complete metric space, Montel's theorem (see [16]) implies the following lemma.

Lemma 2 *A class $\mathcal{F}\subset\mathcal{H}$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded.*

Theorem 10 *The class $\mathcal{S}_{\mathcal{T}}^{m}(k,\eta;A,B)$ is convex and compact subset of $\mathcal{H}$.*

Proof Let $f_1, f_2\in\mathcal{S}_{\mathcal{T}}^{m}(k,\eta;A,B)$ be functions of the form (7), $0\leq\gamma\leq 1$. Since

$$\gamma f_1(z)+(1-\gamma)f_2\left(z\right)=z+\sum_{n=k}^{\infty}\left\{\left(\gamma a_{1,n}+(1-\gamma)\,a_{2,n}\right)z^n+\overline{\left(\gamma b_{1,n}+(1-\gamma)\,b_{2,n}\right)z^n}\right\},$$

and by Theorem 8 we have

$$\begin{aligned}&\sum_{n=k}^{\infty}\left\{\alpha_n\left|\gamma a_{1,n}+(1-\gamma)\,a_{2,n}\right|+\beta_n\left|\gamma b_{1,n}+(1-\gamma)\,b_{2,n}z^n\right|\right\}\\&\quad\leq\gamma\sum_{n=k}^{\infty}\left\{\alpha_n\left|a_{1,n}\right|+\beta_n\left|b_{1,n}\right|\right\}+(1-\gamma)\sum_{n=k}^{\infty}\left\{\alpha_n\left|a_{2,n}\right|+\beta_n\left|b_{2,n}\right|\right\}\\&\quad\leq\gamma\left(B-A\right)+(1-\gamma)\left(B-A\right)=B-A,\end{aligned}$$

the function $\phi=\gamma f_1+(1-\gamma)\,f_2$ belongs to the class $\mathcal{S}_{\mathcal{T}}^{m}(k,\eta;A,B)$. Hence, the class is convex. Furthermore, for $f\in\mathcal{S}_{\mathcal{T}}^{m}(k,\eta;A,B),\ |z|\leq r,\ 0<r<1$, we have

$$|f(z)|\leq r+\sum_{n=k}^{\infty}\left(|a_n|+|b_n|\right)r^n\leq r+\sum_{n=k}^{\infty}\left(\alpha_n\left|a_n\right|+\beta_n\left|b_n\right|\right)\leq r+(B-A)\,. \tag{20}$$

Thus, we conclude that the class $\mathcal{S}_{\mathcal{T}}^{m}(k,\eta;A,B)$ is locally uniformly bounded. By Lemma 2, we only need to show that it is closed *i.e.* if $f_l\in\mathcal{S}_{\mathcal{T}}^{m}(k,\eta;A,B)\quad(l\in\mathbb{N})$ and $f_l\to f$, then $f\in\mathcal{S}_{\mathcal{T}}^{m}(k,\eta;A,B)$. Let f_l and f are given by (7) and (4), respectively. Using Theorem 8 we have

$$\sum_{n=k}^{\infty}\left(\left|\alpha_n a_{l,n}\right|+\left|\beta_n b_{l,n}\right|\right)\leq B-A\quad\left(l\in\mathbb{N}\right). \tag{21}$$

Since $f_l\to f$, we conclude that $\left|a_{l,n}\right|\to|a_n|$ and $\left|b_{l,n}\right|\to|b_n|$ as $l\to\infty\ \ (n\in\mathbb{N})$. The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=k}^{\infty}\left(\alpha_n\left|a_n\right|+\beta\left|_n b_n\right|\right)$ is nondecreasing sequence. Moreover, by (21) it is bounded by $B-A$. Therefore, the sequence $\{S_n\}$

is convergent and

$$\sum_{n=k}^{\infty} \left(\alpha_n |a_n| + \beta_n |b_n|\right) = \lim_{n\to\infty} S_n \le B - A.$$

This gives the condition (12), and, in consequence, $f \in \mathcal{S}_{\mathcal{T}}^m(k, \eta; A, B)$, which completes the proof. □

Theorem 11

$$E\mathcal{S}_{\mathcal{T}}^m(k, \eta; A, B) = \{h_n : n \in \mathbb{N}_{k-1}\} \cup \{g_n : n \in \mathbb{N}_k\},$$

where

$$h_{k-1}(z) = z,\; h_n(z) = z - \frac{B - A}{\alpha_n e^{i(n-1)\eta}} z^n,\;\; g_n(z) = z + \frac{B - A}{\beta_n e^{i(1-n)\eta}} \overline{z}^n \quad (z \in \mathbb{U}). \tag{22}$$

Proof Suppose that $0 < \gamma < 1$ and

$$g_n = \gamma f_1 + (1 - \gamma) f_2,$$

where $f_1, f_2 \in \mathcal{S}_{\mathcal{T}}^m(k, \eta; A, B)$ are functions of the form (7). Then, by (12) we have $\left|b_{1,n}\right| = \left|b_{2,n}\right| = \frac{B-A}{\beta_n}$, and, in consequence, $a_{1,l} = a_{2,l} = 0$ for $l \in \mathbb{N}_k$ and $b_{1,l} = b_{2,l} = 0$ for $l \in \mathbb{N}_k \setminus \{n\}$. It follows that $g_n = f_1 = f_2$, and consequently $g_n \in E\mathcal{S}_{\mathcal{T}}^*(k, \eta; A, B)$. Similarly, we verify that the functions h_n of the form (22) are the extreme points of the class $\mathcal{S}_{\mathcal{T}}^m(k, \eta; A, B)$. Now, suppose that a function f belongs to the set $E\mathcal{S}_{\mathcal{T}}^m(k, \eta; A, B)$ and f is not of the form (22). Then there exists $m \in \mathbb{N}_k$ such that

$$0 < |a_m| < \frac{B - A}{\alpha_m} \text{ or } 0 < |b_m| < \frac{B - A}{\beta_m}.$$

If $0 < |a_m| < \frac{B-A}{\alpha_m}$, then putting

$$\gamma = \frac{\alpha_m |a_m|}{B - A},\; \varphi = \frac{1}{1 - \gamma}\left(f - \gamma h_m\right),$$

we have that $0 < \gamma < 1,\; h_m \ne \varphi$ and

$$f = \gamma h_m + (1 - \gamma)\varphi.$$

Thus, $f \notin E\mathcal{S}_{\mathcal{T}}^m(k, \eta; A, B)$. Similarly, if $0 < |b_m| < \frac{B-A}{\beta_n}$, then putting

$$\gamma = \frac{\beta_m |b_m|}{B - A},\; \phi = \frac{1}{1 - \gamma}\left(f - \gamma g_m\right),$$

we have that $0 < \gamma < 1,\; g_m \ne \phi$ and

$$f = \gamma g_m + (1 - \gamma)\phi.$$

It follows that $f \notin E\mathcal{S}_{\mathcal{T}}^m(k, \eta; A, B)$, and the proof is completed. □

5 Applications

It is clear that if the class

$$\mathcal{F} = \{f_n \in \mathcal{H} : n \in \mathbb{N}\},$$

is locally uniformly bounded, then

$$\overline{co}\mathcal{F}=\left\{\sum_{n=1}^{\infty}\gamma_n f_n:\quad \sum_{n=1}^{\infty}\gamma_n=1,\ \gamma_n\geq 0\ \ (n\in\mathbb{N})\right\}. \tag{23}$$

Thus, by Theorem 6 we have the following corollary.

Corollary 2

$$\mathcal{S}_{\mathcal{T}}^{m}(k,\eta;A,B)=\left\{\sum_{n=k-1}^{\infty}(\gamma_n h_n+\delta_n g_n):\ \sum_{n=k-1}^{\infty}(\gamma_n+\delta_n)=1\ \ (\delta_{k-1}=0,\gamma_n,\delta_n\geq 0)\right\},$$

where h_n, g_n are defined by (22).

For each fixed value of $m,n\in\mathbb{N}_k,\ \ z\in\mathbb{U}$, the following real-valued functionals are continuous and convex on $\mathcal{H}$:

$$\mathcal{J}(f)=|a_n|\,,\_\mathcal{J}(f)=|b_n|\,,\ \mathcal{J}(f)=|f(z)|\ \ \mathcal{J}(f)=|D_{\mathcal{H}}f(z)|\quad (f\in\mathcal{H})\,. \tag{24}$$

Moreover, for $\gamma\geq 1,\ 0<r<1$, the real-valued functional

$$\mathcal{J}(f)=\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{\gamma}d\theta\right)^{1/\gamma}\qquad (f\in\mathcal{H}) \tag{25}$$

is also continuous and convex on $\mathcal{H}$.

Therefore, by Lemma 1 and Theorem 6 we have the following corollaries.

Corollary 3 *Let $f\in\mathcal{S}_{\mathcal{T}}^{m}(k,\eta;A,B)$ be a function of the form* (11)*. Then*

$$|a_n|\leq\frac{B-A}{\alpha_n},\ \ |b_n|\leq\frac{B-A}{\beta_n}\quad (n\in\mathbb{N}_k), \tag{26}$$

where α_n,β_n are defined by (13)*. The result is sharp. The functions h_n, g_n of the form* (22) *are the extremal functions.*

Corollary 4 *Let $f\in\mathcal{S}_{\mathcal{T}}^{m}(k,\eta;A,B)$, $|z|=r<1$. Then*

$$r-\frac{B-A}{\lambda_k\left(k-1+kB-A\right)}r^k\leq|f(z)|\leq r+\frac{B-A}{\lambda_k\left(k-1+kB-A\right)}r^k,$$
$$r-\frac{k\left(B-A\right)}{\lambda_k\left(k-1+kB-A\right)}r^k\leq|D_{\mathcal{H}}f(z)|\leq r+\frac{k\left(B-A\right)}{\lambda_k\left(k-1+kB-A\right)}r^k,$$

where λ_k is defined by (11)*. The result is sharp. The function h_k of the form (22) is the extremal function.*

Corollary 5 *Let $0<r<1,\ \gamma\geq 1$. If $f\in\mathcal{S}_{\mathcal{T}}^{m}(k,\eta;A,B)$, then*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\gamma}d\theta\leq\frac{1}{2\pi}\int_0^{2\pi}\left|h_k(re^{i\theta})\right|^{\lambda}d\theta,$$
$$\frac{1}{2\pi}\int_0^{2\pi}|D_{\mathcal{H}}f(z)|^{\gamma}d\theta\leq\frac{1}{2\pi}\int_0^{2\pi}\left|D_{\mathcal{H}}h_k(re^{i\theta})\right|^{\gamma}d\theta,$$

where h_k is the function defined by (22).

The following covering result follows from Corollary 4.

Corollary 6 *If* $f \in \mathcal{S}^m_{\mathcal{T}}(k, \eta; A, B)$, *then* $\mathbb{U}(r) \subset f(\mathbb{U})$, *where*

$$r = 1 - \frac{B - A}{\lambda_k (k - 1 + kB - A)}.$$

By using Corollary 1 and the results above we obtain corollaries listed below.

Corollary 7 *The class* $\mathcal{R}^m_{\mathcal{T}}(k, \eta; A, B)$ *is convex and compact subset of* $\mathcal{H}$. *Moreover,*

$$E\mathcal{R}^m_{\mathcal{T}}(k, \eta; A, B) = \{h_n : n \in \mathbb{N}_{k-1}\} \cup \{g_n : n \in \mathbb{N}_k\},$$

and

$$\mathcal{R}^m_{\mathcal{T}}(k, \eta; A, B) = \left\{ \sum_{n=k-1}^{\infty} (\gamma_n h_n + \delta_n g_n) : \sum_{n=k-1}^{\infty} (\gamma_n + \delta_n) = 1 \ (\delta_{k-1} = 0, \gamma_n, \delta_n \geq 0) \right\},$$

where

$$h_{k-1}(z) = z, \ h_n(z) = z - \frac{(B - A)\, e^{i(1-n)\eta}}{(1 + B)\,\lambda_n} z^n, \ g_n(z) = z + \frac{(B - A)\, e^{i(n-1)\eta}}{(1 + B)\,\lambda_n} \overline{z}^n \ (z \in \mathbb{U}). \tag{27}$$

Corollary 8 *Let* $f \in \mathcal{R}^m_{\mathcal{T}}(k, \eta; A, B)$ *be a function of the form* (4). *Then*

$$|a_n| \leq \frac{B - A}{(1 + B)\,\lambda_n}, |b_n| \leq \frac{B - A}{(1 + B)\,\lambda_n} \quad (n \in \mathbb{N}_k),$$

$$r - \frac{B - A}{(1 + B)\,\lambda_k} r^k \leq |f(z)| \leq r + \frac{B - A}{(1 + B)\,\lambda_k} r^k \quad (|z| = r < 1),$$

$$r - \frac{(B - A)k}{(1 + B)\,\lambda_k} r^k \leq |D_{\mathcal{H}} f(z)| \leq r + \frac{(B - A)\,k}{(1 + B)\,\lambda_k} r^k \quad (|z| = r < 1),$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\gamma} d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \left| h_k(re^{i\theta}) \right|^{\lambda} d\theta,$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left| D_{\mathcal{H}} f(re^{i\theta}) \right|^{\gamma} d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \left| D_{\mathcal{H}} h_k(re^{i\theta}) \right|^{\gamma} d\theta,$$

where λ_n *is defined by* (11). *The results are sharp. The functions* h_n, g_n *of the form* (27) *are the extremal functions.*

Corollary 9 *If* $f \in \mathcal{R}^m_{\mathcal{T}}(k, \eta; A, B)$, *then* $\mathbb{U}(r) \subset f(\mathbb{U})$, *where*

$$r = 1 - \frac{B - A}{(1 + B)\,\lambda_k}.$$

The classes $\mathcal{S}^n_{\mathcal{H}}(k; A, B)$ and $\mathcal{R}^n_{\mathcal{H}}(k; A, B)$ are related to harmonic starlike functions, harmonic convex functions and harmonic Janowski functions.

The classes $\mathcal{S}_{\mathcal{H}}(\alpha) := \mathcal{S}^0_{\mathcal{H}}(2; 2\alpha - 1, 1)$ and $\mathcal{K}_{\mathcal{H}}(\alpha) := \mathcal{S}^1_{\mathcal{H}}(2; 2\alpha - 1, 1)$ were investigated by Jahangiri [9] (see also [2,18]). They are the classes of starlike and convex functions of order α, respectively. The classes $N_{\mathcal{H}}(\alpha) := \mathcal{R}^1_{\mathcal{H}}(2; 2\alpha - 1, 1)$ and $R_{\mathcal{H}}(\alpha) := \mathcal{R}^2_{\mathcal{H}}(2; 2\alpha - 1, 1)$ were studied in [1] (see also [13]). Finally, the classes $\mathcal{S}_{\mathcal{H}} := \mathcal{S}_{\mathcal{H}}(0)$ and $\mathcal{K}_{\mathcal{H}} := \mathcal{K}_{\mathcal{H}}(0)$

are the classes of functions which are starlike and convex in $\mathbb{U}(r)$, respectively, for all $r \in (0, 1\rangle$. We should notice, that the classes $\mathcal{S}(A, B) := \mathcal{S}_{\mathcal{H}}(2; A, B) \cap \mathcal{A}$ and $\mathcal{R}(A, B) := \mathcal{R}_{\mathcal{H}}(2; A, B) \cap \mathcal{A}$ were introduced by Janowski [12].

Using Theorems 1 or 2 to the classes defined above we obtain corollaries listed below (see also [6]).

Corollary 10

$$\mathcal{S}_{\mathcal{H}}(\alpha) = \left\{\psi_\xi : |\xi| = 1\right\}^*,$$

where

$$\psi_\xi(z) := z\frac{2(1-\alpha)\xi + (1-\xi+2\alpha\xi)z}{(1-z)^2} - \overline{z}\frac{2+2\alpha\xi - (1-\xi+2\alpha\xi)\overline{z}}{(1-\overline{z})^2} \quad (z \in \mathbb{U}).$$

Corollary 11

$$\mathcal{K}_{\mathcal{H}}(\alpha) = \left\{\psi_\xi : |\xi| = 1\right\}^*,$$

where

$$\psi_\xi(z) := z\frac{(1-\alpha)\xi + (1+\alpha\xi)z}{(1-z)^3} + \overline{z}\frac{1+\alpha\xi + (1-\alpha)\xi\overline{z}}{(1-\overline{z})^3} \quad (z \in \mathbb{U}).$$

Corollary 12

$$N_{\mathcal{H}}(\alpha) = \left\{\delta_\xi : |\xi| = 1\right\}^*,$$

where

$$\delta_\xi(z) := z\frac{2(1-\alpha)\xi - (2\alpha\xi - \xi + 1)\left(z^2 - 2z\right)}{(1-z)^2} - \overline{z}\frac{1+\xi}{(1-\overline{z})^2} \quad (z \in \mathbb{U}).$$

Corollary 13

$$\mathcal{S}_{\mathcal{H}} = \left\{\psi_\xi : |\xi| = 1\right\}^*,$$

where

$$\psi_\xi(z) := z\frac{2\xi + (1-\xi)z}{(1-z)^2} - \overline{z}\frac{2 - (1-\xi)\overline{z}}{(1-\overline{z})^2} \quad (z \in \mathbb{U}).$$

Corollary 14

$$\mathcal{K}_{\mathcal{H}} = \left\{\psi_\xi : |\xi| = 1\right\}^*,$$

where

$$\psi_\xi(z) := z\frac{\xi + z}{(1-z)^3} + \overline{z}\frac{1+\xi\overline{z}}{(1-\overline{z})^3} \quad (z \in \mathbb{U}).$$

The class $\mathcal{S}^n_{\mathcal{H}}(k; A, B)$ generalize also classes of starlike functions of complex order. The class $\mathcal{CS}_{\mathcal{H}}(\gamma) := \mathcal{S}_{\mathcal{H}}(2; 1-2\gamma, 1)$ $(\gamma \in \mathbb{C} \setminus \{0\})$ was defined by Yalçin and Öztürk [24]. In particular, if we put $\gamma := \frac{1-\alpha}{1+e^{i\eta}}$, then we obtain the class $\mathcal{RS}_{\mathcal{H}}(\alpha, \eta) := \mathcal{S}_{\mathcal{H}}\left(2; \frac{2\alpha-1+e^{i\eta}}{1+e^{i\eta}}, 1\right)$ studied by Yalçin et al. [25]. It is the class of functions $f \in \mathcal{H}_0$ such that

$$\operatorname{Re}\left\{\left(1+e^{i\eta}\right)\frac{D_{\mathcal{H}}f(z)}{f(z)} - e^{i\eta}\right\} > \alpha \quad (z \in \mathbb{U},\ \eta \in \mathbb{R}).$$

Thus, by Theorem 4 we have the following two corollaries.

Corollary 15

$$\mathcal{CS}_{\mathcal{H}}(\gamma) = \left\{\psi_\xi : |\xi| = 1\right\}^*,$$

where

$$\psi_{\xi}(z) := z\frac{2\gamma\xi + (1+\xi-2\gamma\xi)z}{(1-z)^2} - \overline{z}\frac{2+2(1-\gamma)\xi - (1+\xi-2\gamma\xi)\overline{z}}{(1-\overline{z})^2} \quad (z \in \mathbb{U}). \tag{28}$$

Corollary 16

$$\mathcal{CS}_{\mathcal{H}}^{n}(\alpha, \eta) = \left\{\psi_{\xi} : |\xi| = 1\right\}^{*},$$

where ψ_{ξ} *is defined by* (28) *with* $\gamma := \frac{1-\alpha}{1+e^{i\eta}}$.

Remark 1 By choosing the parameters in the defined classes of functions we can obtain new and also well-known results (see for example [1–3,5–13,18,21–25]) .

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