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ORIGINAL PAPER

Formulas for Poisson–Charlier, Hermite, Milne-Thomson and other type polynomials by their generating functions and p-adic integral approach

Yilmaz Simsek[1]

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Abstract The main propose of this article is to investigate and modify Hermite type polynomials, Milne-Thomson type polynomials and Poisson–Charlier type polynomials by using generating functions and their functional equations. By using functional equations of the generating functions for these polynomials, we not only derive some identities and relations including the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Poisson–Charlier polynomials, the Milne-Thomson polynomials and the Hermite polynomials, but also study some fundamental properties of these functions and polynomials. Moreover, we survey orthogonality properties of these polynomials. Finally, by applying another method which is related to p-adic integrals, we derive some formulas and combinatorial sums associated with some well-known numbers and polynomials.

Keywords Generating function · Functional equation · Orthogonal polynomials · Bernoulli numbers and polynomials · Euler numbers and polynomials · Stirling numbers · Milne-Thomson polynomials · Poisson–Charlier polynomials · Hermite polynomials · Special functions · Special numbers and polynomials · p-adic integral

Mathematics Subject Classification 05A15 · 11B83 · 11B68 · 11S80 · 33C45

1 Introduction

Special polynomials and numbers with their generating functions have many applications in mathematics, physics, engineering and other sciences areas. Polynomials are among the most important tools for constructing mathematical models, computational algorithms, and solving engineering problems. Those polynomials have many basic algebraic operations which are

✉ Yilmaz Simsek
ysimsek@akdeniz.edu.tr

1 Department of Mathematics, Faculty of Science University of Akdeniz, 07058 Antalya, Turkey

finite evaluation schemes, closure under addition, multiplication, differentiation, integration, and composition (*cf.* [1–41]; see also the references cited therein).

By using generating functions with their functional equations and derivative operator and p-adic integral, we give various identities and relations including Bernoulli numbers and polynomials, Euler numbers and polynomials, Stirling numbers, Milne-Thomson polynomials, Poisson–Charlier polynomials, Hermite polynomials, and other special numbers and polynomials. Those identities and relations are of potential usefulness in mathematics, physics, engineering and other research areas.

Throughout this paper, we use the following notations:

$\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$. $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. We assume that $\ln(z)$ denotes the principal branch. Furthermore, $0^n = 1$ if $n = 0$, and, $0^n = 0$ if $n \in \mathbb{N}$.

$$\binom{x}{v} = \frac{x(x-1)\cdots(x-v+1)}{v!} = \frac{\{x\}_v}{v!}$$

(*cf.* [1–41]; see also the references cited therein).

The Bernoulli polynomials are defined by means of the following generating function:

$$F_B(t, x) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1a}$$

Substituting $x = 0$ into (1a), we have the Bernoulli numbers, that is,

$$B_n = B_n(0).$$

The Euler polynomials are defined by means of the following generating function:

$$F_{E1}(t, x) = \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{2}$$

Substituting $x = 0$ into (2), we have Euler numbers of the first kind, that is,

$$E_n = E_n(0).$$

(*cf.* [1–41]; see also the references cited therein).

Euler numbers of the second kind are defined by means of the following generating function:

$$F_{E2}(t) = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!} \tag{3}$$

(*cf.* [1–41]; see also the references cited therein). The numbers E_n^* are related to the polynomials $E_n(x)$, i.e.

$$E_n^* = 2^n E_n\left(\frac{1}{2}\right)$$

(*cf.* [1–41]; see also the references cited therein).

Stirling numbers of the second kind are defined by the following generating function:

$$F_S(t, v) = \frac{\left(e^t - 1\right)^v}{v!} = \sum_{n=0}^{\infty} S_2(n, v) \frac{t^n}{n!}, \quad (v \in \mathbb{N}_0) \tag{4}$$

(cf. [1–41]; see also the references cited in each of these earlier works).

Stirling numbers of the first kind are given by

$$F_{S1}(t,v;\lambda)=\frac{(\ln(1+t))^{v}}{v!}=\sum_{n=0}^{\infty}S_1(n,v)\frac{t^n}{n!},\quad (v\in\mathbb{N}_0) \tag{5}$$

(cf. [1–41]; see also the references cited therein).

Two-variable Hermite polynomials are defined by means of the following generating functions:

$$e^{xt+yt^j}=\sum_{n=0}^{\infty}H_n^{(j)}(x,y)\frac{t^n}{n!} \tag{6}$$

(*cf.* [2,8,9,28,30]).

Using (6), it is easy to see that

$$H_n^{(j)}(x,y)=n!\sum_{r=0}^{[\frac{n}{j}]}\frac{x^{n-jr}y^r}{r!(n-jr)!},\quad (j\in\mathbb{N}\setminus\{1\}) \tag{7}$$

(*cf.* [2,8,9,28,30]).

Setting $y=j=1$ in (7), one has the Hermite polynomials

$$H_n(x)=H_n^{(1)}(x,1)$$

(*cf.* [8,9,28,30]).

The Rodrigues formula for the Hermite polynomials is given by

$$e^{-x^2}H_n(x)=(-1)^n\frac{d^n}{dx^n}\left\{e^{-x^2}\right\},\quad (n\in\mathbb{N}_0)$$

(*cf.* [8,9,28]).

By using the above formula, the orthogonality property of the Hermite polynomials is given as follows:

$$\int_{-\infty}^{\infty}e^{-x^2}H_n(x)H_m(x)\,dx=0,$$

where $m,n\in\mathbb{N}_0$ and $m\neq n$ (*cf.* [8,9,28]).

Setting $x=0$, $y=-1$ and $j=2$ in (6), we have the generating function for the Hermite numbers as follows:

$$F_H(t)=e^{-t^2}=\sum_{n=0}^{\infty}H_n\frac{t^n}{n!} \tag{8}$$

(*cf.* [2,8,9,13,28,30]).

1.1 *p*-adic integral

In order to give combinatorial identities and sums including Bernoulli numbers and polynomials, Euler numbers and polynomials, Stirling numbers, Milne-Thomson polynomials, Poisson–Charlier polynomials, Hermite polynomials, and other special numbers and polynomials, we need p-adic integral and its integral equations.

Let $\mathbb{Z}_p$ be a set of p-adic integers. Let $\mathbb{K}$ be a field with a complete valuation and $C^1(\mathbb{Z}_p\to\mathbb{K})$ be a set of continuous derivative functions. That is $C^1(\mathbb{Z}_p\to\mathbb{K})$ is contained in $\left\{f:\mathbb{X}\to\mathbb{K}: f(x) \text{ is differentiable and } \frac{d}{dx}f(x) \text{ is continuous}\right\}$. Kim [17] introduced and systematically studied the following family of the p-adic q-integral which provides a unification of the Volkenborn integral:

$$I_q(f(x)) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \tag{9}$$

where $q \in \mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$, set of p-adic rational numbers, with $|1-q|_p < 1$, $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$,

$$[x] = [x : q] = \begin{cases} \frac{1-q^x}{1-q}, q \neq 1 \\ x, q = 1 \end{cases}$$

and $\mu_q(x) = \mu_q\left(x + p^N\mathbb{Z}_p\right)$ denotes the q-distribution on $\mathbb{Z}_p$, defined by

$$\mu_q\left(x + p^N\mathbb{Z}_p\right) = \frac{q^x}{\left[p^N\right]_q},$$

(*cf.* [17]).

Remark 1 If $q \to 1$, then (9) reduces to the Volkenborn integral:

$$\lim_{q\to 1} I_q(f(x)) = I_1(f(x))$$

where

$$I_1(f(x)) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x). \tag{10}$$

and $\mu_1(x)$ denotes the Haar distribution. $I_1(f(x))$ is so-called the bosonic integral (*cf.* [31]); see also the references cited therein). This integral has many applications not only in mathematics, but also in mathematical physics. By using this integral and its integral equations, various different generating functions have been constructed.

Remark 2 If $q \to -1$, then (9) reduces to the p-adic fermionic integral:

$$\lim_{q\to -1} I_q(f(x)) = I_{-1}(f(x)),$$

where

$$I_{-1}(f(x)) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x) \tag{11}$$

and

$$\mu_{-1}\left(x + p^N\mathbb{Z}_p\right) = \frac{(-1)^x}{p^N}$$

(*cf.* [16]). By using the p-adic fermionic integral, various different generating functions have been constructed.

The following p-adic integrals formulas are of importance for the following sections.
The p-adic bosonic integral representation of the Bernoulli numbers is given by

$$B_n = \int_{\mathbb{Z}_p} x^n d\mu_1(x), \tag{12}$$

(*cf.* [16,17,31]; see also the references cited therein).

The p-adic fermionic integral representation of the Euler numbers is given by

$$E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x), \tag{13}$$

(*cf.* [10,14–17], [29, p. 45]; see also the references cited therein).

The p-adic integrals representations of the falling factorials are given by

$$\int_{\mathbb{Z}_p} \{x\}_n d\mu_1(x) = \sum_{k=0}^{n} S_1(n,k) B_k = \frac{(-1)^n n!}{n+1}, \tag{14}$$

and

$$\int_{\mathbb{Z}_p} \{x\}_n d\mu_{-1}(x) = \sum_{k=0}^{n} S_1(n,k) E_k = (-1)^n 2^{-n} n!, \tag{15}$$

(*cf.* [10,14–17], [29, p. 45]; see also the references cited therein).

Results of this paper are summarized as follows:

In Sect. 2, we give some survey on the Poisson–Charlier polynomials with their orthogonality property with respect to the Poisson distribution. We also give derivative formulas with a recurrence relation.

In Sect. 3, we construct Milne-Thomson type polynomials including Milne-Thomson base Poisson–Charlier type polynomials, Milne-Thomson base Laguerre polynomials, and Milne-Thomson base Lah numbers. We also give some properties of these polynomials with their generating functions.

In Sect. 4, by using p-adic integral, we derive various identities and relations including Bernoulli numbers and polynomials, Euler numbers and polynomials, Stirling numbers, Milne-Thomson polynomials, Poisson–Charlier polynomials, Hermite polynomials, and other special numbers and polynomials.

2 The Poisson–Charlier polynomials

The Poisson–Charlier polynomials are among the family of Sheffer sequences. These polynomials have been studied, recently, by Jordan, Erdelyi, Szgö, Roman and other mathematicians. These polynomials are defined by means of the following generating function:

$$F_{pc}(t,x;a) = e^{-t}\left(\frac{t}{a}+1\right)^x = \sum_{n=0}^{\infty} C_n(x;a)\frac{t^n}{n!} \tag{16}$$

(*cf.* [30, p. 120]).

We now give explicit formula for the Poisson–Charlier polynomials. Applying binomial theorem for the series to Eq. (16), we obtain

$$\sum_{n=0}^{\infty} C_n(x;a)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \sum_{n=0}^{\infty} \frac{\{x\}_n}{a^n}\frac{t^n}{n!}.$$

By using the Cauchy product rule to the above equation, we have

$$\sum_{n=0}^{\infty} C_n(x;a)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{\{x\}_j}{a^j}\frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have an explicit formula for the polynomials $C_n(x;a)$ as follows:

$$C_n(x;a) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{\{x\}_j}{a^j} \tag{17}$$

(*cf.* [30, p. 120]).

By using (16), we also have the following alternative equation:

$$F_{pc}(t,x;a)e^t = \left(\frac{t}{a}+1\right)^x.$$

By using the above equation, we have

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}C_j(x;a)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\{x\}_n a^{-n}\frac{t^n}{n!}.$$

Therefore, we get the following identity:

Theorem 1

$$a^n\sum_{j=0}^{n}\binom{n}{j}C_j(x;a) = \{x\}_n. \tag{18}$$

2.1 Derivative formulas for the polynomials $C_n(x;a)$

In this section, we give some partial differential equations of the generating function of the function $F_{pc}(t,x;a)$. By using these equations, we derive a recurrence formula and derivative formula for the polynomials $C_n(x;a)$.

We set

$$\frac{\partial^v}{\partial t^v}\left\{F_{pc}(t,x;a)\right\} = \sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}a^{-j}F_{pc}(t,x-j;a)\prod_{l=1}^{j}(x+1-l), \tag{19}$$

where the empty product is understood to be unity, i. e., 1. Here and in the following,

$$\prod_{l=1}^{0}(x+1-l) = 1.$$

By using the above equation, we derive the following recurrence for the polynomials $C_n(x;a)$ relation:

Theorem 2 *Let* $v \in \mathbb{N}$. *Then we have*

$$C_{n+v}(x;a) = \sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}a^{-j}C_n(x-j;a)\prod_{l=1}^{j}(x+1-l). \tag{20}$$

Proof Combining the above equation with (16) and (19), we get

$$\sum_{n=0}^{\infty} C_n(x;a)\frac{t^{n-v}}{(n-v)!} = \sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}a^{-j}\prod_{l=0}^{j}(x+1-l) \times \sum_{n=0}^{\infty} C_n(x-j;a)\frac{t^n}{n!}.$$

Equalizing the coefficients $\frac{t^n}{n!}$ on the both sides of the equation yields the assertion of theorem. □

Remark 3 Substituting $v = 1$ into (20), we get

$$C_{n+1}(x;a) = -C_n(x;a) + a^{-1}xC_n(x-1;a),$$

(*cf.* [30, p.121, Eq-(4.3.10)]).

Theorem 3 *Let $v \in \mathbb{N}$. Then we have*

$$\frac{\partial^v}{\partial x^v}\{C_n(x;a)\} = v!\sum_{j=0}^{n}\binom{n}{j}a^{j-n}C_j(x;a)S_1(n-j,v).$$

Proof We set the following functional equation:

$$\frac{\partial^v}{\partial x^v}\left\{F_{pc}(t,x;a)\right\} = v!F_{pc}(t,x;a)F_{S1}\left(\frac{t}{a},v;\lambda\right).$$

By combining the above equation with (16) and (5), we get

$$\sum_{n=0}^{\infty}\frac{\partial^v}{\partial x^v}\{C_n(x;a)\}\frac{t^n}{n!} = v!\sum_{n=0}^{\infty}C_n(x;a)\frac{t^n}{n!}\sum_{n=0}^{\infty}a^{-n}S_1(n,v)\frac{t^n}{n!}.$$

By Cauchy product in the above equation, we obtain

$$\sum_{n=0}^{\infty}\frac{\partial^v}{\partial x^v}\{C_n(x;a)\}\frac{t^n}{n!} = v!\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}a^{j-n}C_j(x;a)S_1(n-j,v)\frac{t^n}{n!}.$$

Equalizing the coefficients $\frac{t^n}{n!}$ on both sides of the equation yields the assertion of theorem. □

2.2 Orthogonality property of the Poisson–Charlier polynomials with respect to the Poisson distribution

Here, we survey orthogonality property of the Poisson–Charlier polynomials with respect to the Poisson distribution, which is defined as follows:

Let $\lambda > 0$ and let X be a random variable of the Poisson distribution with mean λ. The Poisson distribution is defined by

$$f(x|\lambda) = \begin{cases} e^{-\lambda}\frac{\lambda^x}{x!}, & x \in \mathbb{N}_0 \\ 0, & \text{otherwise} \end{cases} \tag{21}$$

(*cf.* [6, p. 288, Def. 5.4.1]).

Substituting $\lambda = a$ and $x = k$, $(k \in \mathbb{N}_0)$ into (21), and combining with the polynomials $C_n(x; a)$ yields the following relations:

$$f(k|a) = e^{-a}\frac{a^k}{k!}.$$

Therefore, we have the following orthogonality properties:

$$\sum_{k=0}^{\infty} C_n(k; a)C_m(k; a)f(k|a) = a^{-n}n!\delta_{n,m}$$

where

$$\delta_{n,m} = \begin{cases} 0, & m \neq n \\ 1, & m = n. \end{cases}$$

In [35], we give relations between the Bernstein basis functions, the binomial distribution and the Poisson distribution. Let a and b be real numbers. Let $0 \leq \frac{x-a}{b-a} \leq 1$ and $0 \leq \frac{b-x}{b-a} \leq 1$. The Bernstein basis functions, related to a and b, are defined by

$$B_k^n(x; a, b) = \binom{n}{k}\left(\frac{x-a}{b-a}\right)^k\left(\frac{b-x}{b-a}\right)^{n-k}. \tag{22}$$

Substituting $a = 0$ and $b = 1$, (22) reduces to the binomial (Newton) distribution or the Bernstein basis functions:

$$B_k^n(x) = \binom{n}{k}x^k(1-x)^{n-k},$$

(*cf.* [21]).

Let $\mathbf{E}(x; a, b)$ be expected value or mean of the function $B_k^n(x)$. We have

$$\mathbf{E}(x; a, b) = \sum_{k=0}^{n} kB_k^n(x; a, b) = n\left(\frac{x-a}{b-a}\right),$$

(*cf.* [35]). Since

$$\frac{x-a}{b-a} + \frac{b-x}{b-a} = 1,$$

we have

$$B_k^n\left(\frac{b-a}{n}\mathbf{E}(x; a, b) + a; a, b\right) = \frac{\{n\}_k}{k!n^k}\mathbf{E}^k(x; a, b)\left(1 - \frac{\mathbf{E}(x; a, b)}{n}\right)^{n-k}.$$

If $n \to \infty$ in the above equation, we have the well-known Poisson distribution; that is

$$\lim_{n\to\infty} B_k^n\left(\frac{b-a}{n}\mathbf{E}(x; a, b) + a; a, b\right) = \frac{\mathbf{E}^k(x; a, b)e^{-\mathbf{E}(x;a,b)}}{k!}, \tag{23}$$

(*cf.* [6, p. 291], [35]).

3 Milne-Thomson type polynomials and numbers

In this section, we construct Milne-Thomson type polynomials. These polynomials are related to the very well-known polynomials and numbers so-called Hermite base Poisson–Charlier type polynomials, Milne-Thomson base Laguerre polynomials, and Milne-Thomson base Lah numbers. We also give some observations on these polynomials and numbers.

We define the following generating functions for three-variable polynomials $y_6\big(n; x, y, z; a, b, v\big)$:

$$\mathcal{G}(t, x, y, z; a, b, v) = (b + f(t, a))^z e^{xt + yh(t,v)} = \sum_{n=0}^{\infty} y_6(n; x, y, z; a, b, v) \frac{t^n}{n!}. \quad (24)$$

where $f(t, a)$ is a member of family of analytic functions or meromorfic functions, a and b are any real numbers, v is positive integer.

Writing $x = 0$, $y = z = 1$ in (24), we get the following new numbers

$$y_6(n; 0, 1, 1; a, b, v) = y_6(n; a, b, v).$$

Therefore, the numbers $y_6(n; a, b, v)$ are defined by the following generating function:

$$\mathcal{J}(t; a, b, v) = (b + f(t, a)) e^{h(t,v)} = \sum_{n=0}^{\infty} y_6(n; a, b, v) \frac{t^n}{n!}.$$

Some special cases of these numbers give well-known numbers such as the Milne-Thomson numbers, the Hermite numbers, the Bernoulli and Euler numbers, the Lah numbers, and the others.

For example, writing $b = 0$ in the above equation, we have the Milne-Thomson numbers of order a, $\phi_n^{(a)}$, that is

$$y_6(n; a, 0, v) = \phi_n^{(a)}$$

(*cf.* [23, p. 514, Eq-(2)]).

Motivation of the above generating functions is briefly given as follows:

Setting $z = 1$ and $b = 0$ into (24), we get a relation between the $y_6(n; x, y, z; a, b, v)$ and the polynomials $\Psi_n^{(a)}(x, y, v)$:

$$\Psi_n^{(a)}(x, y, v) = y_6(n; x, y, 1; a, 0, v),$$

which is associated with the Hermite base Bernoulli type numbers and polynomials, the generalized Milne-Thomson's polynomials, the two-variable Hermite polynomials, the Laguerre polynomials, and the others (*cf.* [7,11]).

Setting $z = 1$, $b = 0$, $y = 1$ and $h(t, 0) = g(t)$ into (24), we get a relation between the $y_6(n; x, y, z; a, b, v)$ and the Milne-Thomson polynomials $\Phi_n^{(a)}(x)$:

$$y_6(n; x, 1, 1; a, 0, 0) = \Phi_n^{(a)}(x)$$

(*cf.* [23]).

Setting $z = 1$, $b = 0$, $y = 1$, $f(t, a) = 1$ and $h(t, v) = -\frac{vt^2}{2}$ in (24), we get a relation between the $y_6(n; x, y, z; a, b, v)$ and the Hermite polynomials $H_n^{(v)}(x)$:

$$y_6(n; x, 1, 1; a, 0, v) = H_n^{(v)}(x)$$

(*cf.* [13,28,30]) and for $x = 0$ and $v = 2$, the polynomials $H_n^{(v)}(x)$ reduce to the Hermite polynomials. That is

$$y_6(n; 0, 1, 1; a, 0, 2) = H_n$$

where H_n denotes the Hermite numbers which are defined by

$$H_{2n} = \frac{(-1)^n (2n)!}{n!}, \qquad H_{2n+1} = 0$$

for $n \geq 0$ (*cf.* [13,28,30]).

Derivative formula with respect to x of the polynomials $y_6(n; x, y, z; a, b, v)$ is given by the following theorem:

Theorem 4

$$\frac{\partial^k}{\partial x^k} y_6(n; x, y, z; a, b, v) = \{n\}_k y_6(n-k; x, y, z; a, b, v).$$

Proof By applying derivative operator to (24) with respect to x, we get

$$\frac{\partial^k}{\partial x^k} \mathcal{G}(t, x, y, z; a, b, v) = t^k \mathcal{G}(t, x, y, z; a, b, v).$$

By using the above partial derivative formula, we have

$$\sum_{n=0}^{\infty} \frac{\partial^k}{\partial x^k} y_6(n; x, y, z; a, b, v) \frac{t^n}{n!} = \sum_{n=0}^{\infty} y_6(n; x, y, z; a, b, v) \frac{t^{n+k}}{n!}.$$

Equalizing the coefficients $\frac{t^n}{n!}$ on the both sides of the equation yields the assertion of theorem. □

The other derivative formulas with respect to y and z of the polynomials $y_6\big(n; x, y, z; a, b, v\big)$ can be easily obtained by the same method.

3.1 Milne-Thomson base Poisson–Charlier type polynomials

Here, we define Milne-Thomson base Poisson–Charlier type polynomials with help of generating function for the Milne-Thomson type polynomials and numbers. We also give some formulas and identities related to the Milne-Thomson base Poisson–Charlier type polynomials.

Theorem 5

$$y_6(n; -x, 1, z; -a, 1, v) = n! \sum_{j=0}^{\left[\frac{n}{v}\right]} \frac{C_{n-j}(z; a)(-x)^{n-vj}}{j!(n-vj)!},$$

where $[x]$ *denotes the greatest integer function.*

Proof Setting $b = 1$, $y = 1$, $h(t, v) = t^v$, $f(-st, -a) = 1 + \frac{st}{a}$ and $x = -s$ in (24), we get the following functional equation:

$$\mathcal{G}(t, x, y, z; a, b, v) = F_{pc}(st, z; -a)e^{t^v}$$

Combining the above equation with (24) and (16), we obtain

$$\sum_{n=0}^{\infty} y_6(n; -x, 1, z; -a, 1, v) \frac{t^n}{n!} = \sum_{n=0}^{\infty} C_n(z; a)\frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{t^{vn}}{n!}.$$

Again combining the above equation with the following well-known series identity

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(n, k) = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{v}\right]} A(n, n-vk),$$

(*cf.* [28, Lemma 11, Eq-(7)]), we get

$$\sum_{n=0}^{\infty} y_6\left(n; -x, 1, z; -a, 1, v\right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{v}\right]} C_{n-jv}(z; a) \left(-x\right)^{n-jv} \frac{t^n}{j! \left(n-jv\right)!}.$$

Equalizing the coefficients t^n on the both sides of the equation yields the assertion of theorem. □

Theorem 6

$$y_6\left(n; -1, 1, z; a, 1, v\right) = \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} C_{n-2j}(z; a) H_{2j}$$

Proof Substituting $f(t, a) = \frac{t}{a}$, $h(t, v) = t$, $x = -1$, $y = 1$ and $v = 2$ into (24), we obtain the following functional equation:

$$\mathcal{G}\left(t, -1, 1, z; a, 1, 1\right) = F_{pc}(t, z; a) F_H(t).$$

Combining the above functional equation with (24) and (8), we have

$$\sum_{n=0}^{\infty} y_6\left(n; -1, 1, z; a, 1, v\right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} C_n(z; a) \frac{t^n}{n!} \sum_{n=0}^{\infty} H_{2n} \frac{t^{2n}}{(2n)!}.$$

Therefore

$$\sum_{n=0}^{\infty} y_6\left(n; -1, 1, z; a, 1, v\right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{2}\right]} \frac{C_{n-2j}(z; a) H_{2j}}{(2j)! \left(n-2j\right)!} t^n.$$

Equalizing the coefficients t^n on the both sides of the equation yields the assertion of theorem. □

3.2 Milne-Thomson base Laguerre polynomials

Here, we define Milne-Thomson base Laguerre polynomials with help of generating function for the Milne-Thomson type polynomials and numbers. We also give some formulas and identities related to the Milne-Thomson base Laguerre polynomials.

Setting $f(t, a) = \frac{1}{(1-t)^{a+1}}$, $h(t, 1) = \frac{t}{t-1}$ also $x = b = 0$, and $z = 1$ into (24), we have generating function for the Laguerre polynomials as follows:

$$\mathcal{G}\left(t, 0, y, z; a, 0, 1\right) = \frac{1}{(1-t)^{a+1}} e^{\frac{xt}{t-1}} = \sum_{n=0}^{\infty} y_6\left(n; 0, y, 1; a, 0, 1\right) \frac{t^n}{n!}$$

therefore

$$L_n^{(a)}(x) = y_6\left(n; 0, y, 1; a, 0, 1\right).$$

That is

$$F_{Lg}\left(t, x; a\right) = \frac{e^{\frac{xt}{t-1}}}{(1-t)^{a+1}} = \sum_{n=0}^{\infty} L_n^{(a)}(x) \frac{t^n}{n!}. \tag{25}$$

$$L_n^{(a)}(x) = \sum_{k=0}^{n} \frac{n!}{k!} \binom{a+n}{n-k} (-x)^k. \tag{26}$$

The Rodrigues formula for the Laguerre polynomials is given by

$$x^a e^{-x} L_n^{(a)}(x) = \frac{1}{n!}\frac{d^n}{dx^n}\left\{x^{n+a}e^{-x}\right\}, \quad a > -1, x \geq 0, n \geq 0.$$

(*cf.* [4,8,9,20,28]).

By using the above formula, orthogonality properties of the classical Laguerre polynomials is given as follows:

$$\int_{-\infty}^{\infty} x^a e^{-x} L_n^{(a)}(x) L_m^{(a)}(x) dx = 0$$

where $m, n \in \mathbb{N}_0$ and $m \neq n$ (*cf.* [8,9,28]).

Theorem 7

$$L_n^{(x-1)}(a) = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n-1}{n-k} a^k C_k(x,a). \tag{27}$$

Proof Substituting (16) into (25)

$$F_{pc}\left(\frac{az}{1-z}, x; a\right) = F_{Lg}(z, a; x-1).$$

Combining the above equation with (16) and (25), we get

$$\sum_{n=0}^{\infty} C_n(x;a)\frac{\left(\frac{az}{1-z}\right)^n}{n!} = \sum_{n=0}^{\infty} L_n^{(x-1)}(a)\frac{z^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} C_n(x;a)\frac{(az)^n}{n!}\sum_{m=0}^{\infty}\binom{n+m-1}{m} z^m = \sum_{n=0}^{\infty} L_n^{(x-1)}(a)\frac{z^n}{n!}.$$

By combining the above equation with the following series identities (*cf.* [28, p. 56, Lemma 10]):

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k), \tag{28}$$

we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{C_k(x;a)}{k!}a^k\binom{n-1}{n-k} z^n = \sum_{n=0}^{\infty} L_n^{(x-1)}(a)\frac{z^n}{n!}.$$

Equalizing the coefficients $\frac{z^n}{n!}$ on the both sides of the equation yields the assertion of theorem. □

Theorem 8

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\{a+1\}_j L_{n-j}^{(a)}(x) = \sum_{k=0}^{n}(-1)^k\binom{n-1}{n-k}\frac{n!}{k!}x^k.$$

Proof From (25), we have

$$(1-t)^{a+1}\sum_{n=0}^{\infty} L_n^{(a)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\frac{xt}{t-1}\right)^n\frac{1}{n!}. \tag{29}$$

Thus,

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\{a+1\}_j L_{n-j}^{(a)}(x)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\frac{(-xt)^n}{n!}\sum_{k=0}^{\infty}\binom{k-1+n}{k}t^k.$$

By combining the above equation with (28), we get

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\{a+1\}_j L_{n-j}^{(a)}(x)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-x)^k}{k!}\binom{n-1}{n-k}t^n.$$

Equalizing the coefficients t^n on the both sides of the equation yields the assertion of theorem. □

3.3 Milne-Thomson base Lah numbers

Here, we define Milne-Thomson base Lah numbers with help of generating function for the Milne-Thomson type polynomials and numbers. We also give some formulas and identities including the Laguerre polynomials, the Poisson–Charlier polynomials and the Lah numbers.

Setting $f(t,a)=\frac{1}{a!}\left(\frac{t}{1-t}\right)^a$, also $x=b=y=0$, and $z=1$ into (24), we have generating function for the Lah numbers, $L(n,a)$ as follows:

$$\mathcal{G}(t,0,0,k;a,0,1)=\frac{1}{a!}\left(\frac{t}{1-t}\right)^a=\sum_{n=0}^{\infty}y_6(n;0,0,1;a,0,v)\frac{t^n}{n!}$$

where $a\in\mathbb{N}_0$. Therefore,

$$L(n,a)=y_6(n;0,0,1;a,0,1).$$

That is, the Lah numbers are defined by

$$\left(\frac{t}{1-t}\right)^k=k!\sum_{n=0}^{\infty}L(n,k)\frac{t^n}{n!}. \tag{30}$$

A relations between the Lah numbers, the Poisson–Charlier polynomials and the Laguerre polynomials are given as follows:

Theorem 9 *Let $m\in\mathbb{N}$. Then we have*

$$\sum_{k=0}^{n+m+1}(-a)^n L(n+m+1,k+m+1)=\sum_{k=0}^{n}\frac{1}{k!}\binom{n-1}{n-k}a^k C_k(m+1,a).$$

Proof Substituting $a=m\in\mathbb{N}$ into (25) and after some elementary calculation, we get

$$\sum_{v=0}^{\infty}\{v\}_{m+1}L_{v-m-1}^{(m)}(x)\frac{t^v}{v!}=\sum_{v=0}^{\infty}\sum_{n=0}^{v}(n+m+1)!L(v,n+m+1)(-x)^n\frac{t^v}{v!}.$$

Comparing coefficient $\frac{t^v}{v!}$ on both sides of the above equation and setting $x = a$, we have the following presumably known result:

$$L_{v-m-1}^{(m)}(a) = \frac{(n+m+1)!}{\{v\}_{m+1}} \sum_{n=0}^{v} L(v, n+m+1)(-a)^n . \tag{31}$$

On the other hand, by substituting $x = m+1$ into (27), we get

$$L_n^{(m)}(a) = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n-1}{n-k} a^k C_k(m+1, a) . \tag{32}$$

Combining (31) with (32), we get desired results. □

Combining (30) with (29), we get the following result:

$$(1-t)^{a+1} \sum_{n=0}^{\infty} L_n^{(a)}(x) \frac{t^k}{k!} = \sum_{k=0}^{\infty} \sum_{n=0}^{k} x^n L(k, n) \frac{t^n}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \{a+1\}_j L_{n-j}^{(a)}(x) \frac{t^n}{n!} = \sum_{k=0}^{\infty} \sum_{n=0}^{k} x^n L(k, n) \frac{t^n}{n!}$$

Equalizing the coefficients t^n on the both sides of the equation, we obtain the following theorem:

Theorem 10

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \{a+1\}_j L_{n-j}^{(a)}(x) = \sum_{n=0}^{k} x^n L(k, n) .$$

4 Identities and relations including p-adic integrals

In this section, by using p-adic bosonic and fermionic integrals with falling factorials polynomials, we derive some new identities and relations including the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Poisson–Charlier polynomials, the Milne-Thomson polynomials, the Hermite polynomials and also the combinatorial sums.

By applying bosonic p-adic integral to (17), respectively we have

$$\int_{\mathbb{Z}_p} C_n(x; a) d\mu_1(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a^{-k} \int_{\mathbb{Z}_p} \{x\}_k d\mu_1(x)$$

By combining the above equation with (14), we obtain the following results, respectively:

$$\int_{\mathbb{Z}_p} C_n(x; a) d\mu_1(x) = \sum_{k=0}^{n} (-1)^n \binom{n}{k} \frac{k!}{(k+1) a^k}, \tag{33}$$

and

$$\int_{\mathbb{Z}_p} C_n(x;a)d\mu_1(x) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{1}{a^k}\sum_{j=0}^{k}S_1(k,j)B_j. \tag{34}$$

Combining the above equations, we get the following theorem:

Theorem 11

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{a^k}\sum_{j=0}^{k}S_1(k,j)B_j = \sum_{k=0}^{n}\binom{n}{k}\frac{k!}{(k+1)\,a^k}.$$

By applying the fermionic p-adic integral to (17), respectively we have

$$\int_{\mathbb{Z}_p} C_n(x;a)d\mu_{-1}(x) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}a^{-k}\int_{\mathbb{Z}_p}\{x\}_k d\mu_{-1}(x).$$

By combining the above equation with (15), we obtain

$$\int_{\mathbb{Z}_p} C_n(x;a)d\mu_{-1}(x) = \sum_{k=0}^{n}(-1)^{n}\binom{n}{k}\frac{k!}{(2a)^k}, \tag{35}$$

and

$$\int_{\mathbb{Z}_p} C_n(x;a)d\mu_{-1}(x) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{1}{a^k}\sum_{j=0}^{k}S_1(k,j)E_j. \tag{36}$$

Combining the above equations, we get the following theorem:

Theorem 12

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{a^k}\sum_{j=0}^{k}S_1(k,j)E_j = \sum_{k=0}^{n}\binom{n}{k}\frac{k!}{(2a)^k}.$$

Theorem 13 *Let a be an integer. Then*

$$C_n(x;a) = a^{-n}\sum_{j=0}^{a}\binom{a}{j}j!\sum_{k=0}^{n}(-1)^k\binom{n}{k}\{x\}_{n-k}S_2(k,j).$$

Proof We set the following functional equation:

$$F_{pc}(at,x;a) = (t+1)^x\sum_{j=0}^{a}\binom{a}{j}j!F_S(-t,j)$$

Combining the above equation with (4) and (16), we get

$$\sum_{n=0}^{\infty}C_n(x;a)\frac{(at)^n}{n!} = \sum_{j=0}^{a}\binom{a}{j}j!\sum_{n=0}^{\infty}S_2(n,j)\frac{(-t)^n}{n!} \times\sum_{n=0}^{\infty}\{x\}_n\frac{t^n}{n!}.$$

By using the Cauchy product in the above equation, after some elementary calculation, equating coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the desired results of the theorem. □

By applying bosonic and fermionic p-adic integral to Eq. (18) and combining (33), (34), (35), and (36), after some elementary calculation, we get the following theorem:

Theorem 14 *The following identities hold true*

$$\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{j}(-1)^{j}\binom{j}{k}\frac{a^{n-k}k!}{k+1}=\frac{(-1)^{n}n!}{n+1},$$

$$\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{j}(-1)^{j}\binom{j}{k}\frac{a^{n-k}k!}{k+1}=\sum_{k=0}^{n}S_{1}(n,k)B_{k},$$

$$\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{j}(-1)^{j}\binom{j}{k}\frac{a^{n-k}k!}{2^{k}}=(-1)^{n}2^{-n}n!,$$

$$\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{j}(-1)^{j}\binom{j}{k}\frac{a^{n-k}k!}{2^{k}}=\sum_{k=0}^{n}S_{1}(n,k)E_{k},$$

$$\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{1}{a^{k}}\sum_{m=0}^{k}S_{1}(k,m)B_{m}=\sum_{k=0}^{n}S_{1}(n,k)B_{k},$$

$$\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{1}{a^{k}}\sum_{m=0}^{k}S_{1}(k,m)E_{m}=\sum_{k=0}^{n}S_{1}(n,k)E_{k},$$

$$\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{1}{a^{k}}\sum_{m=0}^{k}S_{1}(k,m)B_{m}=\frac{(-1)^{n}n!}{n+1},$$

and

$$\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{1}{a^{k}}\sum_{m=0}^{k}S_{1}(k,m)E_{m}=(-1)^{n}2^{-n}n!.$$

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