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ORIGINAL PAPER

Existence of nontrivial solutions for a system of fractional advection–dispersion equations

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Abstract In this paper, we investigate the existence of nontrivial solutions for a class of fractional advection–dispersion systems. The approach is based on the variational method by introducing a suitable fractional derivative Sobolev space. We take two examples to demonstrate the main results.

Keywords Fractional advection–dispersion equation · Weak solution · Critical point theory · Anomalous diffusion · Variational method

Mathematics Subject Classification 58E05 · 34B15 · 26A33

1 Introduction

Physical models containing fractional differential operators were extensively studied in recent years due to its capacity of simulating anomalous diffusion, i.e., diffusion which can not be accurately modeled by the usual advection–dispersion equation. A fractional advection–dispersion equation (ADE for short) is a generalization of the classical ADE in which the second-order derivative is replaced with a fractional-order derivative. Anomalous diffusion equations have been used in modeling turbulent flow [1–3], chaotic dynamics of classical conservative systems [4], and in contaminant transport of groundwater flow [5]. For more

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background information and applications on the fractional ADE, the reader is referred to [6–10].

Ervin and Loop [1] investigated the following fractional ADE:

$$-\frac{d}{dt}(p\,{}_0\mathcal{D}_t^{-\beta} + (1-p)\,{}_t\mathcal{D}_T^{-\beta})u'(t) + b(t)u'(t) + c(t)u(t) \\ = \nabla F(t, u(t)), \quad \text{a.e.} \quad t \in [0, T] \tag{1.1}$$

where ${}_0\mathcal{D}_t{}^{-\beta}$ and ${}_t\mathcal{D}_T{}^{-\beta}$ are the left and right Riemann–Liouville fractional integral operators respectively, with $0 \le \beta < 1$, $p \in [0, 1]$ is a constant describing the skewness of the transport process, b, c, F are functions satisfying some suitable conditions. A special case of the fractional ADE describes symmetric transitions, where $p = \frac{1}{2}$ in (1.1). In this case,

$$p\,{}_0\mathcal{D}_t{}^{-\beta} + (1-p)\,{}_t\mathcal{D}_T{}^{-\beta} = \frac{1}{2}\,{}_0\mathcal{D}_t{}^{-\beta} + \frac{1}{2}\,{}_t\mathcal{D}_T{}^{-\beta}. \tag{1.2}$$

Another equation for a N-dimensional fractional ADE was given by Fix and Roop [7], and the equation may be written as

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot (\mathbf{v}\phi) - \nabla \cdot (\nabla^{-\beta}(-k\nabla\phi)) + f, \quad \text{in} \quad \Omega, \tag{1.3}$$

where $\phi(t, x)$ is the concentration of a solute at a point x in an arbitrary bounded connected set $\Omega \in \mathbb{R}^n$ at time t, $\mathbf{v}$ is the velocity of the fluid, k is the diffusion constant coefficient, $\mathbf{v}\phi$ and $-k\nabla\phi$ are the mass flux due to advection and dispersion respectively and f is a source term. The operator $\nabla^{-\beta}$ with $0 < \beta < 1$ is a linear combination of the left and right Riemann–Liouville fractional integral operators, and its jth component is defined by

$$\nabla^{-\beta}(-k\nabla\phi)_j = (p\,{}_{-\infty}\mathcal{D}_{x_j}^{-\beta} + (1-p)\,{}_{x_j}\mathcal{D}_{+\infty}^{-\beta})\left(-k\frac{\partial \phi}{\partial x_j}\right), \quad j = 1, 2, \ldots, N,$$

where $p \in [0, 1]$ describes the skewness of the transport process, ${}_{-\infty}\mathcal{D}_{x_j}^{-\beta}$ and ${}_{x_j}\mathcal{D}_{+\infty}^{-\beta}$ are the left and right Riemann–Liouville fractional integral operators, respectively. We take $p = \frac{1}{2}$ in (1.3), and get a special case of the fractional ADE (1.3) which describes symmetric transitions. In this case, the fractional order gradient operator $\nabla^{-\beta}$ reduces to the following symmetric operator

$$(\nabla^{-\beta})_j = \frac{1}{2}\,{}_{-\infty}\mathcal{D}_{x_j}^{-\beta} + \frac{1}{2}\,{}_{x_j}\mathcal{D}_{+\infty}^{-\beta}, \quad j = 1, 2, \ldots, N.$$

Recently, many research results appeared for symmetric fractional ADE. By using the mountain pass theorem and Ekeland's variational principle, Jiao and Zhou [11] established the existence of solution and nontrivial solution for the following symmetric fractional ADE, respectively,

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}({}_0\mathcal{D}_t{}^{-\beta}u')(t) + \frac{1}{2}({}_t\mathcal{D}_T{}^{-\beta}u')(t)\right) + \nabla F(t, u) = 0, \quad \text{a.e.} \ t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.4}$$

where ${}_0\mathcal{D}_t{}^{-\beta}$ and ${}_t\mathcal{D}_T{}^{-\beta}$ denote the left and right Riemann–Liouville fractional integrals of order β with $0 \le \beta < 1$, respectively, $\nabla F(t, x)$ is the gradient of F at $x \in \mathbb{R}^n$. Teng et al. [12] proved the existence and multiplicity of solutions for a similar symmetric case for a class of nonsmooth fractional ADEs by using a variational method based on the nonsmooth critical point theory. Zhang et al. [13] studied the eigenvalue problem for the following symmetric fractional ADE:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}({}_0\mathcal{D}_t{}^{-\beta}u')(t) + \frac{1}{2}({}_t\mathcal{D}_T{}^{-\beta}u')(t)\right) + \lambda\nabla F(t,u) = 0, \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.5}$$

where λ is a real nonnegative parameter. By using the three-critical-point theorem in [14, 15] respectively, several criteria for the existence of multiple nontrivial solutions for the eigenvalue problem (1.5) were established in [13]. For other research results about symmetric fractional ADE, we refer the reader to [16–18] for (1.4) and to [19–23] for the eigenvalue problem (1.5).

Motivated by the above works, in this paper, we will use the critical point theory to study the following symmetric fractional ADE system:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}({}_0\mathcal{D}_t{}^{-\beta_i}u_i')(t) + \frac{1}{2}({}_t\mathcal{D}_T{}^{-\beta_i}u_i')(t)\right) + F'_{u_i}(t,u_1(t),\ldots,u_n(t)) = 0, \quad \text{a.e. } t \in [0,T], \\ u_i(0) = u_i(T) = 0 \end{cases} \tag{1.6}$$

for $1 \leq i \leq n$, where $n \geq 1$, $T > 0$, $0 \leq \beta_i < 1$ for $1 \leq i \leq n$, ${}_0\mathcal{D}_t{}^{-\beta_i}$ and ${}_t\mathcal{D}_T{}^{-\beta_i}$ denote the left and right Riemann–Liouville fractional integrals of order β_i, respectively, $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is a given function. We will establish some conditions on F, which are easily to be verified, to guarantee the existence of a nontrivial solution for (1.6).

Obviously, if we take $\beta_i = \beta \in [0, 1)$ for $1 \leq i \leq n$ in (1.6), then the fractional ADE (1.6) reduces to (1.4). If we take $\beta_i = 0$ for $i = 1, 2, \ldots, n$, then the fractional ADE (1.6) reduces to the classical second-order ADE of the following form

$$\begin{cases} u'' + \nabla F(t, u(t)) = 0, \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.7}$$

where $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is a given function satisfying some assumptions, $n \geq 1$, and $\nabla F(t,u)$ is the gradient of F at $u \in \mathbb{R}^n$. Many excellent results on the existence of solutions for (1.7) have been reported in [24,25].

2 Preliminaries and the fractional derivative space

In this section, we firstly introduce some notations, definitions and preliminary results about fractional derivative which are to be used throughout this paper, then we define a suitable fractional derivative Sobolev space.

Various definitions for the fractional derivative have been introduced over the past years [26]. In this paper, we focus on the Caputo fractional derivative and we refer the reader to [26,27] for details.

For convenience, we denote

$$L^p([0,T],\mathbb{R}) = \left\{u:[0,T]\to\mathbb{R} \;\Big|\; \int_0^T |u(t)|^p \mathrm{d}t < +\infty\right\};$$
$$C([0,T],\mathbb{R}) = \{u:[0,T]\to\mathbb{R} \;\big|\; u(t) \text{ is continuous}\};$$
$$C^k([0,T],\mathbb{R}) = \{u:[0,T]\to\mathbb{R} \;\big|\; u^{(k)}(t) \text{ is continuous}\}, \quad k=1,2,\ldots;$$
$$C_0^\infty([0,T],\mathbb{R}) = \{u \;\big| u\in C^\infty([0,T],\mathbb{R}) \text{ with } u(0)=u(T)=0\};$$
$$\|u\|_\infty = \max_{t\in[0,T]}|u(t)|, \quad \|u\|_{L^p} = \left(\int_0^T |u(t)|^p \mathrm{d}t\right)^{1/p}.$$

Definition 2.1 *(Left and right Riemann–Liouville fractional integrals* [26]) Let g be a function defined on $[a,b]$. The left and right Riemann–Liouville fractional integrals of order $\gamma>0$ for function g, denoted by $({}_aD_t^{-\gamma}g)$ and $({}_tD_b^{-\gamma}g)$ respectively, are defined by

$$\left({}_a\mathcal{D}_t{}^{-\gamma}g\right)(t) = \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}g(s)\mathrm{d}s, \quad \gamma>0, \quad t\in[a,b]$$

and

$$\left({}_t\mathcal{D}_b{}^{-\gamma}g\right)(t) = \frac{1}{\Gamma(\gamma)}\int_t^b (s-t)^{\gamma-1}g(s)\mathrm{d}s, \quad \gamma>0, \quad t\in[a,b]$$

provided that the right-hand sides are pointwise defined on $[a,b]$, where Γ is the "*Gamma Function*" defined by $\Gamma(\gamma)=\int_0^\infty t^{\gamma-1}e^{-t}\mathrm{d}t$, $\gamma>0$.

Definition 2.2 *(Left and right Caputo fractional derivatives* [26]) Let g be a function defined on $[a,b]$, $\gamma\geq 0$ and $n\in\mathbb{N}$. We denote the left and right Riemann–Liouville fractional derivatives of order $\gamma\geq 0$ for function g by $({}_a^cD_t^\gamma g)$ and $({}_t^c\mathcal{D}_b^\gamma g)$ respectively.

(i) If $\gamma\in(n-1,n)$, then

$$\begin{aligned}&({}_a^c\mathcal{D}_t^\gamma g)(t)\\ &= ({}_a\mathcal{D}_t{}^{-(n-\gamma)}g^{(n)})(t)\\ &= \frac{1}{\Gamma(n-\gamma)}\left(\int_a^t (t-s)^{n-\gamma-1}g^{(n)}(s)\mathrm{d}s\right), \quad t\in[a,b]\end{aligned}$$

and

$$\begin{aligned}({}_t^c\mathcal{D}_b^\gamma g)(t) &= (-1)^n({}_t\mathcal{D}_b{}^{-(n-\gamma)}g^{(n)})(t)\\ &= \frac{(-1)^n}{\Gamma(n-\gamma)}\left(\int_t^b (s-t)^{n-\gamma-1}g^{(n)}(s)\mathrm{d}s\right), \quad t\in[a,b].\end{aligned}$$

(ii) If $\gamma=n$, then

$$({}_a^c\mathcal{D}_t^n g)(t) = g^{(n)}(t) \quad \text{and} \quad ({}_t^c\mathcal{D}_b^n g)(t) = (-1)^n g^{(n)}(t), \quad t\in[a,b].$$

Remark 2.1 According to Definition 2.2, if $0<\gamma<1$, then

$$({}_a^c\mathcal{D}_t^\gamma g)(t) = ({}_a\mathcal{D}_t{}^{-(1-\gamma)}g')(t) = \frac{1}{\Gamma(1-\gamma)}\left(\int_a^t (t-s)^{-\gamma}g'(s)\mathrm{d}s\right), \quad t\in[a,b]$$

and

$$({}_t^c\mathcal{D}_b^\gamma g)(t) = -({}_t\mathcal{D}_b{}^{-(1-\gamma)}g')(t) = -\frac{1}{\Gamma(1-\gamma)}\left(\int_t^b (s-t)^{-\gamma}g'(s)\mathrm{d}s\right), \quad t\in[a,b].$$

Property 2.1 [27] *The left and right Riemann–Liouville fractional integral operators have the following property:*

$$ {}_a\mathcal{D}_t^{-\gamma_1}({}_a\mathcal{D}_t^{-\gamma_2} f) = {}_a\mathcal{D}_t^{-\gamma_1-\gamma_2} f \quad \text{and} \quad {}_t\mathcal{D}_b^{-\gamma_1}({}_t\mathcal{D}_b^{-\gamma_2}) = {}_t\mathcal{D}_b^{-\gamma_1-\gamma_2} f, \quad \forall \gamma_1, \gamma_2 > 0 $$

hold in all $t \in [a, b]$ for $f \in C([0, T], \mathbb{R})$.

Property 2.2 [27] *Let $0 < \alpha \le 1$ and $1 \le p < \infty$. For any $f \in L^p([0, T], \mathbb{R})$, we have*

$$ \|{}_a\mathcal{D}_\xi^{-\alpha} f\|_{L^p([0,t])} \le \frac{t^\alpha}{\Gamma(\alpha_1)} \|f\|_{L^p([0,t])}, \quad \xi \in [a, t], t \in [a, b]. $$

Property 2.3 [27] *The left and right Riemann–Liouville fractional integral operators have the following property:*

$$ \int_a^b \left({}_a\mathcal{D}_t^{-\gamma} f\right)(t) g(t) \mathrm{d}t = \int_a^b \left({}_t\mathcal{D}_b^{-\gamma} g\right)(t) f(t) \mathrm{d}t, \quad \gamma > 0, $$

provided that $f \in L^p[0, T], \mathbb{R})$, $g \in L^q[0, T], \mathbb{R})$ and $p \ge 1$, $q \ge 1$, $1/p + 1/q \le 1 + \gamma$ or $p \ne 1$, $q \ne 1$, $1/p + q/1 = 1 + \gamma$.

In order to establish a variational structure for (1.6), we must construct an appropriate function space. By Property 2.2, when $0 < \alpha \le 1$, for any $f \in C^\infty([0, T], \mathbb{R})$, we have $f \in L^p([0, T], \mathbb{R})$ and $({}_0^c\mathcal{D}_t^\alpha f) \in L^p([0, T], \mathbb{R})$. Therefore, we now define the fractional derivative space E^α as the closure of $C_0^\infty([0, T], \mathbb{R})$ with respect to the norm $\|u\|^\alpha = (\|u\|_{L^2}^2 + \|{}_0^c\mathcal{D}_t^\alpha u\|_{L^2}^2)^{1/2}$.

Property 2.4 [11] *For $0 < \alpha \le 1$,*

(i) $E^\alpha = \{u : [0, T] \to \mathbb{R} | u \in L^2([0, T], \mathbb{R}), ({}_p^c\mathcal{D}_t{}^\alpha u) \in L^2([0, T], \mathbb{R}), u(0) = u(T) = 0\}$*;*
(ii) *E^α is compactly embedded in $C([0, T], \mathbb{R})$ and is a reflexive and separable Banach space;*
(iii) $\|u\|_{L^2} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^c\mathcal{D}_t^\alpha u\|_{L^2}, \quad u \in E^\alpha$*;*
(iv) $\|u\|_\infty \le \frac{\sqrt{2}T^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{\alpha+1}} \|{}_0^c\mathcal{D}_t^\alpha u\|_{L^2}, \quad u \in E^\alpha$.

Obviously, for $u \in E^\alpha$, if we define $\|u\|_\alpha = \|{}_0^c\mathcal{D}_t^\alpha u\|_{L^2}$, then by (iii) of Property 2.4, $\|u\|^\alpha$ and $\|u\|_\alpha$ are two equivalent norms. Therefore, we will consider E^α with respect to the norm $\|u\|_\alpha$ for simplicity in the following.

Property 2.5 [11] *if $1/2 < \alpha \le 1$, then for any $u \in E^\alpha$, we have*

$$ |\cos(\pi\alpha)| \|u\|_\alpha^2 \le -\int_0^T ({}_0^c\mathcal{D}_t{}^\alpha u)(t)({}_t^c\mathcal{D}_T{}^\alpha u)(t) \mathrm{d}t \le \frac{\|u\|_\alpha^2}{|\cos(\pi\alpha)|} $$

Let $E = E^{\alpha_1} \times E^{\alpha_2} \times \cdots \times E^{\alpha_n}$ endowed with norm $\|u\|_E = \|(u_1, u_2, \ldots, u_n)\|_E = (\|u_1\|_{\alpha_1}^2 + \|u_2\|_{\alpha_2}^2 + \cdots + \|u_n\|_{\alpha_n}^2)^{1/2}$. Then, E is a reflexive and separable Banach space and compactly embedded in $(C([0, T], \mathbb{R}))^n$.

Definition 2.3 $u \in E$ is a solution of (1.6) on $[0, T]$, if it satisfies (1.6); $u \in E$ is a nontrivial solution of (1.6) on $[0, T]$, if it is a solution satisfying $\|u\|_E \ne 0$.

Definition 2.4 $u = (u_1, u_2, \ldots, u_n) \in E$ is a weak solution of (1.6) if we have

$$-\frac{1}{2}\int_0^T \sum_{i=1}^n [({}_0^c\mathcal{D}_t^{\alpha_i} u_i)(t)({}_t^c\mathcal{D}_T^{\alpha_i} v_i)(t) + ({}_t^c\mathcal{D}_T^{\alpha_i} u_i)(t)({}_0^c\mathcal{D}_t^{\alpha_i} v_i)(t)]\mathrm{d}t$$
$$= \int_0^T \sum_{i=1}^n F'_{u_i}(t, u_1(t), \ldots, u_n(t))v_i(t)\mathrm{d}t$$

for every $v = (v_1, v_2, \ldots, v_n) \in E$.

In order to prove the equivalence between a weak solution and a solution of (1.6), we must rewrite (1.6). To do this, for each $1 \le i \le n$, let $\alpha_i = 1 - \frac{\beta_i}{2}$, then $\alpha_i \in (\frac{1}{2}, 1]$. According to Property 2.1 and Definition 2.2, the fractional ADE (1.6) can be transformed equivalently to the following system:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}{}_0\mathcal{D}_t^{-(1-\alpha_i)}\left({}_0^c\mathcal{D}_t^{\alpha_i} u_i\right)(t) - \frac{1}{2}{}_t\mathcal{D}_T^{-(1-\alpha_i)}\left({}_t^c\mathcal{D}_T^{\alpha_i} u_i\right)(t)\right) \\ \qquad + F'_{u_i}(t, u_1(t), \ldots, u_n(t)) = 0, \quad \text{a.e.} \quad t \in [0, T], \\ u_i(0) = u_i(T) = 0 \end{cases} \tag{2.1}$$

for $1 \le i \le n$.

Lemma 2.1 *If* $u = (u_1, u_2, \ldots, u_n) \in E$ *is a weak solution of* (1.6)*, then* u *must be a solution of* (2.1).

Proof Suppose $u = (u_1, u_2, \ldots, u_n) \in E$ is a weak solution of (1.6). Define

$$w_i(t) = \int_0^t F'_{u_i}(s, u_1(s), \ldots, u_n(s))\mathrm{d}s, \quad t \in [0, T], \quad 1 \le i \le n.$$

For any $v = (v_1, v_2, \ldots, v_n) \in E$, noting that $v_i(T) = 0$ for $1 \le i \le n$, we obtain

$$\begin{aligned}\int_0^T w_i(t)v_i'(t)\mathrm{d}t &= \int_0^T \left\{v_i'(t)\int_0^t F'_{u_i}(s, u_1(s), \ldots, u_n(s))\mathrm{d}s\right\}\mathrm{d}t \\ &= \int_0^T \left\{\int_s^T v_i'(t)\mathrm{d}s\mathrm{d}t\right\} F'_{u_i}(s, u_1(s), \ldots, u_n(s))\mathrm{d}s \\ &= -\int_0^T F'_{u_i}(s, u_1(s), \ldots, u_n(s))v_i(s)\mathrm{d}s, \quad 1 \le i \le n.\end{aligned} \tag{2.2}$$

On the other hand, by Property 2.3, considering $u_i, v_i \in E^{\alpha_i}$, we have

$$\begin{aligned}\int_0^T \left({}_0^c\mathcal{D}_t^{\alpha_i} u_i\right)(t)\left({}_t^c\mathcal{D}_T^{\alpha_i} v_i\right)(t)\mathrm{d}t &= \int_0^T \left({}_0^c\mathcal{D}_t^{\alpha_i} u_i\right)(t)\left(-{}_t\mathcal{D}_T^{-(1-\alpha_i)} v_i'\right)(t)\mathrm{d}t \\ &= -\int_0^T \left({}_0\mathcal{D}_t^{-(1-\alpha_i)}({}_0^c\mathcal{D}_t^{\alpha_i} u_i)\right)(t)v_i'(t)\mathrm{d}t, \quad 1 \le i \le n\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\int_0^T \left({}_t^c\mathcal{D}_T^{\alpha_i} u_i\right)(t)\left({}_0^c\mathcal{D}_t^{\alpha_i} v_i\right)(t)\mathrm{d} &= \int_0^T \left({}_t^c\mathcal{D}_T^{\alpha_i} u_i\right)(t)(-{}_0\mathcal{D}_t^{(1-\alpha_i)} v_i')(t)\mathrm{d}t \\ &= \int_0^T \left({}_t\mathcal{D}_T^{-(1-\alpha_i)}({}_t^c\mathcal{D}_T^{\alpha_i} u_i)\right)(t)v_i'(t)\mathrm{d}t, \quad 1 \le i \le n.\end{aligned} \tag{2.4}$$

From (2.2) to (2.4), considering u is a weak solution of (1.6), we have

$$\int_0^T \Big\{\frac{1}{2}\sum_{i=1}^n [({}_0\mathcal{D}_t{}^{-(1-\alpha_i)}({}_0^c\mathcal{D}_t{}^{\alpha_i}u_i))(t) - ({}_t\mathcal{D}_T{}^{-(1-\alpha_i)}({}_t^c\mathcal{D}_T{}^{\alpha_i}u_i))(t)] + \sum_{i=1}^n w_i(t)\Big\} v_i'(t)\mathrm{d}t = 0. \tag{2.5}$$

Now, for any $v_i \in E^{\alpha_i}(1 \le i \le n)$, we substitute $v = (0, \dots, v_i, \dots, 0) \in E$ into (2.5) to obtain

$$\int_0^T \Big\{\frac{1}{2}[({}_0\mathcal{D}_t{}^{-(1-\alpha_i)}({}_0^c\mathcal{D}_t{}^{\alpha_i}u_i))(t) - ({}_t\mathcal{D}_T{}^{-(1-\alpha_i)}({}_t^c\mathcal{D}_T{}^{\alpha_i}u_i))(t)] + w_i(t)\Big\} v_i'(t)\mathrm{d}t = 0 \tag{2.6}$$

for any $i = 1, 2, \dots, n$. The theory of Fourier series and (2.6) imply that

$$\frac{1}{2}[({}_0\mathcal{D}_t{}^{-(1-\alpha_i)}({}_0^c\mathcal{D}_t{}^{\alpha_i}u_i))(t) - ({}_t\mathcal{D}_T{}^{-(1-\alpha_i)}({}_t^c\mathcal{D}_T{}^{\alpha_i}u_i))(t)] + w_i(t) \equiv C_i, \quad \text{a.e.} \quad t \in [0, T],$$

where $C_i \in \mathbb{R}$ is a constant. Hence, from $u_i \in E^{\alpha_i}$ and the definition of w_i, we have

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}{}_0\mathcal{D}_t{}^{-(1-\alpha_i)}\left({}_0^c\mathcal{D}_t{}^{\alpha_i}u_i\right)(t) - \dfrac{1}{2}{}_t\mathcal{D}_T{}^{-(1-\alpha_i)}\left({}_t^c\mathcal{D}_T{}^{\alpha_i}u_i\right)(t)\right) \\ \qquad + F'_{u_i}(t, u_1(t), \dots, u_n(t)) = 0, \quad \text{a.e.} \quad t \in [0, T], \\ u_i(0) = u_i(T) = 0 \end{cases}$$

for any $i = 1, 2, \dots, n$, which means that $u = (u_1(t), u_2(t), \dots, u_n(t)) \in E$ is a solution of (2.1). □

Lemma 2.2 *Suppose $s_i > 0$ and $c_i \ge 0$ are constants for $1 \le i \le n$ with $\sum_{i=1}^n c_i^2 > n$. Then*

$$\sum_{i=1}^n c_i^{s_i} \le n\left(\sum_{i=1}^n c_i^2\right)^{s_0/2},$$

where $s_0 = \max_{1\le j\le n} s_j$.

Proof Without loss of generality, suppose $c_1 > 1$, then $c_1^{s_0} > 1$. For any $i \in \{1, 2, \dots, n\}$, if $c_i \ge 1$, then

$$c_i^{s_i} \le c_i^{s_0} \le \sum_{i=1}^n c_i^{s_0}; \tag{2.7}$$

if $c_i < 1$, then

$$c_i^{s_i} < 1 < c_1^{s_0} \le \sum_{i=1}^n c_i^{s_0}. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$c_i^{s_i} \le \sum_{i=1}^n c_i^{s_0}, \quad 1 \le i \le n,$$

and thus we have

$$\sum_{i=1}^{n} c_i^{s_i} \le n \sum_{i=1}^{n} c_i^{s_0} \le n \left(\sum_{i=1}^{n} c_i^2 \right)^{s_0/2}.$$

□

3 Existence of nontrivial solutions

We study the existence of nontrivial solutions for the fractional ADE (1.6) in this section. Our tool is a critical point theorem which was developed by Bonanno and D'Aguì [28].

Lemma 3.1 [28] *Let X be a reflexive real Banach space, $\varphi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous functional, and $\psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous functional such that $\varphi - \psi$ is coercive. Assume that there exist a sequentially weakly continuous function $I : X \to \mathbb{R}$ and $r \in (\inf_X(\varphi + I), \sup_X(\psi + I))$ such that*

$$\rho(I, r) := \sup_{(\varphi+I)(y)>r} \frac{(\psi + I)(y) - \sup_{(\varphi+I)(x)\le r}(\psi + I)(x)}{(\varphi + I)(y) - r} > 1.$$

Then the restriction of the function $\varphi - \psi$ to $(\varphi + I)^{-1}(r, +\infty)$ has a global minimum.

Theorem 3.1 *Let $n \ge 1$, $T > 0$, $\frac{1}{2} \le \alpha_i < 1$ for $1 \le i \le n$, $F(\cdot, u_1, u_2, \ldots, u_n) : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is measurable with respect to $t \in [0, T]$ for every $(u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, and $F(t, \cdot, \ldots, \cdot)$ is continuously differentiable with respect to $(u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ for a.e. $t \in [0, T]$. Assume that*

(H1) *$F(t, 0, \ldots, 0) = 0$ for any $t \in [0, T]$;*

(H2) *There exists $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in E$ such that*

$$0 < \|\omega\|_E^2 < 2 \min_{1\le i\le n} |\cos \pi\alpha_i| \int_0^T F(t, \omega_1(t), \ldots, \omega_n(t))\mathrm{d}t;$$

(H3) *There exist $c_i \in [0, |\cos(\pi\alpha_i)|\Gamma^2(\alpha_i + 1)/2T^{2\alpha_i})$, $b_i(t) \in L^{2/(2-s_i)}([0, T], \mathbb{R})$ and $s_i \in (0, 2)$ for $1 \le i \le n$ and $k(t) \in L^1([0, T], \mathbb{R}^+)$ such that*

$$F(t, u_1, u_2, \ldots, u_n) \le \sum_{i=1}^{n} c_i |u_i|^2 + \sum_{i=1}^{n} b_i(t)|u_i|^{s_i} + k(t),$$
$$(t, u_1, u_2, \ldots, u_n) \in [0, T] \times \mathbb{R}^n.$$

Then (1.6) has at least one nontrivial solution $u^* \in E$.

Proof In order to apply Lemma 3.1 to the system (2.1), we introduce the functionals φ, ψ and I for $u \in E$ as follows:

$$\varphi(u) = -\frac{1}{2} \int_0^T \sum_{i=1}^{n} \left({}_0^c\mathcal{D}_t^{\alpha_i} u_i\right)(t) \left({}_t^c\mathcal{D}_T^{\alpha_i} u_i\right)(t)\mathrm{d}t;$$

$$\psi(u) = \int_0^T F(t, u_1(t), \ldots, u_n(t))\mathrm{d}t;$$

$$I(u) = \sum_{i=1}^{n} \|u_i\|_\infty^2.$$

Since E is compactly embedded in $(C([0,T],\mathbb{R}))^n$, it is well known that φ is a sequentially weakly lower semicontinuous function, ψ is a sequentially weakly upper semicontinuous function, and I is a sequentially weakly continuous function. Moreover, both φ and ψ are Gâteaux differentiable functions whose Gâteaux derivatives at the point $u = (u_1, u_2, \ldots, u_n) \in E$ are the functions $\varphi'(u) \in E^*$ and $\psi'(u) \in E^*$ respectively, given by

$$\varphi'(u)(v) = -\frac{1}{2}\int_0^T \sum_{i=1}^n [({}_0^c\mathcal{D}_t^{\alpha_i} u_i)(t)({}_t^c\mathcal{D}_T^{\alpha_i} v_i)(t) + ({}_t^c\mathcal{D}_T^{\alpha_i} u_i)(t)({}_0^c\mathcal{D}_t^{\alpha_i} v_i)(t)]\mathrm{d}t \quad (3.1)$$

and

$$\psi'(u)(v) = \int_0^T \sum_{i=1}^n F'_{u_i}(t, u_1(t), \ldots, u_n(t))v_i(t)\mathrm{d}t \quad (3.2)$$

for every $v = (v_1, v_2, \ldots, v_n) \in E$.

Obviously, from (3.1), (3.2) and Definition 2.4, we get that a critical point $u^* \in E$ of $\varphi - \psi$ must be a weak solution of (1.6). In the following, we will apply Lemma 3.1 to prove the existence of a critical point for $\varphi - \psi$.

For any $u \in E$, from Property 2.5, (H3), the Hölder inequality and (iii) of Property 2.4, we get

$$\begin{aligned}
&\varphi(u) - \psi(u)\\
&= -\frac{1}{2}\int_0^T \sum_{i=1}^n \left({}_0^c\mathcal{D}_t^{\alpha_i} u_i\right)(t)\left({}_t^c\mathcal{D}_T^{\alpha_i} u_i\right)(t)\mathrm{d}t - \int_0^T F(t, u_1(t), \ldots, u_n(t))\mathrm{d}t\\
&\geq \sum_{i=1}^n \frac{|\cos\pi\alpha_i|}{2}\|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n c_i \int_0^T |u_i(t)|^2\mathrm{d}t - \sum_{i=1}^n \int_0^T b_i(t)|u_i(t)|^{s_i}\mathrm{d}t - \int_0^T k(t)\mathrm{d}t\\
&\geq \sum_{i=1}^n \frac{|\cos\pi\alpha_i|}{2}\|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n c_i\|u_i\|_{L^2}^2 - \sum_{i=1}^n \|b_i\|_{L^{2/(2-s_i)}}\|u_i\|_{L^2}^{s_i} - \int_0^T k(t)\mathrm{d}t\\
&\geq \sum_{i=1}^n \frac{|\cos\pi\alpha_i|}{2}\|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n \frac{c_i T^{2\alpha_i}}{\Gamma^2(\alpha_i+1)}\|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n \frac{T^{\alpha_i s_i}\|b_i\|_{L^{2/(2-s_i)}}}{\Gamma^{s_i}(\alpha_i+1)}\|u_i\|_{\alpha_i}^{s_i}\\
&\quad - \int_0^T k(t)\mathrm{d}t\\
&= \sum_{i=1}^n \left(\frac{|\cos\pi\alpha_i|}{2} - \frac{c_i T^{2\alpha_i}}{\Gamma^2(\alpha_i+1)}\right)\|u_i\|_{\alpha_i}^2 - \sum_{i=1}^n \frac{T^{\alpha_i s_i}\|b_i\|_{L^{2/(2-s_i)}}}{\Gamma^{s_i}(\alpha_i+1)}\|u_i\|_{\alpha_i}^{s_i} - \int_0^T k(t)\mathrm{d}t\\
&\geq \mathcal{M}\sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - \mathcal{N}\sum_{i=1}^n \|u_i\|_{\alpha_i}^{s_i} - \int_0^T k(t)\mathrm{d}t, \qquad (3.3)
\end{aligned}$$

where

$$\mathcal{M} := \min_{1\leq i\leq n}\left(\frac{|\cos\pi\alpha_i|}{2} - \frac{c_i T^{2\alpha_i}}{\Gamma^2(\alpha_i+1)}\right) > 0 \quad \text{and} \quad \mathcal{N} := \max_{1\leq i\leq n}\frac{T^{\alpha_i s_i}\|b_i\|_{L^{2/(2-s_i)}}}{\Gamma^{s_i}(\alpha_i+1)}.$$

From Lemma 2.2, when $\sum_{i=1}^n \|u_i\|_{\alpha_i}^2 > n$, we have

$$\sum_{i=1}^n \|u_i\|_{\alpha_i}^{s_i} \leq n\left(\sum_{i=1}^n \|u_i\|_{\alpha_i}^2\right)^{s_0/2}, \quad (3.4)$$

where $s_0 = \max_{1\le i\le n} s_i \in (0,2)$. Hence, when $\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^2 > n$, we substitute (3.4) into (3.3) to obtain

$$\varphi(u) - \psi(u) \ge \mathcal{M}\|u\|_E^2 - n\mathcal{N}\|u\|_E^{s_0} - \int_0^T k(t)\mathrm{d}t.$$

Thus, by $\mathcal{M} > 0$ and $s_0 \in (0,2)$, we have

$$\lim_{\|u\|_E\to+\infty}(\varphi(u) - \psi(u)) = +\infty,$$

which means $\varphi(u) - \psi(u)$ is coercive.

Next, we will prove $\rho(I, r) > 1$ for some $r \in (\inf_E(\varphi + I), \sup_E(\psi + I))$.

Firstly, by Property 2.5 and (H2), we have

$$\begin{aligned}(\psi - \varphi)(\omega) &= \int_0^T F(t, \omega_1(t), \ldots, \omega_n(t))\mathrm{d}t \\ &\quad - \left(-\frac{1}{2}\int_0^T \sum_{i=1}^{n} \left({}_0^c\mathcal{D}_t^{\alpha_i}\omega_i\right)(t)\left({}_t^c\mathcal{D}_T^{\alpha_i}\omega_i\right)(t)\mathrm{d}t\right) \\ &\ge \int_0^T F(t, \omega_1(t), \ldots, \omega_n(t))\mathrm{d}t - \left(\frac{1}{2}\sum_{i=1}^{n}\frac{\|\omega_i\|_{\alpha_i}^2}{|\cos(\pi\alpha_i)|}\right) \\ &\ge \int_0^T F(t, \omega_1(t), \ldots, \omega_n(t))\mathrm{d}t - \left(\frac{\|\omega\|_E^2}{2\min\limits_{1\le i\le n}|\cos(\pi\alpha_i)|}\right) > 0 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}(\varphi + I)(\omega) &= -\frac{1}{2}\int_0^T \sum_{i=1}^{n}\left({}_0^c\mathcal{D}_t^{\alpha_i}\omega_i\right)(t)\left({}_t^c\mathcal{D}_T^{\alpha_i}\omega_i\right)(t)\mathrm{d}t + \sum_{i=1}^{n}\|u_i\|_\infty^2 \\ &\ge \frac{1}{2}\sum_{i=1}^{n}|\cos(\pi\alpha_i)|\|\omega_i\|_{\alpha_i}^2 + \sum_{i=1}^{n}\|u_i\|_\infty^2 \\ &\ge \frac{1}{2}\min_{1\le i\le n}|\cos(\pi\alpha_i)|\|\omega\|_E^2 + \sum_{i=1}^{n}\|u_i\|_\infty^2 > 0. \end{aligned} \tag{3.6}$$

From (H1), (3.5) and (3.6), we have

$$\begin{aligned}&\lim_{r\to 0}\frac{(\psi + I)(\omega) - \int_0^T \max\limits_{\sum_{i=1}^{n}|\xi_i|^2 \le \frac{r}{1+c_0}} F(t, \xi_1, \ldots, \xi_n)\mathrm{d}t - \frac{r}{1+c_0}}{(\varphi + I)(\omega) - r} = \frac{(\psi + I)(\omega)}{(\varphi + I)(\omega)} \\ &= 1 + \frac{(\psi - \varphi)(\omega)}{(\varphi + I)(\omega)} > 1, \end{aligned} \tag{3.7}$$

where $c_0 = \min_{1\le i\le n}\frac{|\cos(\pi\alpha_i)|\Gamma^2(\alpha_i)(\alpha_i+1)}{4T^{2\alpha_i-1}}$.

Secondly, from $(\varphi + I)(\omega) > 0$, combining (3.7), we may choose a constant $r_0 \in \mathbb{R}$ satisfying

$$0 < r_0 < (\varphi + I)(\omega) \tag{3.8}$$

and

$$\frac{(\psi+I)(\omega)-\int_0^T \max_{\sum_{i=1}^{n}|\xi_i|^2\le\frac{r_0}{1+c_0}} F(t,\xi_1,\ldots,\xi_n)\mathrm{d}t-\frac{r}{1+c_0}}{(\varphi+I)(\omega)-r_0}>1. \tag{3.9}$$

For $x \in \{x\big|x \in E, (\varphi+I)(x)\le r_0\}$, by Property 2.5 and (iii) of Property 2.4, we have

$$\begin{aligned} r_0 \ge (\varphi+I)(x) &= -\frac{1}{2}\int_0^T \sum_{i=1}^{n}\left({}_0^c\mathcal{D}_t^{\alpha_i}x_i\right)(t)\left({}_t^c\mathcal{D}_T^{\alpha_i}x_i\right)(t)\mathrm{d}t+\sum_{i=1}^{n}\|x_i\|_\infty^2 \\ &\ge \frac{1}{2}\sum_{i=1}^{n}|\cos(\pi\alpha_i)|\|x_i\|_{\alpha_i}^2+\sum_{i=1}^{n}\|x_i\|_\infty^2 \\ &\ge \frac{1}{2}\sum_{i=1}^{n}|\cos(\pi\alpha_i)|\frac{\Gamma^2(\alpha_i)((\alpha_i-1)/2+1)}{T^{2\alpha_i-1}}\|x_i\|_\infty^2+\sum_{i=1}^{n}\|x_i\|_\infty^2 \\ &\ge (1+c_0)\sum_{i=1}^{n}\|x_i\|_\infty^2. \end{aligned} \tag{3.10}$$

By (3.10), we conclude

$$\{x\big|x\in E, (\varphi+I)(x)\le r_0\}\subseteq\left\{x\Big|x\in E, \sum_{i=1}^{n}\|x_i\|_\infty^2\le\frac{r_0}{1+c_0}\right\}.$$

Then,

$$\begin{aligned} \sup_{(\varphi+I)(x)\le r_0}(\psi+I)(x) &= \sup_{(\varphi+I)(x)\le r_0}\left\{\int_0^T F(t,x_1(t),\ldots,x_n(t))\mathrm{d}t+\sum_{i=1}^{n}\|x_i\|_\infty^2\right\} \\ &\le \sup_{(\varphi+I)(x)\le r_0}\int_0^T F(t,x_1(t),\ldots,x_n(t))\mathrm{d}t+\sup_{(\varphi+I)(x)\le r_0}\sum_{i=1}^{n}\|x_i\|_\infty^2 \\ &\le \int_0^T \max_{\sum_{i=1}^{n}|\xi_i|^2\le\frac{r}{1+c_0}} F(t,\xi_1,\ldots,\xi_n)\mathrm{d}t+\frac{r_0}{1+c_0}. \end{aligned} \tag{3.11}$$

Therefore, by (3.8), (3.11) and (3.9), we get

$$\begin{aligned} \rho(I,r_0) &= \sup_{(\varphi+I)(y)>r_0}\frac{(\psi+I)(y)-\sup_{(\varphi+I)(x)\le r_0}(\psi+I)(x)}{(\varphi+I)(y)-r_0} \\ &\ge \frac{(\psi+I)(\omega)-\sup_{(\varphi+I)(x)\le r_0}(\psi+I)(x)}{(\varphi+I)(\omega)-r_0} \\ &\ge \frac{(\psi+I)(\omega)-\int_0^T \max_{\sum_{i=1}^{n}|\xi_i|^2\le\frac{r}{1+c_0}} F(t,\xi_1,\ldots,\xi_n)\mathrm{d}t-\frac{r_0}{1+c_0}}{(\varphi+I)(\omega)-r_0}>1. \end{aligned} \tag{3.12}$$

So Lemma 3.1 guarantees that $\varphi - \psi$ has a critical point $u^* = (u_1^*, u_2^*, \ldots, u_n^*) \in E$ such that $(\varphi + I)(u^*) > r_0$. By Property 2.5 and (iv) of Property 2.4, we get

$$\begin{aligned} r_0 < (\varphi + I)(u^*) &= -\frac{1}{2}\int_0^T \sum_{i=1}^n \left({}_0^c\mathcal{D}_t^{\alpha_i} u_i^*\right)(t)\left({}_t^c\mathcal{D}_T^{\alpha_i} u_i^*\right)(t)\mathrm{d}t + \sum_{i=1}^n \|u_i^*\|_\infty^2 \\ &\le \frac{1}{2}\sum_{i=1}^n \frac{\|u_i^*\|_{\alpha_i}^2}{|\cos(\pi\alpha_i)|} + \sum_{i=1}^n \frac{2T^{2\alpha_i - 1}}{\Gamma^2(\alpha_i)(\alpha_i+1)}\|u_i^*\|_{\alpha_i}^2 \\ &= \sum_{i=1}^n \left(\frac{1}{2|\cos(\pi\alpha_i)|} + \frac{2T^{2\alpha_i-1}}{\Gamma^2(\alpha_i)(\alpha_i+1)}\right)\|u_i^*\|_{\alpha_i}^2 \\ &\le \|u^*\|_E^2 \max_{1\le i\le n}\left(\frac{1}{2|\cos(\pi\alpha_i)|} + \frac{2T^{2\alpha_i-1}}{\Gamma^2(\alpha_i)(\alpha_i+1)}\right), \end{aligned}$$

which means that

$$\|u^*\|_E > r_0^{\frac{1}{2}} / \max_{1\le i\le n}\left(\frac{1}{2|\cos(\pi\alpha_i)|} + \frac{2T^{2\alpha_i-1}}{\Gamma^2(\alpha_i)(\alpha_i+1)}\right)^{\frac{1}{2}},$$

and thus $u^* \in E$ is a nontrivial solution of (1.6). □

Now we deduce a particular but verifiable consequence of Theorem 3.1 where the test function ω is specified. For convenience, put

$$\begin{aligned} \mathcal{B}(\alpha_i, T) =& \frac{16}{T^2}\int_0^T t^{2(1-\alpha_i)}\mathrm{d}t + \frac{16}{T^2}\int_{T/4}^T \left(\left(t - \frac{T}{4}\right)^{2(1-\alpha_i)} - 2\left(t^2 - \frac{T}{4}t\right)^{1-\alpha_i}\right)\mathrm{d}t \\ &+ \frac{16}{T^2}\int_{3T/4}^T \left(t - \frac{3T}{4})^{2(1-\alpha_i)} - 2\left(t^2 - \frac{3T}{4}t\right)^{1-\alpha_i}\right. \\ &\left. +2\left(t^2 - Tt + \frac{3T^2}{16}\right)^{1-\alpha_i}\right)\mathrm{d}t \end{aligned}$$

for $1 \le i \le n$.

Corollary 3.1 *Let F be as that defined in Theorem 3.1, and both (H1) and (H3) of Theorem 3.1 hold. Assume that*

(H4) *There exist $d_i > 0$ for $1 \le i \le n$ such that*

- (i) *$F(t, \xi_1, \ldots, \xi_n) \ge 0$ for all $(t, \xi_1, \ldots, \xi_n) \in ([0, \frac{T}{4})\bigcup[\frac{3T}{4}, T]) \times [0, d_1\Gamma(2 - \alpha_1)] \times [0, d_2\Gamma(2-\alpha_2)] \times \cdots \times [0, d_n\Gamma(2-\alpha_n)]$;*
- (ii) *$2\min_{1\le i\le n}|\cos\pi\alpha_i| \int_{T/4}^{3T/4} F(t, \Gamma(2-\alpha_1)d_1, \Gamma(2-\alpha_2)d_2, \ldots, \Gamma(2-\alpha_n)d_n) > \sum_{i=1}^n d_i^2\mathcal{B}(\alpha_i, T)$.*

Then (1.6) has at least one nontrivial solution $u^* \in E$.

Proof We only need to show that (H2) of Theorem 3.1 are fulfilled by choosing $\omega = (\omega_1(t), \omega_2(t), \ldots, \omega_n(t))$ with

$$\omega_i(t) = \begin{cases} \dfrac{4d_i\Gamma(2-\alpha_i)}{T}t, & t \in \left[0, \frac{T}{4}\right], \\ d_i\Gamma(2-\alpha_i), & t \in \left[\frac{T}{4}, \frac{3T}{4}\right], \\ \dfrac{4d_i\Gamma(2-\alpha_i)}{T}(T-t), & t \in \left[\frac{3T}{4}, T\right] \end{cases} \tag{3.13}$$

for $1 \le i \le n$.

We calculate directly that

$$\left({}_{a}^{c}\mathcal{D}_{t}^{\alpha_i}\omega_i\right)(t)=\frac{4d_i}{T}\begin{cases}t^{1-\alpha_i}, & t\in\left[0,\frac{T}{4}\right],\\ t^{1-\alpha_i}-\left(t-\frac{T}{4}\right)^{1-\alpha_i}, & t\in\left[\frac{T}{4},\frac{3T}{4}\right],\\ t^{1-\alpha_i}-\left(t-\frac{T}{4}\right)^{1-\alpha_i}-\left(t-\frac{3T}{4}\right)^{1-\alpha_i}, & t\in\left[\frac{3T}{4},T\right]\end{cases}$$

for $1\le i\le n$.

Obviously, $\omega_i\in L^2([0,T],\mathbb{R})$ and $({}_{0}^{c}\mathcal{D}_{t}^{\alpha}\omega_i)\in L^2([0,T],\mathbb{R})$ for $1\le i\le n$. Noting $\omega_i(0)=\omega_i(T)=0$ for $1\le i\le n$, we conclude $\omega_i\in E^{\alpha_i}$, and thus $\omega=(\omega_1,\omega_2,\ldots,\omega_n)\in E$.

Furthermore, we have

$$\begin{aligned}\|\omega_i\|_{\alpha_i}^2&=\int_0^T\left|\left({}_{a}^{c}\mathcal{D}_{t}^{\alpha_i}\omega_i\right)(t)\right|^2\mathrm{d}t\\ &=\int_0^{T/4}+\int_{T/4}^{3T/4}+\int_{3T/4}^{T}\left|\left({}_{a}^{c}\mathcal{D}_{t}^{\alpha_i}\omega_i\right)(t)\right|^2\mathrm{d}t\\ &=\frac{16d_i^2}{T^2}\int_0^T t^{2(1-\alpha_i)}\mathrm{d}t+\frac{16d_i^2}{T^2}\int_{T/4}^{T}\left(\left(t-\frac{T}{4}\right)^{2(1-\alpha_i)}-2\left(t^2-\frac{T}{4}t\right)^{1-\alpha_i}\right)\mathrm{d}t\\ &\quad+\frac{16d_i^2}{T^2}\int_{3T/4}^{T}\left(t-\frac{3T}{4}\right)^{2(1-\alpha_i)}-2\left(t^2-\frac{3T}{4}t\right)^{1-\alpha_i}+2\left(t^2-Tt+\frac{3T^2}{16}\right)^{1-\alpha_i}\Bigg)\mathrm{d}t\\ &=d_i^2\mathcal{B}(\alpha_i,T).\end{aligned}$$

On the other hand, according to (3.13), we get $0\le\omega_i(t)\le d_i\Gamma(2-\alpha_i)$ $(i=1,2,\ldots,n)$ for all $t\in[0,T]$. Then condition (i) of (H4) ensures that

$$\begin{aligned}\int_0^T F(t,\omega_1(t),\ldots,\omega_n(t))\mathrm{d}t&=\int_0^{T/4}+\int_{T/4}^{3T/4}+\int_{3T/4}^{T}F(t,\omega_1(t),\ldots,\omega_n(t))\mathrm{d}t\\ &\ge\int_{T/4}^{3T/4}F(t,d_1\Gamma(2-\alpha_1),\ldots,d_n\Gamma(2-\alpha_n))\mathrm{d}t.\end{aligned}\quad(3.14)$$

Condition (ii) of (H4) and (3.14) ensure that

$$\begin{aligned}&2\min_{1\le i\le n}|\cos\pi\alpha_i|\int_0^T F(t,\omega_1(t),\ldots,\omega_n(t))\mathrm{d}t\\ &\quad\ge 2\min_{1\le i\le n}|\cos\pi\alpha_i|\int_{T/4}^{3T/4}F(t,d_1\Gamma(2-\alpha_1),\ldots,d_n\Gamma(2-\alpha_n))\mathrm{d}t\\ &\quad>\sum_{i=1}^n d_i^2\mathcal{B}(\alpha_i,T)=\sum_{i=1}^n\|\omega_i\|_{\alpha_i}^2=\|\omega\|_E^2,\end{aligned}$$

which means that ω satisfies (H2) of Theorem 3.1. □

Remark 3.1 Other candidates for the test function ω in (3.13) can take other forms. For example,

$$\overline{\omega_i}(t)=\begin{cases}\dfrac{16d_i\Gamma(1-\alpha_i)}{T^2}\left(\dfrac{T}{2}-t\right)t, & t\in\left[0,\frac{T}{4}\right],\\ d_i\Gamma(1-\alpha_i), & t\in\left[\frac{T}{4},\frac{3T}{4}\right],\\ \dfrac{16d_i\Gamma(1-\alpha_i)}{T^2}\left(\dfrac{T}{2}-t\right)(t-T), & t\in\left[\frac{3T}{4},T\right]\end{cases} \tag{3.15}$$

for $1\le i\le n$.

In this case,

$$\begin{aligned}\left({}_a^c\mathcal{D}_t^{\alpha_i}\overline{\omega_i}\right)(t)&=\frac{32d_i}{T^2}\\ &\times\begin{cases}\dfrac{t^{2-\alpha_i}}{2-\alpha_i}+\dfrac{(T-4t)t^{1-\alpha_i}}{4(1-\alpha_i)}, & t\in\left[0,\frac{T}{4}\right],\\ \dfrac{t^{2-\alpha_i}}{2-\alpha_i}-\dfrac{(4t-T)t^{1-\alpha_i}}{4(1-\alpha_i)}+\dfrac{(4t-T)^{2-\alpha_i})}{(1-\alpha_i)(2-\alpha_i)4^{2-\alpha_i}}, & t\in\left[\frac{T}{4},\frac{3T}{4}\right],\\ \dfrac{t^{2-\alpha_i}}{2-\alpha_i}-\dfrac{(4t-T)t^{1-\alpha_i}}{4(1-\alpha_i)}+\dfrac{(4t-T)^{2-\alpha_i}-(4t-3T)^{2-\alpha_i}}{(1-\alpha_i)(2-\alpha_i)4^{2-\alpha_i}}, & t\in\left[\frac{3T}{4},T\right]\end{cases}\end{aligned}$$

and

$$\begin{aligned}\overline{\mathcal{B}}(\alpha_i,T)&=\frac{32^2d_i^2}{T^4}\int_0^T\left(\frac{t^{2(2-\alpha_i)}}{(2-\alpha_i)^2}+\frac{(T/4-t)^2t^{2(2-\alpha_i)}}{(2-\alpha_i)^2}+\frac{2(T/4-t)t^{3-2\alpha_i}}{(1-\alpha_i)(2-\alpha_i)}\right)\mathrm{d}t\\ &+\frac{32^2d_i^2}{T^4}\int_{T/4}^T\left(\frac{(t-T/4)^{2(2-\alpha_i)}}{(1-\alpha_i)^2(2-\alpha_i)^2}+\frac{2(t^2-T/4t)^{2-\alpha_i}}{(1-\alpha_i)(2-\alpha_i)^2}-\frac{2(t-T/4)^{3-\alpha_i}t^{1-\alpha_i}}{(1-\alpha_i)^2(2-\alpha_i)}\right)\mathrm{d}t\\ &+\frac{32^2d_i^2}{T^4}\int_{3T/4}^T\left(\frac{(t-3/4T)^{2(2-\alpha_i)}}{(1-\alpha_i)^2(2-\alpha_i)^2}-\frac{2(t^2-3/4Tt)^{2-\alpha_i}}{(1-\alpha_i)(2-\alpha_i)^2}\right.\\ &\left.+\frac{2(t-T/4)(t-3/4T)^{2-\alpha_i}t^{1-\alpha_i}}{(1-\alpha_i)^2(2-\alpha_i)}-\frac{2(t^2-Tt+3/16T^2)^{2-\alpha_i}}{(1-\alpha_i)^2(2-\alpha_i)^2}\right)\mathrm{d}t\end{aligned}$$

for $1\le i\le n$.

Remark 3.2 If we take $\beta_i=\beta\in[0,1)$ for $1\le i\le n$ in (1.6), then (1.6) reduces to (1.4). The result about (1.4) in [11, Theorem 5.1] is that (1.4) has at least one solution $u^*\in E$ if

$$|F(t,u)|\le\overline{a}|u|^2+\overline{b}(t)|u|^s+\overline{c}(t), \tag{3.16}$$

where $\overline{a}\in[0,|\cos(\pi\alpha|\Gamma^2(\alpha+1/2T^{2\alpha}),\ s\in(0,2),\ \overline{b}\in L^{2/(2-s)}([0,T],\mathbb{R})$, and $\overline{c}\in L^1([0,T],\mathbb{R})$.

The restriction on F in (3.16) is similar to the (H3) in Theorem 3.1. Obviously, comparing with the fact that (1.6) has a nontrivial solution $u^*\in E$ satisfying

$$\|u^*\|_E>r_0^{\frac{1}{2}}\Big/\max_{1\le i\le n}\left(\frac{1}{2|\cos(\pi\alpha_i)|}+\frac{2T^{2\alpha_i-1}}{\Gamma^2(\alpha_i)(\alpha_i+1)}\right)^{\frac{1}{2}}$$

in our paper, it can not be ruled out that u^* is a zero solution for (1.4) in [11].

Example 3.1 Consider the following fractional advection-dispersion equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left({}_0\mathcal{D}_t^{-0.5}u_1'\right)(t)+\frac{1}{2}\left({}_t\mathcal{D}_T^{-0.5}u_1'\right)(t)\right)+F_{u_1}(t,u_1,u_2)=0, \quad \text{a.e. } t\in[0,1], \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left({}_0\mathcal{D}_t^{-0.4}u_2'\right)(t)+\frac{1}{2}\left({}_t\mathcal{D}_T^{-0.4}u_2'\right)(t)\right)+F_{u_2}(t,u_1,u_2)=0, \quad \text{a.e. } t\in[0,1], \\ u_1(0)=u_1(1)=0, \quad u_2(0)=u_2(1)=0, \end{cases} \tag{3.17}$$

where $F:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is the function defined by

$$F(t,u_1,u_2)=|1-2t|\left\{\frac{1}{4}\left(u_1^2+u_2^2\right)\sin\sqrt{u_1^2+u_2^2}+2\sqrt[4]{u_1^2+u_2^2}e^{-\left(u_1^2+u_2^2\right)}\right\}. \tag{3.18}$$

Comparing (3.17) with (1.6), we have $T=1$, $n=2$, $\beta_1=0.5$ and $\beta_2=0.4$, which lead to $\alpha_1=0.75$ and $\alpha_2=0.8$ in (2.1).

Obviously, both (H1) and (H3) of Theorem 3.1 are satisfied since $F(t,0,0)=0$ and

$$F(t,u_1,u_2)\le c_1u_1^2+c_2u_2^2+b_1(t)\sqrt{|u_1|}+b_2(t)\sqrt{|u_2|},$$

where $c_1=\frac{1}{4}\in[0,\frac{|\cos(\pi\alpha_1)|\Gamma^2(\alpha_1+1)}{2})\approx[0,0.2986)$, $c_2=\frac{1}{4}\in[0,\frac{|\cos(\pi\alpha_2)|\Gamma^2(\alpha_2+1)}{2})\approx[0,0.3509)$ and $b_1(t)=b_2(t)=2|1-2t|\in L^2([0,1],\mathbb{R})$.

Next, we will show that (H2) of Theorem 3.1 is also satisfied.

We choose

$$\omega_1(t)=\Gamma(1.25)t(1-t) \quad \text{and} \quad \omega_2(t)=\Gamma(1.2)t(1-t), \quad t\in[0,1],$$

then, one has $\omega=(\omega_1,\omega_2)\in E=E_{0.75}\times E_{0.8}$. We then have

$$({}_0^c\mathcal{D}_t^{0.75}\omega_1)(t)=t^{0.25}-\frac{8}{5}t^{1.25} \quad \text{and} \quad ({}_0^c\mathcal{D}_t^{0.8}\omega_1)(t)=t^{0.2}-\frac{5}{3}t^{1.2},$$

and thus we have $\|\omega_1\|_{0.75}^2\approx 0.1181$ and $\|\omega_2\|_{0.8}^2\approx 0.1424$. Hence

$$\|\omega\|_E^2=\|\omega_1\|_{0.75}^2+\|\omega_2\|_{0.8}^2\approx 0.2605. \tag{3.19}$$

Moreover,

$$\begin{aligned} &2\min_{1\le i\le 2}|\cos\pi\alpha_i|\int_0^1 F(t,\omega_1(t),\omega_2(t))\mathrm{d}t \\ &\quad=2\cos\frac{\pi}{4}\int_0^1|1-2t|\frac{2\sqrt[4]{\Gamma^2(1.25)+\Gamma^2(1.2)}\sqrt{t(1-t)}}{e^{(\Gamma^2(1.25)+\Gamma^2(1.2))t^2(1-t)^2}}\mathrm{d}t \\ &\qquad+2\cos\frac{\pi}{4}\int_0^1|1-2t|\frac{\Gamma^2(1.25)+\Gamma^2(1.2)}{4}t^2(1-t)^2\sin t(1-t)\mathrm{d}t \\ &\quad\approx 0.5123. \end{aligned} \tag{3.20}$$

Hence, we conclude that (H2) of Theorem 3.1 is satisfied from (3.19) and (3.20).

Therefore, from Theorem 3.1, the fractional ADE (3.17) has a nontrivial solution $u^*=(u_1^*,u_2^*)\in E_{0.75}\times E_{0.8}$.

Example 3.2 Consider the following fractional advection-dispersion equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left({}_0\mathcal{D}_t{}^{-0.6}u_1'\right)(t)+\frac{1}{2}\left({}_t\mathcal{D}_T{}^{-0.6}u_1'\right)(t)\right)+F_{u_1}(t,u_1,u_2)=0, & \text{a.e. } t\in[0,1],\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left({}_0\mathcal{D}_t{}^{-0.8}u_2'\right)(t)+\frac{1}{2}\left({}_t\mathcal{D}_T{}^{-0.8}u_2'\right)(t)\right)+F_{u_2}(t,u_1,u_2)=0, & \text{a.e. } t\in[0,1],\\ u_1(0)=u_1(1)=0,\quad u_2(0)=u_2(1)=0, \end{cases} \tag{3.21}$$

where $F:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is the function defined by

$$F(t,u_1,u_2)=\frac{|1-2t|}{10}\ln\left(1+u_1^2+u_2^2\right)+8\left(u_1^2+u_2^2\right)^{\frac{3}{4}}(t-t^2)^{\frac{1}{2}}. \tag{3.22}$$

Comparing (3.21) with (1.6), we have $T=1$, $n=2$, $\beta_1=0.6$ and $\beta_2=0.8$, which lead to $\alpha_1=0.7$ and $\alpha_2=0.6$ in (2.1).

Obviously, both (H1) and (H3) of Theorem 3.1 are satisfied since $F(t,0,0)=0$ and

$$F(t,u_1,u_2)\le c_1u_1^2+c_2u_2^2+b_1(t)|u_1|^{\frac{3}{2}}+b_2(t)|u_2|^{\frac{3}{2}},$$

where $c_1=\frac{1}{10}\in[0,\frac{|\cos(\pi\alpha_1)|\Gamma^2(\alpha_1+1)}{2})\approx[0,0.2425)$, $c_2=\frac{1}{10}\in[0,\frac{|\cos(\pi\alpha_2)|\Gamma^2(\alpha_2+1)}{2})\approx[0,0.1233)$ and $b_1(t)=b_2(t)=8(t-t^2)^{\frac{1}{2}}\in L^2([0,1],\mathbb{R})$.

Next, we will show that (H2) of Theorem 3.1 is also satisfied.

We choose

$$\omega_1(t)=\Gamma(1.3)t(1-t)\quad\text{and}\quad\omega_2(t)=\Gamma(1.4)t(1-t),\quad t\in[0,1],$$

then, one has $\omega=(\omega_1,\omega_2)\in E=E_{0.7}\times E_{0.6}$. We then have

$$({}_0^c\mathcal{D}_t{}^{0.7}\omega_1)(t)=t^{0.3}-\frac{20}{13}t^{1.3}\quad\text{and}\quad({}_0^c\mathcal{D}_t{}^{0.6}\omega_1)(t)=t^{0.4}-\frac{10}{7}t^{1.4},$$

and thus we have $\|\omega_1\|_{0.7}^2\approx0.0989$ and $\|\omega_2\|_{0.6}^2\approx0.0723$. Hence

$$\|\omega\|_E^2=\|\omega_1\|_{0.7}^2+\|\omega_2\|_{0.6}^2\approx0.1712. \tag{3.23}$$

Moreover,

$$\begin{aligned} &2\min_{1\le i\le 2}|\cos\pi\alpha_i|\int_0^1F(t,\omega_1(t),\omega_2(t))\mathrm{d}t\\ &=2\cos\frac{2\pi}{5}\int_0^1\frac{|1-2t|}{10}\ln[1+(\Gamma^2(1.3)+\Gamma^2(1.4))(t-t^2)^2]\mathrm{d}t\\ &\quad+16\cos\frac{2\pi}{5}\int_0^1[\Gamma^2(1.3)+\Gamma^2(1.4)]^{\frac{3}{4}}(t-t^2)^2\mathrm{d}t\\ &\approx0.2346. \end{aligned} \tag{3.24}$$

Hence, we conclude that (H2) of Theorem 3.1 is satisfied from (3.23) and (3.24).

Therefore, from Theorem 3.1, the fractional ADE (3.21) has a nontrivial solution $u^*=(u_1^*,u_2^*)\in E_{0.7}\times E_{0.6}$.

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