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ORIGINAL PAPER

Radial extensions in fractional Sobolev spaces

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Abstract Given $f : \partial(-1,1)^n \to \mathbb{R}$, consider its radial extension $Tf(X) := f(X/\|X\|_\infty)$, $\forall\, X \in [-1,1]^n\setminus\{0\}$. Brezis and Mironescu (RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 95:121–143, 2001), stated the following auxiliary result (Lemma D.1). If $0 < s < 1$, $1 < p < \infty$ and $n \geq 2$ are such that $1 < sp < n$, then $f \mapsto Tf$ is a bounded linear operator from $W^{s,p}(\partial(-1,1)^n)$ into $W^{s,p}((-1,1)^n)$. The proof of this result contained a flaw detected by Shafrir. We present a correct proof. We also establish a variant of this result involving higher order derivatives and more general radial extension operators. More specifically, let B be the unit ball for the standard Euclidean norm $|\ |$ in $\mathbb{R}^n$, and set $U_a f(X) := |X|^a f(X/|X|)$, $\forall\, X \in \overline{B}\setminus\{0\}$, $\forall\, f : \partial B \to \mathbb{R}$. Let $a \in \mathbb{R}$, $s > 0$, $1 \leq p < \infty$ and $n \geq 2$ be such that $(s-a)p < n$. Then $f \mapsto U_a f$ is a bounded linear operator from $W^{s,p}(\partial B)$ into $W^{s,p}(B)$.

Keywords Sobolev spaces · Fractional Sobolev spaces · Radial extensions

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In [1], the first two authors stated the following

Lemma 1 [1, Lemma D.1] *Let* $0 < s < 1$, $1 < p < \infty$ *and* $n \geq 2$ *be such that* $1 < sp < n$. *Let*

$$Q := (-1, 1)^n. \tag{1}$$

Set

$$Tf(X) := f(X/\|X\|_\infty), \quad \forall X \in \overline{Q}\setminus\{0\}, \quad \forall f : \partial Q \to \mathbb{R}; \tag{2}$$

here, $\| \|_\infty$ *is the sup norm in* $\mathbb{R}^n$. *Then* $f \mapsto Tf$ *is a bounded linear operator from* $W^{s,p}(\partial Q)$ *into* $W^{s,p}(Q)$.

The argument presented in [1] does not imply the conclusion of Lemma 1. Indeed, it is established in [1] (see estimate (D.3) there) that

$$|Tf|^p_{W^{s,p}(Q)} \leq C \int_{\partial Q}\int_{\partial Q} \frac{|f(x) - f(y)|^p}{\|x - y\|_\infty^{n+sp}}\, d\sigma(x) d\sigma(y).$$

However, this does not imply the desired conclusion in Lemma 1, for which we need the stronger estimate

$$|Tf|^p_{W^{s,p}(Q)} \leq C \int_{\partial Q}\int_{\partial Q} \frac{|f(x) - f(y)|^p}{\|x - y\|_\infty^{n-1+sp}}\, d\sigma(x) d\sigma(y).$$

In what follows, we establish the following slight generalization of Lemma 1.

Lemma 2 *Let* $0 < s \leq 1$, $1 \leq p < \infty$ *and* $n \geq 2$ *be such that* $sp < n$. *Let* Q, T *be as in* (1), (2). *Then* $f \mapsto Tf$ *is a bounded linear operator from* $W^{s,p}(\partial Q)$ *into* $W^{s,p}(Q)$.

Lemma 2 can be generalized beyond one derivative, but for this purpose it is necessary to work on unit spheres arising from norms smoother that $\| \|_\infty$. We consider for example maps $f : \partial B \to \mathbb{R}$, with

$$B := \text{ the Euclidean unit ball in } \mathbb{R}^n. \tag{3}$$

For $a \in \mathbb{R}$, set

$$U_a f(X) := |X|^a f(X/|X|), \quad \forall X \in \overline{B}\setminus\{0\}, \quad \forall f : \partial B \to \mathbb{R}; \tag{4}$$

here, $| |$ is the standard Euclidean norm in $\mathbb{R}^n$.
We will prove the following

Lemma 3 *Let* $a \in \mathbb{R}$, $s > 0$, $1 \leq p < \infty$ *and* $n \geq 2$ *be such that* $(s - a)p < n$. *Then* $f \mapsto U_a f$ *is a bounded linear operator from* $W^{s,p}(\partial B)$ *into* $W^{s,p}(B)$.

It is possible to establish directly Lemma 2 by adapting some arguments presented in Step 3 in the proof of Lemma 4.1 in [2]. However, we will derive it from Lemma 3.

Proof of Lemma 2 using Lemma 3 Let

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi(X) := \begin{cases} \dfrac{|X|}{\|X\|_\infty} X, & \text{if } X \neq 0 \\ 0, & \text{if } X = 0 \end{cases}, \quad \Lambda := \Phi_{|\overline{B}} \text{ and } \Psi := \Phi_{|\partial B}.$$

Clearly,

$$\Lambda : \overline{B} \to \overline{Q},\ \Psi : \partial B \to \partial Q \text{ are bi-Lipschitz homeomorphisms} \tag{5}$$

and

$$Tf = [U_0(f \circ \Psi)] \circ \Lambda^{-1}. \tag{6}$$

Using (5) and the fact that $0 < s \le 1$, we find that

$$f \mapsto f \circ \Psi \text{ is a bounded linear operator from } W^{s,p}(\partial Q) \text{ into } W^{s,p}(\partial B) \tag{7}$$

and

$$g \mapsto g \circ \Lambda^{-1} \text{ is a bounded linear operator from } W^{s,p}(B) \text{ into } W^{s,p}(Q). \tag{8}$$

We obtain Lemma 2 from (6)–(8) and Lemma 3 (with $a = 0$). The same argument shows that the conclusion of Lemma 2 holds for the unit sphere and ball of any norm in $\mathbb{R}^n$. □

Proof of Lemma 3 Consider a, s, p and n such that

$$a \in \mathbb{R},\ s > 0, \quad 1 \le p < \infty, \quad n \ge 2 \ \text{ and } \ (s-a)p < n. \tag{9}$$

Considering spherical coordinates on B, we obtain that

$$\begin{aligned} \|U_a f\|^p_{L^p(B)} &= \int_0^1 \int_{\partial B} r^{n-1} |U_a f(r\,x)|^p \, d\sigma(x) dr \\ &= \int_0^1 \int_{\partial B} r^{n-1+ap} |f(x)|^p \, d\sigma(x) dr \\ &= \frac{1}{n+ap} \|f\|^p_{L^p(\partial B)}. \end{aligned} \tag{10}$$

Here, we have used the fact that, by (9), we have $n + ap > n - (s-a)p > 0$.
In view of (10), it suffices to establish the estimate

$$|U_a f|^p_{W^{s,p}(B)} \le C \, \|f\|^p_{W^{s,p}(\partial B)}, \quad \forall f \in W^{s,p}(\partial B), \tag{11}$$

for some appropriate $C = C_{a,s,p,n}$ and semi-norm $|\ |_{W^{s,p}}$ on $W^{s,p}(B)$.
Step 1. Proof of (11) *when* $0 < s < 1$. We consider the standard Gagliardo semi-norm on $W^{s,p}(B)$. We have

$$\begin{aligned} |U_a f|^p_{W^{s,p}(B)} &= \int_B \int_B \frac{|U_a f(X) - U_a f(Y)|^p}{|X-Y|^{n+sp}} \, dX dY \\ &= \int_0^1 \int_0^1 \int_{\partial B} \int_{\partial B} r^{n-1} \rho^{n-1} \frac{|U_a f(r\,x) - U_a f(\rho\,y)|^p}{|r\,x - \rho\,y|^{n+sp}} \, d\sigma(x) d\sigma(y) dr d\rho \\ &= \int_0^1 \int_0^1 \int_{\partial B} \int_{\partial B} r^{n-1} \rho^{n-1} \frac{|r^a\, f(x) - \rho^a\, f(y)|^p}{|r\,x - \rho\,y|^{n+sp}} \, d\sigma(x) d\sigma(y) dr d\rho \\ &= 2 \int_{\partial B} \int_{\partial B} \int_0^1 \int_0^r r^{n-1} \rho^{n-1} \frac{|r^a\, f(x) - \rho^a\, f(y)|^p}{|r\,x - \rho\,y|^{n+sp}} \, d\rho dr d\sigma(x) d\sigma(y). \end{aligned}$$

With the change of variable $\rho = t\,r$, $t \in [0, 1]$, we find that

$$\begin{aligned} |U_a f|^p_{W^{s,p}(B)} &= 2 \int_0^1 r^{n-(s-a)p-1} \, dr \int_{\partial B} \int_{\partial B} \int_0^1 t^{n-1} \frac{|f(x) - t^a\, f(y)|^p}{|x - t\,y|^{n+sp}} \, dt d\sigma(x) d\sigma(y) \\ &= \frac{2}{n-(s-a)\,p} \int_{\partial B} \int_{\partial B} \int_0^1 k(x, y, t) \, dt d\sigma(x) d\sigma(y), \end{aligned}$$

with

$$k(x, y, t) := t^{n-1} \frac{|f(x) - t^a f(y)|^p}{|x - t y|^{n+sp}}, \quad \forall x, y \in \partial B, \forall t \in [0, 1].$$

In order to complete this step, it thus suffices to establish the estimates

$$I_1 := \int_{\partial B} \int_{\partial B} \int_0^{1/2} k(x, y, t)\, dt d\sigma(x) d\sigma(y) \le C \|f\|^p_{L^p(\partial B)}, \tag{12}$$

$$I_2 := \int_{\partial B} \int_{\partial B} \int_{1/2}^1 \frac{|f(x) - f(y)|^p}{|x - t y|^{n+sp}}\, dt d\sigma(x) d\sigma(y) \le C |f|^p_{W^{s,p}(\partial B)}, \tag{13}$$

$$I_3 := \int_{\partial B} \int_{\partial B} \int_{1/2}^1 \frac{|(1 - t^a) f(y)|^p}{|x - t y|^{n+sp}}\, dt d\sigma(x) d\sigma(y) \le C \|f\|^p_{L^p(\partial B)}; \tag{14}$$

here, $|\ |_{W^{s,p}(\partial B)}$ is the standard Gagliardo semi-norm on ∂B.

In the above and in what follows, C denotes a generic finite positive constant independent of f, whose value may change with different occurrences.

Using the obvious inequalities

$$|x - t y| \ge 1 - t \ge 1/2, \quad \forall x, y \in \partial B, \forall t \in [0, 1/2],$$
$$|f(x) - t^a f(y)| \le (1 + t^a)(|f(x)| + |f(y)|),$$

and the fact that, by (9), we have $n + ap > 0$, we find that

$$I_1 \le C \int_0^{1/2} (t^{n-1} + t^{n-1+ap})\, dt\, \|f\|^p_{L^p(\partial B)} \le C \|f\|^p_{L^p(\partial B)},$$

so that (12) holds.

In order to obtain (13), it suffices to establish the estimate

$$\int_{1/2}^1 \frac{1}{|x - t y|^{n+sp}}\, dt \le \frac{C}{|x - y|^{n-1+sp}}, \quad \forall x, y \in \partial B. \tag{15}$$

Set $A := \langle x, y\rangle \in [-1, 1]$. If $A \le 0$, then $|x - t y| \ge 1, \forall t \in [1/2, 1]$, and then (15) is clear. Assuming $A \ge 0$ we find, using the change of variable $t = A + (1 - A^2)^{1/2}\, \tau$,

$$\begin{aligned}
\int_{1/2}^1 \frac{1}{|x - t y|^{n+sp}}\, dt &\le \int_{\mathbb{R}} \frac{1}{|x - t y|^{n+sp}}\, dt \\
&= \int_{\mathbb{R}} \frac{1}{(t^2 + 1 - 2A\, t)^{(n+sp)/2}}\, dt \\
&= \frac{1}{(1 - A^2)^{(n-1+sp)/2}} \int_{\mathbb{R}} \frac{1}{(\tau^2 + 1)^{(n+sp)/2}}\, d\tau \\
&= \frac{C}{(1 - A^2)^{(n-1+sp)/2}} \le \frac{C}{(2 - 2A)^{(n-1+sp)/2}} \\
&= \frac{C}{|x - y|^{n-1+sp}},
\end{aligned}$$

and thus (15) holds again. This completes the proof of (13).

In order to prove (14), we note that

$$|1 - t^a|^p \le C(1 - t)^p, \quad \forall t \in [1/2, 1],$$

and that the integral

$$J := \int_{1/2}^{1}\int_{\partial B} \frac{(1-t)^p}{|x-t\,y|^{n+sp}}\,d\sigma(x)dt$$

does not depend on $y \in \partial B$.

By the above, we have

$$\begin{aligned} I_3 &\le C\int_{1/2}^{1}\int_{\partial B}\int_{\partial B} \frac{(1-t)^p|f(y)|^p}{|x-t\,y|^{n+sp}}\,d\sigma(x)d\sigma(y)dt \\ &= C\,J\,\|f\|^p_{L^p(\partial B)}, \end{aligned}$$

and thus (14) amounts to proving that $J < \infty$. Since J does not depend on y, we may assume that $y = (0, \ldots, 0, 1)$. Expressing J in spherical coordinates and using the change of variable $t = 1 - \tau$, $\tau \in [0, 1/2]$, we find that

$$J = C\int_{1/2}^{1}\int_{0}^{\pi} \frac{\tau^p\ \sin^{n-1}\theta}{(\tau^2+4(1-\tau)\sin^2\theta/2)^{(n+sp)/2}}\,d\theta d\tau.$$

When $\tau \in [0, 1/2]$ and $\theta \in [0, \pi]$, we have

$$\begin{aligned} \frac{\tau^p\ \sin^{n-1}\theta}{(\tau^2+4(1-\tau)\sin^2\theta/2)^{(n+sp)/2}} &\le C\frac{\tau^p\ \sin^{n-1}\theta}{(\tau+\sin\theta/2)^{n+sp}} \\ &\le C\frac{\tau^p\ \sin^{n-1}\theta/2\ \cos\theta/2}{(\tau+\sin\theta/2)^{n+sp}} \\ &\le C(\tau+\sin\theta/2)^{p-sp-1}\ \cos\theta/2. \end{aligned}$$

Inserting the last inequality into the formula of J, we find that

$$\begin{aligned} J &\le C\int_{0}^{1/2}\int_{0}^{\pi}(\tau+\sin\theta/2)^{p-sp-1}\ \cos\theta/2\,d\theta d\tau \\ &= C\int_{0}^{1/2}\int_{0}^{1}(\tau+\xi)^{p-sp-1}\,d\xi d\tau < \infty, \end{aligned}$$

the latter inequality following from $p - sp > 0$. This completes the proof of (14) and Step 1.

Step 2. Proof of (11) *when* $s \ge 1$. We will reduce the case $s \ge 1$ to the case $0 \le s < 1$. Using the linearity of $f \mapsto U_a f$ and a partition of unity, we may assume with no loss of generality that supp f is contained in a spherical cap of the form $\{x \in \partial B;\ |x - \mathbf{e}| < \varepsilon\}$ for some $\mathbf{e} \in \partial B$ and sufficiently small ε. We may further assume that $\mathbf{e} = (0, 0, \ldots, 0, 1)$, and thus

$$f \in W^{s,p}(\partial B; \mathbb{R}), \quad \text{supp}\ f \subset \mathcal{E} := \{x \in \partial B;\ |x-(0,0,\ldots,0,1)| < \varepsilon\}. \tag{16}$$

Let

$$\mathcal{S} := \{x \in \partial B;\ |x-(0,0,\ldots,0,1)| \le 2\varepsilon\} \quad \text{and} \quad \mathcal{H} := \mathbb{R}^{n-1}\times\{1\}.$$

Consider the projection Θ with vertex 0 of

$$\mathbb{R}^n_+ := \{X = (X', X_n) \in \mathbb{R}^{n-1}\times\mathbb{R};\ X_n > 0\}$$

onto $\mathcal{H}$, given by the formula $\Theta(X', X_n) = (X'/X_n, 1)$. The restriction Π of Θ to $\mathcal{S}$ maps $\mathcal{S}$ onto $\mathcal{N} := \mathcal{B} \times \{1\}$, with

$$\mathcal{B} := \{X' \in \mathbb{R}^{n-1};\ |X'| \le r := 2\varepsilon\sqrt{1-\varepsilon^2}/(1-2\varepsilon^2)\},$$

and is a smooth diffeomorphism between these two sets. We choose ε such that $r = 1/2$, and thus $\mathcal{B} \subset \{X' \in \mathbb{R}^{n-1};\ \|X'\|_\infty \le 1/2\}$.

Set

$$g(X') := \begin{cases} |(X', 1)|^a f(\Pi^{-1}(X', 1)), & \text{if } X' \in \mathcal{B} \\ 0, & \text{otherwise} \end{cases}. \tag{17}$$

By the above, there exist $C, C' > 0$ such that for every $f \in W^{s,p}(\partial B)$ satisfying (16), the function g defined in (17) satisfies

$$C\|g\|_{W^{s,p}(\mathbb{R}^{n-1})} \le \|f\|_{W^{s,p}(\partial B)} \le C'\|g\|_{W^{s,p}(\mathbb{R}^{n-1})}. \tag{18}$$

On the other hand, set $\mathcal{C} := \{(tY', t);\ Y' \in \mathcal{B},\ t > 0\}$ and

$$V_a g(X', X_n) := \begin{cases} (X_n)^a g(X'/X_n), & \text{if } (X', X_n) \in \mathcal{C} \\ 0, & \text{otherwise} \end{cases}.$$

Then we have $U_a f(X', X_n) = V_a g(X', X_n), \forall (X', X_n) \in \overline{B}\backslash\{0\}$.

Write now $s = m + \sigma$, with $m \in \mathbb{N}$ and $0 \le \sigma < 1$. When $s = m$, we consider, on $W^{s,p}(B)$, the semi-norm

$$|F|^p_{W^{s,p}(B)} = \sum_{\substack{\alpha\in\mathbb{N}^n\backslash\{0\} \\ |\alpha|\le m}} \|\partial^\alpha F\|^p_{L^p(B)}. \tag{19}$$

When s is not an integer, we consider the semi-norm

$$|F|^p_{W^{s,p}(B)} = \sum_{\substack{\alpha\in\mathbb{N}^n\backslash\{0\} \\ |\alpha|\le m}} \|\partial^\alpha F\|^p_{L^p(B)} + \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|= m}} |\partial^\alpha F|^p_{W^{\sigma,p}(B)} \tag{20}$$

(the semi-norm on $W^{\sigma,p}(B)$ is the standard Gagliardo one.)

By the above discussion, in order to obtain (11) it suffices to establish the estimate

$$|V_a g|^p_{W^{s,p}(B)} \le C\, \|g\|^p_{W^{s,p}(\mathbb{R}^{n-1})}, \quad \forall\, g \in W^{s,p}(\mathbb{R}^{n-1}) \text{ with } \operatorname{supp} g \subset \mathcal{B}. \tag{21}$$

Let $\alpha \in \mathbb{N}^n\backslash\{0\}$ be such that $|\alpha| \le m$. By a straightforward induction on $|\alpha|$, the distributional derivative $\partial^\alpha[V_a g]$ satisfies

$$\partial^\alpha[V_a g](X', X_n) = \sum_{|\beta'|\le|\alpha|} V_{a-|\alpha|}[P_{\alpha,\beta'}\, \partial^{\beta'} g](X', X_n) \quad \text{in } \mathcal{D}'(B\backslash\{0\}), \tag{22}$$

for some appropriate polynomials $P_{\alpha,\beta'}(Y')$, $Y' \in \mathbb{R}^{n-1}$, depending only on $a \in \mathbb{R}$, $\alpha \in \mathbb{N}^n$ and $\beta' \in \mathbb{N}^{n-1}$.

Thanks to the fact that $g(X'/X_n) = 0$ when $(X', X_n) \notin \mathcal{C}$, we find that for any such α we have

$$\begin{aligned} \int_B |\partial^\alpha[V_a g]|^p\, dx &\le C \sum_{|\beta'|\le|\alpha|} \int_{\mathcal{C}\cap Q} (X_n)^{(a-|\alpha|)p} |\partial^{\beta'} g(X'/X_n)|^p\, dX' dX_n \\ &= \frac{C}{n + (a-|\alpha|)p} \sum_{|\beta'|\le|\alpha|} \int_{\mathcal{B}} |\partial^{\beta'} g(Y')|^p\, dY'. \end{aligned} \tag{23}$$

Here, we rely on

$$\int_0^1 (X_n)^{n-1+(a-|\alpha|)p}\, dX_n = \frac{1}{n+(a-|\alpha|)p} < \infty,$$

thanks to the assumption (9), which implies that $(|\alpha| - a)p < n$.

Using (23), the fact that $V_a g \in W^{m,p}_{loc}(B\setminus\{0\})$ and the assumption that $n \geq 2$, we find that the equality (22) holds also in $\mathcal{D}'(B)$, that $V_a g \in W^{m,p}(B)$ and that

$$\|V_a g\|^p_{W^{m,p}(B)} \leq C\|g\|^p_{W^{m,p}(\mathbb{R}^{n-1})}, \quad \forall\, g \in W^{m,p}(\mathbb{R}^{n-1}) \text{ with } \operatorname{supp} g \subset \mathcal{B}. \tag{24}$$

In particular, Eq. (21) holds when s is an integer.

Assume next that s is not an integer. In view of (18), (22) and (24), estimate (21) will be a consequence of

$$|V_b[Ph]|^p_{W^{\sigma,p}(B)} \leq C\,\|h\|^p_{W^{\sigma,p}(\mathbb{R}^{n-1})}, \quad \forall\, h \in W^{\sigma,p}(\mathbb{R}^{n-1}) \text{ with } \operatorname{supp}\, h \subset \mathcal{B}, \tag{25}$$

under the assumptions

$$0 < \sigma < 1, \quad 1 \leq p < \infty, \quad n \geq 2, \quad (\sigma - b)p < n \tag{26}$$

and

$$P \in C^\infty(\mathbb{R}^{n-1}). \tag{27}$$

(Estimate (25) is applied with $b := a - m$, $P := P_{\alpha,\beta'}$ and $h := \partial^{\beta'} g$.)

In turn, estimate (25) follows from Step 1. Indeed, consider $k : \partial B \to \mathbb{R}$ such that $\operatorname{supp}\ k \subset \mathcal{B}$ and $U_b k = V_b[Ph]$. (The explicit formula of k can be obtained by "inverting" the formula (17).) By Step 1 and (18), we have

$$\begin{aligned}|V_b[Ph]|^p_{W^{s,p}(B)} &= |U_b k|^p_{W^{s,p}(B)} \leq C\|k\|^p_{W^{s,p}(\partial B)} \leq C\|Ph\|^p_{W^{s,p}(\mathbb{R}^{n-1})}\\ &\leq C\|h\|^p_{W^{s,p}(\mathbb{R}^{n-1})}.\end{aligned}$$

This completes Step 2 and the proof of Lemma 3.

□

Finally, we note that the assumptions of Lemma 3 are optimal in order to obtain that $U_a f \in W^{s,p}(B)$.

Lemma 4 *Let $a \in \mathbb{R}$, $s > 0$, $1 \leq p < \infty$ and $n \geq 2$. Assume that for some measurable function $f : \partial B \to \mathbb{R}$ we have $U_a f \in W^{s,p}(B)$. Then:*

1. *$f \in W^{s,p}(\partial B)$.*
2. *If, in addition, $U_a f$ is not a polynomial, we deduce that $(s-a)p < n$.*

Proof 1. Let $G : (1/2, 1) \times \partial B \to \mathbb{R}$, $G(r, x) := r^{-a}\, U_a f(r\,x)$. If $U_a f \in W^{s,p}(B)$, then $G \in W^{s,p}((1/2, 1) \times \partial B)$. In particular, we have $G(r, \cdot) \in W^{s,p}(\partial B)$ for a.e. r. Noting that $G(r, x) = f(x)$, we find that $f \in W^{s,p}(\partial B)$.

2. Let

$$\Omega_j := \{X \in \mathbb{R}^n;\ 2^{-j-1} < |X| < 2^{-j}\}, \quad j \in \mathbb{N}.$$

We consider on each Ω_j a semi-norm as in (19), (20). Assuming that $U_a f$ is not a polynomial, we have $|U_a f|_{W^{s,p}(\Omega_0)} > 0$. By scaling and the homogeneity of $U_a f$, we have

$$|U_a f|^p_{W^{s,p}(\Omega_j)} = 2^{j[(s-a)p-n]}|U_a f|^p_{W^{s,p}(\Omega_0)}.$$

Assuming that $U_a f \in W^{s,p}(B)$, we find that

$$\infty > |U_a f|^p_{W^{s,p}(B)} \geq \sum_{j\geq 0} |U_a f|^p_{W^{s,p}(\Omega_j)} = \sum_{j\geq 0} 2^{j[(s-a)p-n]} |U_a f|^p_{W^{s,p}(\Omega_0)} > 0,$$

so that $(s-a)p < n$. □

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