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ORIGINAL PAPER

On generalization of midpoint type inequalities with generalized fractional integral operators

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Abstract The Hermite–Hadamard inequality is the first principal result for convex functions defined on a interval of real numbers with a natural geometrical interpretation and a loose number of applications for particular inequalities. In this paper we proposed the Hermite–Hadamard and midpoint type inequalities for functions whose first and second derivatives in absolute value are s-convex through the instrument of generalized fractional integral operator and a considerable amount of results for special means which can naturally be deduced.

Keywords Hermite–Hadamard inequality · Midpoint inequality · Fractional integral operators · Convex function

Mathematics Subject Classification 26D15 · 26B25 · 26D10

1 Introduction

The Hermite–Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. A number of mathematicians have devoted their efforts

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to generalise, refine, counterpart and extend it for different classes of functions such as using convex mappings.

The inequalities discovered by Hermite and Hadamard for convex functions are considerable significant in the literature (see, e.g., [6,8], [13, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [2,3,5,7,10,12,15,17–20,22]) and the references cited therein.

The overall structure of the study takes the form of five sections including introduction. The remainder of this work is organized as follows: In Sect. 2, the generalised version of fractional integral operator are summarised, along with the very first results. In Sect. 3 the Hermite–Hadamard type inequalities for generalized fractional integral operators are introduced while in Sects. 4 and 5 midpoint type inequalities for functions whose first and second derivatives in absolute value are s-convex with generalized fractional integral operators are presented and we also provide some corollary for theorems. Some conclusions and further directions of research are discussed in Sect. 6.

2 Preliminaries

Now we reviewed some definitions and theorems which will be used in the proof of our main cumulative results.

Definition 1 [4] Let $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex (in the second sense), or that f belongs to the class K_s^2, if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \in [0, \infty)$ and $\lambda \in [0, 1]$.

An s-convex function was introduced in Breckner's paper [4] and a number of properties and connections with s-convexity in the first sense were discussed in paper [9].

In addition to this, Raina [14] defined the following results connected with the general class of fractional integral operators.

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < \mathcal{R}), \tag{2.1}$$

where the coefficients $\sigma(k)$ ($k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$) is a bounded sequence of positive real numbers and $\mathcal{R}$ is the set of real numbers. With the help of (2.1), Raina [14] and Agarwal et al. [1] defined the following left-sided and right-sided fractional integral operators, respectively, as follows:

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega} f(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(x-t)^{\rho}\right] f(t)dt, \quad x > a, \tag{2.2}$$

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega (t-x)^{\rho}\right] f(t)dt, \quad x < b, \tag{2.3}$$

where $\lambda, \rho > 0, \omega \in \mathbb{R}$, and $f(t)$ is such that the integrals on the right side exists.

It is easy to verify that $\mathcal{J}^{\alpha}_{\rho,\lambda,a+;\omega}f(x)$ and $\mathcal{J}^{\alpha}_{\rho,\lambda,b-;\omega}f(x)$ are bounded integral operators on $L(a,b)$, if

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega (b-a)^{\rho}\right] < \infty. \tag{2.4}$$

In fact, for $f \in L(a,b)$, we have

$$\left\| \mathcal{J}^{\alpha}_{\rho,\lambda,a+;\omega}f(x)\right\|_1 \leq \mathfrak{M}(b-a)^{\lambda} \|f\|_1 \tag{2.5}$$

and

$$\left\| \mathcal{J}^{\alpha}_{\rho,\lambda,b^-;\omega}f(x)\right\|_1 \leq \mathfrak{M}(b-a)^{\lambda} \|f\|_1, \tag{2.6}$$

where

$$\|f\|_p := \left(\int_a^b |f(t)|^p\, dt\right)^{\frac{1}{p}}.$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Here, we just point out that the classical Riemann–Liouville fractional integrals I^{α}_{a+} and I^{α}_{b-} of order α defined by (see [11])

$$\left(I^{\alpha}_{a^+}f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt \quad (x > a; \alpha > 0) \tag{2.7}$$

and

$$\left(I^{\alpha}_{b^-}f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt \quad (x < b; \alpha > 0) \tag{2.8}$$

follow easily by setting

$$\lambda = \alpha, \;\; \sigma(0) = 1, \;\; and w = 0 \tag{2.9}$$

in (2.2) and (2.3), and the boundedness of (2.7) and (2.8) on $L(a,b)$ is also inherited from (2.5) and (2.6) (see [1]).

Yaldiz and Sarikaya [21] gave the following Hermite–Hadamard inequality for the generalized fractional integral operators:

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ with $a < b$, then the following inequalities for fractional integral operators hold*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2(b-a)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega (b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b) + \mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)\right] \\ &\leq \frac{f(a)+f(b)}{2} \end{aligned}$$

with $\lambda > 0$.

The main purpose of this paper is to introduce new type Hermite Hadamard and midpoint integral inequalities with the aid of generalized fractional integral operators and establish some results connected with the them.

3 Hermite–Hadamard type inequalities for generalized fractional integral operators

In this section, we will present a theorem for Hermite–Hadamard type inequalities with generalized fractional integral operators which is the generalization of previous work.

Theorem 2 *Let $f:[a,b]\to\mathbb{R}$ be a function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a s-convex function on $[a,b]$, then we have the following inequalities for generalized fractional integral operators:*

$$2^s f\left(\frac{a+b}{2}\right)\leq\frac{2^{\lambda}}{(b-a)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]$$
$$\leq\frac{2^{-s}}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[A_1(\lambda-1,s)+\mathcal{F}^{\sigma_{0,s}}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)\right]\right][f(a)+f(b)] \tag{3.1}$$

where $\sigma_{0,s}(k)=\frac{\sigma(k)}{\rho k+s+\lambda}$, $k=0,1,2,\ldots$ and

$$A_1(\lambda,s)=\int_0^1 t^{\lambda}\,(2-t)^s\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]dt.$$

Proof Since f is s-convex function on $[a,b]$, we have for $x,y\in[a,b]$

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2^s}.$$

For $x=\frac{t}{2}a+\frac{2-t}{2}b$ and $y=\frac{2-t}{2}a+\frac{t}{2}b$, we obtain

$$2^s f\left(\frac{a+b}{2}\right)\leq f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b\right). \tag{3.2}$$

Multiplying both sides of (3.2) by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]$, then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$2^s f\left(\frac{a+b}{2}\right)\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]dt$$
$$\leq\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt$$
$$+\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt.$$

For $u=\frac{t}{2}a+\frac{2-t}{2}b$ and $v=\frac{2-t}{2}a+\frac{t}{2}b$, we obtain

$$
\begin{aligned}
&2^s f\left(\frac{a+b}{2}\right)\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]\\
&\le \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\left(\frac{2}{b-a}(b-u)\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\left(\frac{2}{b-a}(b-u)\right)^{\rho}\right]f(u)\,du\\
&\quad+\frac{2}{b-a}\int_{a}^{\frac{a+b}{2}}\left(\frac{2}{b-a}(v-a)\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\left(\frac{2}{b-a}(v-a)\right)^{\rho}\right]f(v)\,dv\\
&=\left(\frac{2}{b-a}\right)^{\lambda}\int_{\frac{a+b}{2}}^{b}(b-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-u)^{\rho}\right]f(u)\,du\\
&\quad+\left(\frac{2}{b-a}\right)^{\lambda}\int_{a}^{\frac{a+b}{2}}(v-a)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-a)^{\rho}\right]f(v)\,dv\\
&=\left(\frac{2}{b-a}\right)^{\lambda}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]
\end{aligned}
$$

and the first inequality is proved.

For the proof of the second inequality (3.1), we first note that if f is a s-convex function, it yields

$$f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\le\frac{t^s}{2^s}f(a)+\frac{(2-t)^s}{2^s}f(b)$$

and

$$f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\le\frac{(2-t)^s}{2^s}f(a)+\frac{t^s}{2^s}f(b).$$

By adding these inequalities together, one has the following inequality:

$$f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\le 2^{-s}\left[f(a)+f(b)\right]\left[t^s+(2-t)^s\right]. \tag{3.3}$$

Then multiplying both sides of (3.3) by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$
\begin{aligned}
&\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\\
&\quad+\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\\
&\le 2^{-s}\left[f(a)+f(b)\right]\int_0^1\left[t^s+(2-t)^s\right]t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]dt
\end{aligned}
$$

$$= 2^{-s}\left[A_1(\lambda-1,s)+\mathcal{F}^{\sigma_{0,s}}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)\right]\right][f(a)+f(b)].$$

That is,

$$\left(\frac{2}{b-a}\right)^{\lambda}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]$$
$$\leq 2^{-s}\left[A_1(\lambda-1,s)+\mathcal{F}^{\sigma_{0,s}}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)\right]\right][f(a)+f(b)].$$

Hence, the proof is completed. □

Corollary 1 *If we take $s=1$ in Theorem 2, then we have the following inequality*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\lambda}}{(b-a)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]$$
$$\leq\frac{f(a)+f(b)}{2}.$$

Corollary 2 *If we take $\lambda=\alpha,\ \sigma(0)=1,\ w=0$ in Theorem 2, then we have the following inequality for Riemann–Lioville fractional integral operators*

$$2^{s}f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]$$
$$\leq\alpha 2^{-s}\left[B_1(\alpha-1,s)+\frac{1}{\alpha+s}\right][f(a)+f(b)]$$

where

$$B_1(\alpha,s)=\int_0^1 t^{\alpha}(2-t)^{s}\,dt.$$

Remark 1 Choosing $s=1$ in Corollary 2, then we have the following inequality for Riemann–Lioville fractional integral operators

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\leq\frac{f(a)+f(b)}{2}$$

which was given by Sarikaya and Yıldırım [16].

4 Midpoint type inequalities for differentiable functions with generalized fractional integral operators

In this section, firstly we need to give a lemma for differentiable functions which will help us to prove our main theorems. Then, we present some theorems which are the generalization of those given in earlier works.

Lemma 1 *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $f'\in L[a,b]$, then we have the following identity for generalized fractional integral operators:*

$$\frac{2^{\lambda-1}}{(b-a)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]-f\left(\frac{a+b}{2}\right)$$
$$=\frac{b-a}{4\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\right.$$
$$\left.-\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right]. \tag{4.1}$$

Proof Integrating by parts gives

$$\begin{aligned}I_1&=\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\\&=-\frac{2}{b-a}t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\Bigg|_0^1\\&\quad+\frac{2}{b-a}\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\\&=-\frac{2}{b-a}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]f\left(\frac{a+b}{2}\right)+\left(\frac{2}{b-a}\right)^{\lambda+1}\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)\end{aligned} \tag{4.2}$$

and similarly we get

$$\begin{aligned}I_2&=\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\\&=\frac{2}{b-a}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]f\left(\frac{a+b}{2}\right)-\left(\frac{2}{b-a}\right)^{\lambda+1}\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a).\end{aligned} \tag{4.3}$$

By subtracting equation (4.3) from (4.2), we have

$$\begin{aligned}I_1-I_2=&-\frac{4}{b-a}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]f\left(\frac{a+b}{2}\right)\\&+\left(\frac{2}{b-a}\right)^{\lambda+1}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right].\end{aligned}$$

By re-arranging the last equality above, we get the desired result. □

Theorem 3 *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $|f'|$ is s-convex function in the second sense, then we have the following inequality for generalized fractional integral operators:*

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{2^{s+2}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[A_1(\lambda,s)+\mathcal{F}^{\sigma_{1,s}}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)\right]\right]\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \tag{4.4}$$

where $\sigma_{1,s}(k)=\frac{\sigma(k)}{\rho k+s+\lambda+1}$, $k=0,1,2,\ldots$ *and* $A_1(\lambda,s)$ *is defined as in Theorem 2.*

Proof Taking modulus of (4.1) and using s-convexity of $\left|f'\right|$, we have

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right.$$
$$\left.+\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right]$$
$$\leq\frac{b-a}{4\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]\left[\left(\frac{t}{2}\right)^{s}\left|f'(a)\right|\right.\right.$$
$$\left.+\left(\frac{2-t}{2}\right)^{s}\left|f'(b)\right|\right]dt$$
$$\left.+\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]\left[\left(\frac{2-t}{2}\right)^{s}\left|f'(a)\right|+\left(\frac{t}{2}\right)^{s}\left|f'(b)\right|\right]dt\right]$$
$$=\frac{b-a}{2^{s+2}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[A_1(\lambda,s)+\int_0^1 t^{\lambda+s}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]dt\right]\left[\left|f'(a)\right|\right.$$
$$\left.+\left|f'(b)\right|\right]$$
$$=\frac{b-a}{2^{s+2}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[A_1(\lambda,s)+\mathcal{F}^{\sigma_{1,s}}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)\right]\right]\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

where $\sigma_{1,s}(k)$ and $A_1(\lambda,s)$ are defined above. Thus, the proof is completed. □

Corollary 3 *If we take $s=1$ in Theorem 3, then we have*

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4}\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[\omega\left(\frac{b-a}{2}\right)\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{4.5}$$

Corollary 4 *If we take $\lambda = \alpha,\ \sigma(0) = 1,\ w = 0$ in Theorem 3, then we have the following inequality for Riemann–Lioville fractional integral operators*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{2^{s+2}}\left[B_1(\alpha,s)+\frac{1}{s+\alpha+1}\right]\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

where $B_1(\alpha, s)$ is defined as in Corollary 2.

Remark 2 Chosing $s = 1$ in Corollary 4, we obtain following inequality

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4(\alpha+1)}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

which was given by Sarikaya and Yıldırım in [16].

Theorem 4 *Let $f : [a, b] \to \mathbb{R}$ be differentiable function on (a, b) with $a < b$. If $\left|f'\right|^{q}$, $q > 1$, is s-convex function in the second sense, then we have the following inequality for generalized fractional integral operators:*

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)\,C_1(\lambda,p)}{2^{2+\frac{s}{q}}(s+1)^{\frac{1}{q}}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]} \times\left[\left(\left|f'(a)\right|^{q}+\left[2^{s+1}-1\right]\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}+\left(\left[2^{s+1}-1\right]\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right] \tag{4.6}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$C_1(\lambda,p)=\left(\int_0^1 t^{\lambda p}\left[\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]\right]^{p}dt\right)^{\frac{1}{p}}.$$

Proof Taking modulus of (4.1) and using well-known Hölder inequality, we obtain

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt + \int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right]$$

$$\leq \frac{b-a}{4\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left(\int_0^1 t^{\lambda p}\left[\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]\right]^p dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]. \quad (4.7)$$

Since $\left|f'\right|^q$, $q>1$, is s-convex, we have

$$\int_0^1\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt \leq \int_0^1\left[\left(\frac{t}{2}\right)^s\left|f'(a)\right|^q+\left(\frac{2-t}{2}\right)^s\left|f'(b)\right|^q\right]dt$$

$$=\frac{1}{2^s(s+1)}\left[\left|f'(a)\right|^q+\left[2^{s+1}-1\right]\left|f'(b)\right|^q\right] \quad (4.8)$$

and similarly

$$\int_0^1\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt \leq \frac{1}{2^s(s+1)}\left[\left[2^{s+1}-1\right]\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]. \quad (4.9)$$

By substituting inequalities (4.8) and (4.9) into (4.7), we get the desired result (4.6). □

Corollary 5 *If we take $s=1$ in Theorem 4, then we get*

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)\,C_1(\lambda,p)}{2^{2+\frac{2}{q}}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\left(\left|f'(a)\right|^q+3\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(3\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{(b-a)\,C_1(\lambda,p)}{2^{\frac{2}{q}}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \quad (4.10)$$

Proof The proof of the first inequality in (4.10) is obvious. For the proof of second inequality, let $a_1=3\left|f'(a)\right|^q$, $b_1=\left|f'(b)\right|^q$, $a_2=\left|f'(a)\right|^q$ and $b_2=3\left|f'(b)\right|^q$. Using the fact that,

$$\sum_{k=1}^{n}(a_k+b_k)^s\leq\sum_{k=1}^{n}a_k^s+\sum_{k=1}^{n}b_k^s,\ 0\leq s<1$$

the desired result can be obtained straightforwardly. □

Corollary 6 *If we take $\lambda=\alpha$, $\sigma(0)=1$, $w=0$ in Theorem 4, then we have the following inequality for Riemann–Liouville fractional integral operators*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)}{2^{2+\frac{s}{q}}(s+1)^{\frac{1}{q}}}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\left[\left|f'(a)\right|^{q}+\left[2^{s+1}-1\right]\left|f'(b)\right|^{q}\right]\right)^{\frac{1}{q}}+\left(\left[\left[2^{s+1}-1\right]\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}\right]\right)^{\frac{1}{q}}\right].$$

Remark 3 Choosing $\lambda=\alpha,\ \sigma(0)=1,\ w=0$ in Corollary 5, we have the following inequality

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(a)\right|^{q}+3\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

which is the same result given by Sarikaya and Yıldırım [16].

Theorem 5 *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$. If $\left|f'\right|^{q}$, $q\geq 1$, is s-convex function in the second sense, then we have the following inequality for generalized fractional integral operators:*

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{2^{2+\frac{s}{q}}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left(\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\mathcal{F}^{\sigma_{1,s}}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)\right]\left|f'(a)\right|^{q}+A_1(\lambda,s)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(A_1(\lambda,s)\left|f'(a)\right|^{q}+\mathcal{F}^{\sigma_{1,s}}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)\right]\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right]$$

where $\sigma_{1,s}(k)$, $k=0,1,2,\ldots$ and $A_1(\lambda,s)$ are defined as in Theorem 3.

Proof Taking modulus of (4.1), using well-known power mean inequality and s -convexity of $\left|f'\right|^{q}$, we obtain

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^{\rho}t^{\rho}\right]\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right.$$

$$+\int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^\rho t^\rho\right]\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\Bigg]$$

$$\leq \frac{b-a}{4\mathcal{F}^\sigma_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^\rho\right]}\left(\int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^\rho t^\rho\right]dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^\rho t^\rho\right]\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^\rho t^\rho\right]\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{4\mathcal{F}^\sigma_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^\rho\right]}\left(\mathcal{F}^\sigma_{\rho,\lambda+2}\left[\omega\left(\frac{b-a}{2}\right)^\rho\right]\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^\rho t^\rho\right]\left[\left(\frac{t}{2}\right)^s\left|f'(a)\right|^q+\left(\frac{2-t}{2}\right)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^\rho t^\rho\right]\left[\left(\frac{2-t}{2}\right)^s\left|f'(a)\right|^q+\left(\frac{t}{2}\right)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right]$$

$$=\frac{b-a}{2^{2+\frac{s}{q}}\mathcal{F}^\sigma_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^\rho\right]}\left(\mathcal{F}^\sigma_{\rho,\lambda+2}\left[\omega\left(\frac{b-a}{2}\right)^\rho\right]\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)\right]\left|f'(a)\right|^q+A_1(\lambda,s)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(A_1(\lambda,s)\left|f'(a)\right|^q+\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)\right]\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]$$

which completes the proof. □

Corollary 7 *If we take $s=1$ in Theorem 5, then we get*

$$\left|\frac{2^{\lambda-1}}{(b-a)^\lambda\mathcal{F}^\sigma_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^\rho\right]}\left[\mathcal{J}^\sigma_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)+\mathcal{J}^\sigma_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{b-a}{2^{2+\frac{1}{q}}\mathcal{F}^\sigma_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)^\rho\right]}\left(\mathcal{F}^\sigma_{\rho,\lambda+2}\left[\omega\left(\frac{b-a}{2}\right)^\rho\right]\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\mathcal{F}^{\sigma_{1,1}}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)\right]\left|f'(a)\right|^q+\left(\mathcal{F}^\sigma_{\rho,\lambda+2}\left[\omega\left(\frac{b-a}{2}\right)\right]\right.\right.\right.$$

$$-\mathcal{F}^{\sigma_{1,1}}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)\right]\right)\left|f'(b)\right|^{q}\Bigg)^{\frac{1}{q}}$$
$$+\left(\left(\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[\omega\left(\frac{b-a}{2}\right)\right]-\mathcal{F}^{\sigma_{1,1}}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)\right]\right)\left|f'(a)\right|^{q}\right.$$
$$\left.+\mathcal{F}^{\sigma_{1,1}}_{\rho,\lambda+1}\left[\omega\left(\frac{b-a}{2}\right)\right]\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\Bigg].$$

Corollary 8 *If we take $\lambda=\alpha,\ \sigma(0)=1,\ w=0$ in Theorem 5, then we have the following inequality for Riemann–Liouville fractional integral operators*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)}{2^{2+\frac{s}{q}}}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}}$$
$$\times\left[\left(\left|f'(a)\right|^{q}+(\alpha+s+1)\,B_1(\alpha,s)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left((\alpha+s+1)\,B_1(\alpha,s)\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right]$$

where $B_1(\alpha,s)$ is defined as in Corollary 4.

Remark 4 Choosing $s=1$ in Corollary 8, we have the following inequality

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4}\left(\frac{1}{\alpha+1}\right)\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\left[\left((\alpha+1)\left|f'(a)\right|^{q}+(\alpha+3)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left((\alpha+3)\left|f'(a)\right|^{q}+(\alpha+1)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right].$$

which is the same result given by Sarikaya and Yıldırım [16].

5 Midpoint type inequalities for twice differentiable functions with generalized fractional integral operators

In Sect. 5, firstly we need to give a lemma for twice differentiable functions which will help us to prove our main theorems. Then, we present some theorems which are the generalization of those given in earlier works.

Lemma 2 *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable function on (a,b) with $a<b$. If $f''\in L[a,b]$, then we have the following identity for generalized fractional integral operators:*

$$\frac{2^{\lambda-1}}{(b-a)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)\right]-f\left(\frac{a+b}{2}\right)$$

$$
\begin{aligned}
&= \frac{(b-a)^2}{8\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\int_0^1 (1-t)^{\lambda+1}\,\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\right.\\
&\quad\times f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
&\quad\left.+\int_0^1 (1-t)^{\lambda+1}\,\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right]. \qquad (5.1)
\end{aligned}
$$

Proof

$$
\begin{aligned}
I &= \int_0^1 (1-t)^{\lambda+1}\,\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
&\quad+\int_0^1 (1-t)^{\lambda+1}\,\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\\
&= I_1+I_2. \qquad (5.2)
\end{aligned}
$$

Integrating by parts we have

$$
\begin{aligned}
I_1 &= \int_0^1 (1-t)^{\lambda+1}\,\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
&= -\frac{2}{b-a}(1-t)^{\lambda+1}\,\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\Bigg|_0^1\\
&\quad-\frac{2}{b-a}\int_0^1 (1-t)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
&= \frac{2}{b-a}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]f'\left(\frac{a+b}{2}\right)\\
&\quad+\frac{4}{(b-a)^2}(1-t)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\Bigg|_0^1\\
&\quad+\frac{4}{(b-a)^2}\int_0^1 (1-t)^{\lambda-1}\,\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
&= \frac{2}{b-a}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]f'\left(\frac{a+b}{2}\right)\\
&\quad-\frac{4}{(b-a)^2}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]f\left(\frac{a+b}{2}\right)\\
&\quad+\left(\frac{2}{b-a}\right)^{\lambda+2}\int_a^{\frac{a+b}{2}} (x-a)^{\lambda-1}\,\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-a)^{\rho}\right]f(x)\,dx. \qquad (5.3)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
I_2 &= \int_0^1 (1-t)^{\lambda+1} \mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right] f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt \\
&= -\frac{2}{b-a}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right] f'\left(\frac{a+b}{2}\right) \\
&\quad -\frac{4}{(b-a)^2}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(b-a\right)^{\rho}\right] f\left(\frac{a+b}{2}\right) \\
&\quad +\left(\frac{2}{b-a}\right)^{\lambda+2}\int_{\frac{a+b}{2}}^{b}(b-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-x\right)^{\rho}\right] f(x)\,dx. \qquad (5.4)
\end{aligned}
$$

Combining of (5.2), (5.3) and (5.4), we obtain

$$
\begin{aligned}
I &= \left(\frac{2}{b-a}\right)^{\lambda+2}\int_a^{\frac{a+b}{2}}(x-a)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(x-a\right)^{\rho}\right] f(x)\,dx \\
&\quad +\left(\frac{2}{b-a}\right)^{\lambda+2}\int_{\frac{a+b}{2}}^{b}(b-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-x\right)^{\rho}\right] f(x)\,dx \\
&\quad -\frac{8}{(b-a)^2}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right] f\left(\frac{a+b}{2}\right) \\
&= \left(\frac{2}{b-a}\right)^{\lambda+2}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega_1} f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega_1} f(b)\right] \\
&\quad -\frac{8}{(b-a)^2}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right] f\left(\frac{a+b}{2}\right). \qquad (5.5)
\end{aligned}
$$

Mutiplying both sides of (5.5) by $\frac{(b-a)^2}{8\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}$ completes the proof. □

Theorem 6 *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable function on (a,b) with $a<b$. If $\left|f''\right|$ is s-convex function in the second sense, then we have the following inequality for generalized fractional integral operators:*

$$
\begin{aligned}
&\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega} f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega} f(b)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^2}{2^{s+3}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[A_2(\lambda,s)+\mathcal{F}^{\sigma_{2,s}}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\right]\left[\left|f''(a)\right|+\left|f''(b)\right|\right]
\end{aligned}
\qquad (5.6)
$$

where $\sigma_{2,s}(k)=\frac{\sigma(k)}{\rho k+\lambda+s+2}$, $k=0,1,2,\ldots$ and $A_2(\lambda,s)$ is defined by

$$
A_2(\lambda,s)=\int_0^1(1-t)^{\lambda+1}(1+t)^s\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]dt. \qquad (5.7)
$$

Proof Taking modulus both sides of (3.1) and using s-convexity of $\left|f''\right|$, we obtain

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^2}{8\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right.$$
$$\left.+\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt\right]$$
$$\leq\frac{(b-a)^2}{8\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\right.$$
$$\times\left[\left(\frac{1+t}{2}\right)^{s}\left|f''(a)\right|+\left(\frac{1-t}{2}\right)^{s}\left|f''(b)\right|\right]dt$$
$$\left.+\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\left[\left(\frac{1-t}{2}\right)^{s}\left|f''(a)\right|+\left(\frac{1+t}{2}\right)^{s}\left|f''(b)\right|\right]dt\right]$$
$$=\frac{(b-a)^2}{2^{s+3}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[A_2(\lambda,s)+\int_0^1(1-t)^{s+\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]dt\right]$$
$$\times\left[\left|f''(a)\right|+\left|f''(b)\right|\right]$$
$$=\frac{(b-a)^2}{2^{s+3}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[A_2(\lambda,s)+\mathcal{F}^{\sigma_{2,s}}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\right]\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

Thus, the proof is completed. □

Corollary 9 *If we choose* $s=1$ *Theorem 6, then we have the following inequality*

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^2}{8}\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

Corollary 10 *If we take* $\lambda=\alpha,\ \sigma(0)=1,\ w=0$ *in Theorem 6, then we have the following inequality for Riemann–Liouville fractional integral operators*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^2}{2^{s+3}(\alpha+1)}\left[B_2(\alpha,s)+\frac{1}{s+\alpha+2}\right]\left[\left|f''(a)\right|+\left|f''(b)\right|\right]$$

where

$$B_2(\alpha,s)=\int_0^1(1-t)^{\alpha+1}(1+t)^{s}\,dt.$$

which is the same result in [12]

Theorem 7 *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable function on (a,b) with $a<b$. If $\left|f''\right|^q$, $q>1$, is s-convex function in the second sense, then we have the following inequality for generalized fractional integral operators:*

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^2C_2(\lambda,p)}{2^{3+\frac{s}{q}}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}$$
$$\times\left\{\left(\left(2^{s+1}-1\right)\left|f''(a)\right|^q+\left|f''(b)\right|^q\right)^{\frac{1}{q}}+\left(\left|f''(a)\right|^q+\left(2^{s+1}-1\right)\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right\} \tag{5.8}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $C_2(\lambda,p)$ is defined by

$$C_2(\lambda,p)=\left(\int_0^1(1-t)^{p(\lambda+1)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\right]^p dt\right)^{\frac{1}{p}}. \tag{5.9}$$

Proof Taking modulus both sides of (3.1) and using well-known Hölder inequality, we have

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^2}{8\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right.$$
$$\left.+\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt\right]$$
$$\leq\frac{(b-a)^2}{8\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left(\int_0^1(1-t)^{p(\lambda+1)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\right]^p dt\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]. \tag{5.10}$$

Since $\left|f''\right|^q$ is s-convex function in the second sense, we get

$$\int_0^1\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\leq\left|f''(a)\right|^q\int_0^1\left(\frac{1+t}{2}\right)^s dt+\left|f''(b)\right|^q\int_0^1\left(\frac{1-t}{2}\right)^s dt$$
$$=\frac{1}{2^s(s+1)}\left[\left(2^{s+1}-1\right)\left|f''(a)\right|^q+\left|f''(b)\right|^q\right] \tag{5.11}$$

and similarly

$$\int_0^1 \left| f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt \le \frac{1}{2^s(s+1)}\left[\left|f''(a)\right|^q+\left(2^{s+1}-1\right)\left|f''(b)\right|^q\right]. \tag{5.12}$$

If we substitute the inequalities (5.11) and (5.12) in (5.10), we obtain desired result (5.8). □

Corollary 11 *If we choose $s=1$ Theorem 7, then $\left|f''\right|^q$, $q>1$, is convex and we have the following inequality*

$$\begin{aligned}&\left|\frac{2^{\lambda-1}}{(b-a)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^\rho\right]}\left[\mathcal{J}^\sigma_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)+\mathcal{J}^\sigma_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\le\frac{(b-a)^2C_2(\lambda,p)}{2^{1+\frac{2}{q}}\mathcal{F}^\sigma_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^\rho\right]}\left\{\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}\\&\le\frac{(b-a)^2C_2(\lambda,p)}{2^{1+\frac{2}{q}}\mathcal{F}^\sigma_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^\rho\right]}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]\end{aligned}$$

$\frac{1}{p}+\frac{1}{q}=1$ and $C_2(p,\lambda)$ is defined as in (5.9).

Proof The proof can be done by following the similar steps with Corollary 5. □

Corollary 12 *If we take $\lambda=\alpha,\ \sigma(0)=1,\ w=0$ in Theorem 7, then we have the following inequality for Riemann–Liouville fractional integral operators*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)+I^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\le\frac{(b-a)^2}{2^{3+\frac{s}{q}}(\alpha+1)}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{1}{p(\lambda+1)+1}\right)^{\frac{1}{p}}\\&\quad\times\left\{\left(\left[\left(2^{s+1}-1\right)\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]\right)^{\frac{1}{q}}+\left(\left[\left|f''(a)\right|^q+\left(2^{s+1}-1\right)\left|f''(b)\right|^q\right]\right)^{\frac{1}{q}}\right\}\end{aligned}$$

which is the same result in [12].

Corollary 13 *If we take $\lambda=\alpha,\ \sigma(0)=1,\ w=0$ in Corollary 11, we have*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)+I^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\le\frac{(b-a)^2}{2^{1+\frac{2}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\\&\quad\times\left\{\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}\\&\le\frac{(b-a)^2}{2^{\frac{2}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right].\end{aligned}$$

Theorem 8 *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable function on (a,b) with $a < b$. If $\left|f''\right|^q$, $q \geq 1$, is s-convex function in the second sense, then we have the following inequality for generalized fractional integral operators:*

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^2}{2^{3+\frac{s}{q}}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left(\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[A_2(\lambda,s)\left|f''(a)\right|^q+\mathcal{F}^{\sigma_{2,s}}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\mathcal{F}^{\sigma_{2,s}}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\left|f''(a)\right|^q+A_2(\lambda,s)\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\} \tag{5.13}$$

where $\sigma_{2,s}(k)$, $k=0,1,2,\ldots$ and $A_2(\lambda,s)$ are defined as in Theorem 6.

Proof Taking modulus both sides of (3.1) and using well-known power mean inequality, we have

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^2}{8\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right.$$
$$\left.+\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt\right]$$
$$\leq \frac{(b-a)^2}{8\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left(\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]dt\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

By a simple computation, we obtain

$$\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]dt=\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w(b-a)^{\rho}\right]. \tag{5.14}$$

Using the equality (5.14) and s-convexity of $\left|f''\right|^q$, we have

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega_1}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega_1}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^2}{8\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left(\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\right.\right.$$
$$\left.\times\left[\left(\frac{1+t}{2}\right)^{s}\left|f''(a)\right|^{q}+\left(\frac{1-t}{2}\right)^{s}\left|f''(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}(1-t)^{\rho}\right]\left[\left(\frac{1-t}{2}\right)^{s}\left|f''(a)\right|^{q}\right.\right.$$
$$\left.\left.\left.+\left(\frac{1+t}{2}\right)^{s}\left|f''(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]$$
$$=\frac{(b-a)^2}{2^{3+\frac{s}{q}}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left(\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[A_2(\lambda,s)\left|f''(a)\right|^{q}+\mathcal{F}^{\sigma_{2,s}}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\left|f''(b)\right|^{q}\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\mathcal{F}^{\sigma_{2,s}}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\left|f''(a)\right|^{q}+A_2(\lambda,s)\left|f''(b)\right|^{q}\right]^{\frac{1}{q}}\right\}$$

which completes the proof. □

Corollary 14 *If we choose $s=1$ Theorem 5, then $\left|f''\right|^{q}$, $q\geq 1$, is convex and we have the following inequality*

$$\left|\frac{2^{\lambda-1}}{(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)-;\omega}f(a)+\mathcal{J}^{\sigma}_{\rho,\lambda,\left(\frac{a+b}{2}\right)+;\omega}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^2}{2^{3+\frac{1}{q}}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}\left(\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\left(\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]-\mathcal{F}^{\sigma_{2,1}}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\right)\left|f''(a)\right|^{q}\right.\right.$$
$$\left.+\mathcal{F}^{\sigma_{2,1}}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\left|f''(b)\right|^{q}\right]^{\frac{1}{q}}$$
$$+\left[\mathcal{F}^{\sigma_{2,1}}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\left|f''(a)\right|^{q}+\left(\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\right.\right.$$
$$\left.\left.\left.-\mathcal{F}^{\sigma_{2,1}}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\right)\left|f''(b)\right|^{q}\right]^{\frac{1}{q}}\right\}.$$

Corollary 15 *If we take $\lambda = \alpha$, $\sigma(0) = 1$, $w = 0$ in Theorem 7, then we have the following inequality for Riemann–Liouville fractional integral operators*

$$\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} f(b) + I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) \right] - f\left(\frac{a+b}{2}\right) \right|$$
$$\leq \frac{(b-a)^2}{2^{3+\frac{s}{q}}(\alpha+1)} \left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times \left\{ \left[B_2(\alpha,s) \left|f''(a)\right|^q + \frac{1}{s+\alpha+2} \left|f''(b)\right|^q \right]^{\frac{1}{q}} \right.$$
$$\left. + \left[\frac{1}{s+\alpha+2} \left|f''(a)\right|^q + B_2(\alpha,s) \left|f''(b)\right|^q \right]^{\frac{1}{q}} \right\}$$

which is the same result in [12].

6 Concluding remarks

In this study, we consider the Hermite–Hadamard and midpoint type inequalities for functions whose first and second derivatives in absolute value are s-convex and related results to establish new type inequalities involving generalised fractional integral operator. The results presented in this study would provide generalizations of those given in earlier works.

References

1. Agarwal, R.P., Luo, M.-J., Raina, R.K.: On Ostrowski Type Inequalities, Fasciculi Mathematici, vol. 204. De Gruyter, Berlin (2016). https://doi.org/10.1515/fascmath-2016-0001
2. Anastassiou, G. A.: General fractional Hermite–Hadamard inequalities using m-convexity and (s, m)-convexity. Front. Time Scales Inequal., 237–255 (2016)
3. Azpeitia, A.G.: Convex functions and the Hadamard inequality. Rev. Colomb. Math. **28**, 7–12 (1994)
4. Breckner, W.W.: Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen. Pupl. Inst. Math. **23**, 13–20 (1978)
5. Chen, H., Katugampola, U.N.: Hermite–Hadamard and Hermite–Hadamard-Fejér type inequalities for generalized fractional integrals. J. Math. Anal. Appl. **446**, 1274–1291 (2017)
6. Dragomir, S.S., Pearce, C.E.M.: Selected Topics on Hermite–Hadamard Inequalities and Applications. Victoria University, RGMIA Monographs (2000)
7. Gorenflo, R., Mainardi, F.: Fractional calculus: integral and differential equations of fractional order, pp. 223–276. Springer, Wien (1997)
8. Hadamard, J.: Etude sur les proprietes des fonctions entieres en particulier d'une fonction consideree par Riemann. J. Math. Pures Appl. **58**, 171–215 (1893)
9. Hudzik, H., Maligranda, L.: Some remarks on s-convex functions. Aequ. Math. **48**, 100–111 (1994)
10. Iqbal, M., Qaisar, S., Muddassar, M.: A short note on integral inequality of type Hermite–Hadamard through convexity. J. Comput. Anal. Appl. **21**(5), 946–953 (2016)
11. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 204. Elsevier Science, Amsterdam (2006)
12. Noor, M.A., Awan, M.U.: Some integral inequalities for two kinds of convexities via fractional integrals. TJMM **5**(2), 129–136 (2013)
13. Pečarić, J.E., Proschan, F., Tong, Y.L.: Convex Functions, Partial Orderings and Statistical Applications. Academic Press, Boston (1992)
14. Raina, R.K.: On generalized Wright's hypergeometric functions and fractional calculus operators. East Asian Math. J. **21**(2), 191–203 (2005)

15. Sarikaya, M.Z., Ogunmez, H.: On new inequalities via Riemann–Liouville fractional integration. Abstr. Appl. Anal. 2012, 10 (2012) (Article ID 428983). https://doi.org/10.1155/2012/428983
16. Sarikaya, M.Z., Yildirim, H.: On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. Miskolc Math. Notes **17**(2), 1049–1059 (2017)
17. Sarikaya, M.Z., Budak, H.: Generalized Hermite–Hadamard type integral inequalities for fractional integrals. Filomat **30**(5), 1315–1326 (2016)
18. Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N.: Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Modell. **57**, 2403–2407 (2013). https://doi.org/10.1016/j.mcm.2011.12.048
19. Tunc, T., Sarikaya, M.Z.: On Hermite–Hadamard type inequalities via fractional integral operators (2016) (submitted)
20. Wanga, J., Lia, X., Zhou, Y.: Hermite–Hadamard inequalities involving Riemann–Liouville fractional integrals via s-convex functions and applications to special means. Filomat **30**(5), 1143–1150 (2016)
21. Yaldiz, H., Sarikaya, M.Z.: On Hermite–Hadamard type inequalities for fractional integral operators (2016) (submitted)
22. Yaldiz, H., Sarikaya, M.Z.: On the Midpoint Type Inequalities Via Generalized Fractional Integral Operators, Xth International Statistics Days Conference. Giresun, Turkey, pp. 181–189 (2016)