RACSAM (2018) 112:893–914
https://doi.org/10.1007/s13398-018-0502-3

SURVEY

# Lens spaces among 3-manifolds and quotient surface singularities

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Received: 27 March 2017 / Accepted: 17 January 2018 / Published online: 19 February 2018


**Abstract** This is a pseudo-historical survey about some aspects of lens spaces and their relations with cyclic quotient singularities. References are ordered by the year of publication. Their list is not exhaustive.

**Keywords** 3-Manifold · Surface singularity · Lens space

## 1 Introduction: definition of lens spaces as a quotient

Let me begin with a remark. Sometimes, it is the case that topologists write Lens spaces as if a mathematician named Lens had existed and had invented them. This is not the case. The name comes from the German "Linsen Räume", introduced by Threlfall–Seifert [8]. A Linse is an optical lens. In German, names begin with a capital letter. Not in English, nor in French where "espace lenticulaire" is used. So I will write lens spaces.

**Definition 1** Let $n \geq 2$ be an integer and let $q$ be a residue mod $n$, prime to $n$. Let $\zeta$ be a $n$-th root of unity. Consider the linear automorphism of $\mathbf{C}^2$ given by $(z_1, z_2) \mapsto (\zeta z_1, \zeta^q z_2)$ for $(z_1, z_2) \in \mathbf{C}^2$. The set of all these linear automorphisms as $\zeta$ varies among $n$-th roots of unity make up the cyclic group $C_{n,q} \subset U(2) \subset GL_2(\mathbf{C})$.

The sphere $S^3 \subset \mathbf{C}^2$ is invariant by the action of $C_{n,q}$ and since $q$ is prime to $n$, the action on $S^3$ is free.

**Definition 2** The lens space $L(n, q)$ is the quotient of $S^3$ by this action.

This definition has the advantage of endowing $L(n, q)$ with a (canonical) orientation as follows. We orient $\mathbf{C}^2$ with the orientation given by its complex structure. This orients the

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unit ball $B^4 \subset \mathbf{C}^2$ and $S^3$ has the boundary orientation. Since the action of $C_{n,q}$ preserves the orientation of $S^3$, the quotient inherits one.

This definition is due to Hopf [6], de Rham [7], William Threlfall and Seifert [8,9], without taking too much care of orientations. But we maintain that lens spaces are oriented as above. Lens spaces were known earlier under different disguises. Here is a list of five instances where lens spaces appear.

(1) As we have seen, as a quotient of $S^3$ by the action of a cyclic group of $GL_2(\mathbf{C})$. In other words as a spherical space form.
(2) As a quotient of a "lens", i.e. of a flattened 3-ball, with identifications on the boundary. As de Rham and Threlfall-Seifert realized, the lens is a fundamental domain for the action given in (1).
(3) As a manifold with a Heegaard decomposition of genus 1. This includes $S^3$ and $S^1 \times S^2$. From the action of a cyclic group as in (1) one gets a decomposition of the quotient in two solid tori by observing that the action preserves the canonical decomposition of $S^3$.
(4) As a finite covering of $S^3$ ramified over the Hopf link.
(5) As a twofold covering of $S^3$ ramified over a 2-bridge knot or link. This is due to Horst Schubert, following a hint by Seifert. I shall not investigate this approach.

A generalization of lens spaces to odd dimensions exists, by copying the above definition as the quotient of the sphere $S^{2m+1}$ by a free action of a cyclic group. See, for instance, several papers by de Rham, typically [14,52], who was very fond of this subject and also Franz [12,17]. Milnor [34] presents an extremely detailed study of lens spaces in higher dimensions in his paper on Whitehead's Torsion §12.

## 2 A short history

### 2.1 Lens spaces as unions of two solid torii according to Poul Heegaard

Heegaard went to Göttingen and Paris between 1893 and 1898. He met Felix Klein and attended lectures by Emile Picard and Camille Jordan, but not by Henri Poincaré. He found himself the subject of his thesis and worked without thesis see ref [2] director. He gives the impression to have been both a mathematically clever (even bold) young man and a kind of ill-mannered person. But let us begin. We are at the end of the nineteenth century. The theory of algebraic functions of one variable is now well established. But what about two variables? A formidable obstacle is dressed on the road. The hypersurface in $\mathbf{C}^3$ defined by the algebraic equation $P(w,z) = 0$ has singularities, where

$$P(w,z) = w^d + a_1(z)w^{d-1} + \cdots + a_{d-1}(z)w + a_d(z)$$

and

$$a_i(z) = a_i(z_1, z_2) \in \mathbf{C}[z_1, z_2]$$

Singularities were already present in the one variable case, but for them the ramification produced by the singularity is easily handled by Riemann cuts.

For algebraic functions of two variables, Heegaard has a fertile idea. To proceed, one should first understand what the "shape" of a singularity is. But, what does shape mean? Well, Heegaard means the topology. But topology does not exist yet, except for a paper [1] by Henri Poincaré entitled "Analysis Situs". Heegaard distinguishes between topology (he means topology in the sense introduced by Johann Benedikt Listing, close to Carl Friedrich

Gauss' Geometria Situs) and Analysis Situs (in the sense of Riemann and Poincaré), and adds stupidly that Poincaré does not understand topology. So, he undertakes to do it his own way. First, let us observe that he has to understand a 4-dimensional object. This seems impossible to do. Remember that we are in 1898 or so and that combinatorial surfaces have only been classified some years ago. Heegaard says that (in the case of interest) a singularity is locally homeomorphic (in modern words) to a cone over a 3-dimensional manifold. That it is a cone is true, and was proved much later. But the basis of the cone is in general only a pseudo-manifold and rarely a genuine manifold. However, this is true for the examples that Heegaard investigates. He maybe imagines that this is always the case. He goes on and he wishes to describe the topology of a 3-manifold. It is to achieve this goal that he invents what is now called the Heegaard decomposition of a 3-manifold. He sketches the proof that every (orientable) 3-manifold has a decomposition into two handlebodies with a common boundary: the Heegaard surface. The genus of the decomposition is the genus of that surface. Lens spaces appear as 3-manifolds with a decomposition of genus 1, so with a decomposition into two solid torii.

As a basic example, he computes the Poincaré homology of the lens space we now denote by $L(2, 1)$ (which is nothing else but the 3-dimensional real projective space). He discovers that $H_1 = \mathbf{Z}/2$ but that $H_2 = 0$. This contradicts what Poincaré said in "Analysis Situs" when he first stated his duality. Heegaard wrote to Poincaré who corrected his definition by writing his first two Compléments. Poincaré said rightly that "his" duality must be expressed differently for the torsion part of homology and he modified the definitions accordingly. As we shall see, the duality for the torsion part played an important role in the history of lens spaces.

An important point in Heegaard's thesis is the observation that lens spaces (in his sense) are coverings (called Riemann spaces) of $S^3$ ramified over the Hopf link. He was motivated by his desire to study the topology of the singularity (often called the ordinary double point, later a key object in the Picard-Lefschetz transformation) in $\mathbf{C}^3$ given by the equation $w^2 - (z_1^2 + z_2^2) = 0$. He found out that the base of the local cone is precisely $L(2, 1)$, since it is the twofold covering of $S^3$ ramified over the Hopf link. Since the homology of $L(2, 1)$ is not isomorphic to the homology of $S^3$ he says that the ordinary double point is topologically singular. So Heegaard was the first to introduce topology in the theory of singularities of algebraic surfaces. He is the precursor of Friedrich Hirzebruch and David Mumford.

**Caution.** What I just wrote represents the "transcendent" approach to algebraic functions. The "algebraic" approach (which goes back to Richard Dedekind and Heinrich Weber (1882) in the case of curves) makes little use (if any) of topology. An important contribution to algebraic functions of two variables in the algebraic spirit was given by Jung [3].

I said above that Heegaard was bold, but he was also shy. He apparently did not meet Poincaré, who would have been happy to talk with him.

Poincaré was not affected by the naughty remarks about him contained in the thesis. On the contrary he was pleased by the decomposition constructed by Heegaard for 3-manifolds. He saw that the decomposition rests on what is today called Morse Theory and he devoted the whole 5th Complément to an exploration of that theory. This is very well described in Cameron Gordon's beautiful paper [61] on 3-manifolds before 1960. Certainly Poincaré saw in Morse Theory a way to construct new 3-manifolds, for instance from a presentation of the fundamental group.

At the end of the 5th Complément Poincaré says that he had thought that a 3-dimensional manifold with trivial homology is necessarily homeomorphic to the 3-sphere. He provides a counterexample, the famous Poincaré sphere (the spherical dodecahedral space). Many people believe that he constructed a quotient of $S^3$ by an action of the binary icosahedral group, but this is not the case. He constructs explicitly a genus two Heegaard diagram, which is the first such diagram ever drawn. From the presentation given by Poincaré for the fundamental group it is clear that he knew the icosahedral group! At the very end he asks the question: “There remains to see if a 3-manifold with trivial fundamental group is homeomorphic to the 3-sphere” (translation with vocabulary adapted to modern times). The last sentence is prophetic: “But this would lead us too far”. I have the feeling that maybe he thought that with a lot of work he could get the answer. Probably by what John Stallings called a way how NOT to prove the conjecture, i.e. by handling Morse functions.

It is striking to note that lens spaces and singularities were present together at the origin of our story.

## 2.2 Three definitions of lens spaces by Heinrich Tietze

It is now time to introduce Wilhelm Wirtinger. He was close to Felix Klein and already in the 1890’s he tried to overcome the difficulties caused by singularities in the study of algebraic functions of two complex variables. See the well-documented paper by Epple [59]. The paper by Epple is extremely useful, since Wirtinger, who had a broad range of mathematical interests, published nothing about the subject we are talking about. But he influenced many mathematicians! Wirtinger read Heegaard’s thesis and very likely he realized that a first step to go on was to rewrite Poincaré’s papers on Analysis Situs, by putting Poincaré’s ideas on a solid ground. He convinced Heinrich Tietze to do the job. This was done with great care by Tietze in his Habilitationschrift [4]. At the end of the paper, Tietze presents “ein Beispiel” (an example in §20, 21, 22). The example consists in lens spaces (of course Wirtinger had told him to read Heegaard). He gives for them three definitions.

The first one is by a “lens” (a flattened 3-ball) with identifications on the boundary. The construction is presented by words, with no picture (the first one was given by Seifert-Threlfall). The name ”lens“ is not used. The second one is by a Heegaard decomposition of genus 1. The third one is by cyclic coverings of $S^3$ ramified over the Hopf link. Tietze proves that his three definitions produce the same objects.

The §22 is very interesting. The title of the section is “On manifolds for which the invariants defined so far coincide”. The invariants are homology and the fundamental group. The same investigation is also presented by Poincaré in §14 of ”Analysis Situs“. Tietze adds that maybe the lens spaces $L(5, 1)$ and $L(5, 2)$ are examples of 3-manifolds with the same fundamental group which are not homeomorphic. This is true as Alexander proved in 1919. See below, Sect. 2.3.

It is striking to note that a century later, after the efforts of many topologists, it is known that two irreducible 3-manifolds with the same fundamental group are homeomorphic, the only exception being given by lens spaces. See below, Sect. 3.4.

## 2.3 James Alexander and the linking form

We stand back and consider the following situation. $M^{2q+1}$ is a closed, connected and oriented manifold of dimension $(2q + 1)$. We denote by $T_q(M)$ or simply by $T_q$ the torsion subgroup of $H_q(M^{2q+1}; \mathbf{Z})$.

**Proposition 1** *There exists a form* $L_q : T_q \times T_q \longrightarrow \mathbf{Q}/\mathbf{Z}$ *which is* $(-1)^{q+1}$*-symmetric and non-degenerate.*

This form is often called the linking form on the $q$-th dimensional torsion. A systematic presentation was given by Seifert-Threlfall in their book §77, under the name “Verschlingungszahlen" (= linking numbers). Its existence is the expression of Poincaré duality for the torsion in dimension $q$. Modern proofs use cohomology, the Universal Coefficient Theorem and homology with values in $\mathbf{Q}/\mathbf{Z}$. In terms only of homology and intersection it goes roughly as follows.

Let $x \in T_q$ and let $d$ be its order. Let $c_{q+1}$ be an integral chain such that $\partial c_{q+1} = dx$. If one replaces the integral chain $c_{q+1}$ by another such chain, then the image of the rational number $(c_{q+1} \cdot y)/d$ in $\mathbf{Q}/\mathbf{Z}$ does not change, where the dot denotes the intersection number with value in $\mathbf{Z}$. This is by definition the value of $L_q$ on $(x, y)$.

Certainly the linking form $L_1$ is an appropriate object to consider for lens spaces. This is what Alexander [5] did in the case of lens spaces $L(n, q)$ with $n = 5$, but his approach is valid for all lens spaces. The presentation of Alexander does not use formally the form just defined, but all the necessary ingredients are there. See Cameron Gordon's paper [61] p. 464–465. The formal definition was given by de Rham and Seifert-Threlfall in the beginning of the thirties. Alexander's result is the following.

**Proposition 2** *There is a generator $g$ in $H_1(L(n, q); \mathbf{Z})$ such that $L_1(g, g) = q/n$.*

A geometric generator is the core of one of the solid torii which decompose the lens space. If the other core is chosen $q$ is replaced by $q'$ where $qq' \equiv 1 \bmod n$. A wrong belief would be that the linking form $L_1$ determines $q$. This is not true, but another algebraic generator is equal to $kg$ with $k$ prime to $n$ mod $n$. Now $L_1(kg, kg) = k^2q/n$. Hence we get the corollary:

**Corollary 1** *Let $f : L(n, q) \longrightarrow L(n, q^*)$ be a map which induces an isometry on the linking forms. Then we have $q^* \equiv k^2 q \bmod n$, for some integer $k$ prime to $n$.*

Let us remark that a homeomorphism, or a homotopy equivalence, preserving orientations induces an isometry on linking forms.

With such a tool we can easily prove that $L(5, 1)$ and $L(5, 2)$ are not homeomorphic (even if we accept to reverse orientations), as Alexander did. In fact the linking forms are not isometric since 1 is a square while 2 is not a square mod 5. If we change the orientation of $L(5, 2)$ we get $L(5, 3)$ and 3 also is not a square.

### 2.4 The homotopy type of lens spaces

**Lemma 1** *Let $M^k$ and $N^k$ be two closed, connected, oriented k-manifolds. Let $f : M^k \longrightarrow N^k$ be a degree 1 map. Then $f$ induces a surjective homomorphism on the fundamental groups.*

The proof is easy, by contradiction. If this were not the case, $f$ would factorize through a covering of $N^k$ and hence it would not be of degree 1.

**Proposition 3** *A map $f : L(n, q) \longrightarrow L(n, q^*)$ is a homotopy equivalence if and only if it is of degree 1.*

*Proof* A homotopy equivalence is certainly of degree 1. So let us prove the converse. Since $f$ is of degree 1, the lemma above implies that it induces an isomorphism between the fundamental groups. The map $\widehat{f} : S^3 \longrightarrow S^3$ between the universal coverings is also of degree 1, as an obvious commutative diagram shows. Therefore, the pair of maps $(f, \widehat{f})$ satisfies the conditions of Whitehead's Theorem about homotopy equivalences. □

Since I try to follow the history, the proof of the proposition is somewhat anachronistic, since Whitehead's theorem was only available in 1948. See Thm 3 p. 1135 of Whitehead's paper [18].

I make a little parenthesis to recall what is Whitehead's theorem. I wish to call it the Full Whitehead Theorem.

**Theorem 1** (Full whitehead theorem) *Let X and Y be two connected CW complexes. A map $\varphi : X \longrightarrow Y$ is a homotopy equivalence if and only if the following two conditions are satisfied.*

1. *The map $\varphi$ induces an isomorphism on fundamental groups.*
2. *The map between universal coverings $\widehat{\varphi} : \widehat{X} \longrightarrow \widehat{Y}$ which covers $\varphi$ induces an isomorphism on the homology groups with integer coefficients.*

The chronology looks a little strange. The homeomorphism problem for lens spaces was solved (for PL homeomorphisms) in 1935 by Kurt Reidemeister, as we shall see in the next subsection. The classification up to homotopy type was harder to solve. An important step was taken by Rueff in his [15] paper. Here it is.

**Theorem 2** *Let $L(n, q)$ and $L(n, q^*)$ be two lens spaces. There exists a degree 1 map $L(n, q) \longrightarrow L(n, q^*)$ if and only if there exists an integer k prime to n such that $k^2q \equiv q^* \bmod n$. In this case there also exists a map $L(n, q^*) \longrightarrow L(n, q)$ of degree 1.*

We see that Rueff essentially proved the following theorem, except for the vocabulary (the notion of homotopy type was missing at the time). **Caution**: Maybe I am going too fast, but in any case Rueff was the precursor.

**Theorem 3** *Let $L(n, q)$ and $L(n, q^*)$ be two lens spaces. They have the same homotopy type (given by a homotopy equivalence which preserves the orientations) if and only if there exists an integer k such that $k^2q \equiv q^* \bmod n$. This is the case if and only if the two lens spaces have isometric linking forms.*

With these words, the theorem was stated and proved by Whitehead in [16] and by Franz in [17].

### 2.5 The classification of lens spaces up to homeomorphism

The main result is due to Kurt Reidemeister. It reads as follows [13].

**Theorem 4** *The lens spaces $L(n, q)$ and $L(n, q')$ are homeomorphic by a piecewise linear (in short PL) homeomorphism, preserving the orientations if and only if $q = q'$ or $qq' \equiv 1 \bmod n$.*

**On the proof.** It was known very early that the condition $qq' \equiv 1 \bmod n$ implies homeomorphism, by considering the linear map $\mathbf{C}^2 \longrightarrow \mathbf{C}^2$ which exchanges the factors. The obvious question was: is this condition necessary? The answer yes came from Kurt Reidemeister in 1935. But there was something a little bit unpleasant in the proof. Reidemeister's result states that the condition is necessary to have a PL homeomorphism. So a new question arose: Can we get rid of PL? Without entering into details, let us say that Reidemeister handles chain complexes which are modules over the group ring $\mathbf{C}C_n$ (where $C_n$ denotes a cyclic group of order $n$). He needs to have bases for the chain modules in order to take determinants. The bases are provided by the simplices of a triangulation. This leads to the Reidemeister Torsion for the lens space $L(n, q)$. It is an invariant $\Delta(T(n, q)) \in \mathbf{C}C_n$ which determines the lens space up to PL homeomorphism. For more details see Milnor's paper [34] on Whitehead Torsion §12.

The arguments to eliminate the PL condition came from different horizons. Historically Edwin Moise [19] came first in 1952. He proved that for 3-manifolds, the PL and TOP categories are equivalent. Roughly, topological 3-manifolds can be triangulated and if there is a homeomorphism between two PL 3-manifolds there is also a PL one. In 1959 R.H. Bing simplified Moise's proof.

In 1969, another proof was provided by Kirby and Siebenmann [42] (see p. 744), who proved that Reidemeister Torsion is a topological invariant. The idea of their proof is explained by Burlet and Milnor [43].

Still another proof can be obtained by the theory of Hilbert cube manifolds of James West and Thomas Chapman around 1972. See also Siebenmann's Bourbaki Seminar [47].

Another approach without any torsion nor simple homotopy type was provided by Bonahon in [55]. From a geometrical point of view, I think that it concludes beautifully the discussion.

**Theorem 5** (Francis Bonahon) *A Heegaard torus in a lens space $L(n, q)$ is unique up to an orientation preserving homeomorphism.*

By definition, a **Heegaard torus** is an embedded torus which bounds a solid torus on each side. From the theorem it is easy to get the classification result for lens spaces up to homeomorphism. See p. 336 in Bonahon [55].

Implicitly, in the approach "à la Bonahon" it is good to know that the there categories TOP, PL, DIFF are equivalent as far as 3-manifolds are involved. For the equivalence PL $\longleftrightarrow$ DIFF the arguments rest on Whitehead [32] who proved that differential manifolds can be triangulated and on [29] who proved that PL $n$-manifolds in dimension $n \leq 3$ can be smoothed (in fact up to dimension 7 for existence and 6 for uniqueness).

On the other hand, the equivalence TOP $\longleftrightarrow$ PL in dimension three is originally due to Moise as we have seen.

## 2.6 Orientation reversal

**Proposition 4** *Let $L(n, q)$ be a lens space oriented (as said before) by the canonical orientation. There is an orientation-preserving homeomorphism from $L(n, q)$ to $L(n, n - q)$ with $L(n, n - q)$ equipped with the orientation opposite to the canonical one. [In short: $L(n, n - q) = -L(n, q)$]*

*Proof* Let $\mathbf{C}_1^2$ and $\mathbf{C}_2^2$ be two copies of $\mathbf{C}^2$. Let $\phi$ be the orientation reversing $\mathbf{R}$-linear map $\mathbf{C}_1^2 \longrightarrow \mathbf{C}_2^2$ given by $\phi(z_1, z_2) = (z_1, \bar{z}_2)$. Let $C_{n,q}$ act on $\mathbf{C}_1^2$ and $C_{n,n-q}$ act on $\mathbf{C}_2^2$. The map $\phi$ is equivariant with respect to these actions since $\bar{\zeta^q} = \zeta^{n-q}$. Therefore, $\phi$ induces an orientation-reversing homeomorphism from $L(n, q)$ to $L(n, n - q)$. □

**Definition 3** Let us define a lens space $L(n, q)$ to be **achiral** if there exists an orientation-reversing homeomorphism $L(n, q) \longrightarrow L(n, q)$. Otherwise a lens space is said to be **chiral**.

The fact is that most lens spaces are chiral. Historically it seems that it was a novelty to meet chiral manifolds, maybe because surfaces are all achiral. But some lens spaces are achiral.

**Proposition 5** *The lens space $L(n, q)$ is achiral if and only if $q^2 \equiv -1 \bmod n$.*

*Proof* From Reidemeister's theorem and the proposition above we deduce that $L(n, q)$ is achiral if and only if $-q \equiv q^{-1} \bmod n$, i.e. if and only if $q^2 \equiv -1 \bmod n$.

So, let us ask for which $n$ there exists a $q$ such that $q^2 \equiv -1 \bmod n$. If this is the case, one says that $-1$ is a **quadratic residue mod** $n$. This is a well-known subject in elementary number theory and the result is the following. See for instance Jean-Pierre Serre's "Cours d'Arithmétique". □

**Proposition 6** *The integer $-1$ is a quadratic residue mod n if and only if:*

*(1) either $n = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$ with each $p_i$ an odd prime congruent to 1 mod 4;*
*(2) or n is twice such an integer.*

It is hence easy to find the achiral lens spaces. For small values of $n$ they are: $L(2,1), L(5,2), L(10,3), L(13,5), L(17,4), \cdots$

## 3 Lens spaces among 3-manifolds

We shall see in this section that lens spaces are exceptional among 3-manifolds in many ways.

### 3.1 A reminder on 3-manifolds

The reader will find in Hempel's book [48] a basic handbook on 3-manifolds. Hatcher's notes [63] adopt a different point of view, full of new ideas. In this reminder a 3-manifold $M^3$ is compact, connected and without boundary. Embeddings are always supposed to be PL or DIFF, to avoid wildness.

We first say a few words about irreducibility.

**Definition 4** A 3-manifold $M^3$ is **irreducible** if every 2-sphere embedded in $M^3$ bounds a 3-ball in $M^3$.

A classical result is that an irreducible 3-manifold is "prime", i.e. admits only trivial decompositions as a connected sum (we should say indecomposable) or is homeomorphic to $S^1 \times S^2$ or to the non-orientable 2-sphere bundle over $S^1$.

If we consider non-orientable 3-manifolds another notion of irreducibility is necessary, as was shown by Epstein [30].

**Definition 5** A non orientable 3-manifold $M^3$ is $P^2$**-irreducible** if it is irreducible and does not contain a projective plane $P^2$ with trivial normal bundle (one often says "does not contain a 2-sided projective plane").

The sphere + projective plane Theorem says the following.

**Theorem 6** (Papakyriakopoulos [26] + Epstein [30]) *Suppose that $M^3$ is irreducible and orientable or $P^2$-irreducible and non-orientable. Then the second homotopy group $\pi_2(M^3)$ vanishes.*

We now proceed towards sufficiently large 3-manifolds.

**Definition 6** Let $F^2 \subset M^3$ be an embedded surface. A **compression disc** for $F$ in $M$ is an embedded 2-disc $\Delta \subset M$ such that:

(i) $\Delta \cap F = \partial\Delta$

(ii) $\partial\Delta$ is essential in $F$, i.e. it does not bound a 2-disc in $F$.

**Definition 7** A surface $F^2 \subset M^3$ is **incompressible** if there exists no compression disc for $F$ in $M$.

**Definition 8** A 3-manifold $M^3$ is **sufficiently large** if it contains an incompressible surface, distinct from a 2-sphere.

Irreducible and sufficiently large orientable 3-manifolds have nice properties as shown by Waldhausen [40]. For instance a homeomorphism $M^3 \longrightarrow M^3$ homotopic to the identity is in fact isotopic to the identity.

### 3.2 Seifert foliated 3-manifolds

**Definition 9** A compact, connected and oriented 3-manifold $M^3$ is a **Seifert manifold** if it has a foliation in circles with coherently orientable leaves.

The foliation is called a **Seifert foliation**. The leaves are coherently orientable, but an orientation is not specified. If this is the case, the leaves are the orbits of an effective, fixed-point free action of $SO(2)$ and the leaves are oriented by the orientation of $SO(2)$. An orbit is **generic** if its isotropy subgroup is trivial. If it is not trivial, it is a finite, cyclic subgroup of $SO(2)$ of order $\alpha \geq 2$. The corresponding orbit is called **exceptional**. There are a finite number of them.

There are more general kinds of Seifert foliations, but in this paper I only consider foliations which satisfy the orientability condition stated above. Moreover, I assume that the boundary $\partial M$ is empty.

We denote by $B$ the space of leaves of the Seifert foliation. It is a compact, connected, differentiable surface with empty boundary. It is orientable, since the foliation is orientable. Since $M$ is oriented, an orientation of $B$ determines an orientation of the leaves and conversely.

**Fact.** Among closed Seifert foliated manifolds, lens spaces together with $S^3$ and $S^1 \times S^2$ are the only ones to have more than one (in fact infinitely many) Seifert foliations. They are the 3-manifolds which admit a Heegaard splitting of genus 1. All the other orientable Seifert foliated manifolds have a unique Seifert foliation with orientable base space. For an idea of the proof see Jaco's book [51] p. 96–97 and observe that besides his case (a) the other cases can be discarded either since a second Seifert foliation is non-orientable or since the manifold has a boundary.

It would take too much space to define the Seifert invariants: the obstruction $e$ and the $\beta$'s. Here I use the normalized Seifert invariants, introduced originally by Seifert. See his 1932 paper [10].

### 3.3 Seifert manifolds with finite fundamental group

This is an extremely interesting subject. So I spend some time on it.

Let us begin with **Clifford-Klein space forms**, space forms for short. The early days of the subject are presented in detail by Epple [64].

Epple's paper ends with Heinz Hopf's 1925 paper. In this paper, Hopf makes clear that a space form is a $n$-dimensional Riemannian complete manifold of constant curvature $K$, with $K = +1, 0, -1$. Then Hopf says that for every dimension $n$ and curvature $K$ there exists a model space $M^n_K$ which is simply connected and unique up to isometry. Then every (Riemannian and complete) $n$-manifold of constant curvature is isometric to a quotient $M^n_K/\Gamma$ where $\Gamma$ is a group of isometries of the model space, acting freely and properly discontinuously.

For the necessary prerequisite in differential geometry see Postnikov's book [60].

In the paper quoted above, Hopf undertakes the determination of the finite subgroups of $SO(4)$ which act freely on the 3-sphere $S^3$.

The exhaustive list of such groups was obtained by Threlfall–Seifert in [8,9]. In their work, they represented a fundamental domain of the action of the cyclic groups by a flattened 3-ball and they called this ball a "lens". The first published picture of such a flattened ball with identifications on the boundary made explicit is in their book [11]. In this book, lens spaces appear several times to provide examples.

The title of the papers makes clear that the goal of the authors is to describe the topology of the quotient, i.e. the topology of spherical space forms in dimension 3.

The determination of the subgroups follows the path of Hopf (although the authors do not quote Hopf, except in the very last pages of the second paper). It is presented beautifully by Scott in his [57] paper. The determination of the quotient (this is the so-called "Discontinuitätsbereich") caused some difficulties. In fact the authors say that it is precisely in order to obtain a complete description of the quotient that they discovered and studied Seifert manifolds. The second part of the paper is devoted to the description of the quotient via Seifert foliations. The main result of Part II is the following theorem. See p. 568.

**Theorem 7** *Spherical space forms in dimension 3 coincide with Seifert 3-manifolds with finite fundamental group (of course up to isomorphism in the adequate category).*

Let us first recall that a 3-dimensional manifold with finite fundamental group is necessarily orientable. This observation was made by Hopf in his 1925 paper. We argue by contradiction and suppose that the manifold is non-orientable. The universal cover is a homotopy 3-sphere (there is no need to use Perelman here). The fundamental group acts freely by Galois transformations on the homotopy 3-sphere. But the action cannot be free, since an orientation-reversing homeomorphism of a 3-dimensional homotopy sphere has fixed points, by the Hopf-Lefschetz Fixed Point Formula.

The admissible Seifert invariants are as follows. See Threlfall–Seifert [9] Part II Section 8. See also Seifert's paper on "his" manifolds [10] Section 10.

(1) Base space (space of leaves) $S^2$ and no more than 2 exceptional points. These are the lens spaces including $S^3 = L(1.0)$.
(2) Base space $S^2$ and 3 exceptional points with $(\alpha_1, \alpha_2, \alpha_3)$ equal to $(2, 2, c)$ and $c \geq 2$. These are the prism manifolds. They are known (should we say notorious?) for having another Seifert foliation with a non-orientable base.
(3) Base space $S^2$ and 3 exceptional points with $(\alpha_1, \alpha_2, \alpha_3)$ equal to $(2, 3, 3)$.
(4) Base space $S^2$ and 3 exceptional points with $(\alpha_1, \alpha_2, \alpha_3)$ equal to $(2, 3, 4)$.
(5) Base space $S^2$ and 3 exceptional points with $(\alpha_1, \alpha_2, \alpha_3)$ equal to $(2, 3, 5)$.

These triples of integers $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_i \geq 2$ are the only triples to satisfy $\Sigma_i 1/\alpha_i > 1$. Finiteness of the fundamental group depends only on the value of the $\alpha_i$'s.

Seifert manifolds which correspond to the cases (2)–(5) have a unique Seifert foliation with orientable base. Now we have to be careful about orientations. The unit sphere $S^3$ in $\mathbf{R}^4$ has no canonical orientation. Hence the quotient does not have a canonical orientation. But the Seifert invariants $\beta_i$ and the Euler number, say $e = -b$ where $b$ is Seifert's obstruction do require an orientation of the 3-manifold. Therefore we have two possible sets of Seifert's invariants for each finite (non-cyclic) group. A change of orientation changes the sign of $e$ and of the $\beta_i$. At this point it is good to introduce the **rational Euler number** $e_0 = e - \Sigma\beta_i/\alpha_i$. We see that a change of orientation modifies $e_0$ to $-e_0$.

Another consequence of Threlfall–Seifert's work is the following rigidity result. See Part II bottom p. 565. This rigidity theorem also follows from the arguments given by Scott [57].

**Theorem 8** *Let $G = SO(4)$ and $\mathcal{F}$ be the family of finite groups in $SO(4)$ acting freely on $S^3$, excluding cyclic groups. Then the family $\mathcal{F}$ is rigid.*

Since rigidity will appear again later, I recall what it means, with a definition of my own.

**Definition 10** Let $G$ be a "large" group (in the two cases we shall meet, $G$ is a Lie group). Let $\mathcal{F}$ be a family of subgroups of $G$. We say that $\mathcal{F}$ is **rigid** if any time two elements $G_1$ and $G_2$ in $\mathcal{F}$ are abstractly isomorphic (i.e. isomorphic as abstract groups) then they are conjugate in $G$.

**Comment.** The cyclic subgroups cannot constitute a rigid family. The fundamental group $C_{n,q}$ of $L(n, q)$ is determined abstractly by $n$, but two lens spaces $L(n, q)$ and $L(n', q')$ are isometric if and only if $n = n'$ and $q = q'$ or $q^{-1} = q'$.

Besides Scott's paper [57], another recommended reading is Orlik's book [46] on Seifert manifolds Chap.6 §1 and §2. Orlik determines completely the subgroups of $SO(4)$ which act freely on $S^3$ and recovers the presentation of these groups given by Milnor in his paper [25] on groups which act freely on spheres. Orlik proves also that the quotients are Seifert fibered. Then he can make the correlation between on the one hand the group (via its presentation) and the Seifert invariants on the other hand.

**The fantastic news brought by Gregory Perelman is that there is no other 3-manifold with finite fundamental group.**

For instance a 3-manifold with finite cyclic group is a lens space. This is valid in particular for the trivial group. Therefore, by Perelman, a homotopy sphere is a lens space. The classification of lens spaces shows that a lens space with trivial fundamental group is homeomorphic to the standard sphere. So Perelman obtains a proof of the Poincaré Conjecture as a special case !!!!!

### 3.4 The fundamental group determines 3-manifolds, except for lens spaces

The story begins with Poincaré himself in his series of papers "Analysis Situs" [1] and the five "Compléments" (the last one [1]). The story ends after more than a century with the three papers by Perelman [65] and full proofs provided a bit later by several authors. See for instance Morgan [67].

Let us begin with Poincaré. In "Analysis Situs" §14 he investigates whether the invariants he has just introduced (the fundamental group and homology) classify manifolds. Note that Poincaré is only interested by manifolds, which are for him both triangulated and differentiable. He shows that homology classifies surfaces. Then he studies 3-dimensional manifolds. He provides several examples for them. For a lucid and modern presentation see Cameron Gordon's paper [61]. He shows that there are 3-dimensional manifolds with the same homology but with different fundamental group. And then, at the end of §14, he explicitly asks the following question, which we quote in French: "Deux variétés d'un même nombre de dimension qui ont même groupe $G$ sont-elles toujours homéomorphes ?". Of course we should specify "dimension 3" in the beginning of the question, and that $G$ is the fundamental group. The question is tackled again in the famous Fifth Complement. There he says that he had thought that a 3-dimensional manifold with trivial homology is necessarily homeomorphic to the 3-sphere. He provides a counterexample, the famous Poincaré sphere (i.e. the spherical dodecahedral space). See also the web-site http://analysis-situs.math.cnrs.fr

We have seen that the lens spaces $L(5,1)$ and $L(5,2)$ are historically the first example of a pair of 3-manifolds with the same fundamental group and which are not homeomorphic (without taking orientations into account). It is easy to find many such pairs among lens spaces. But the efforts to construct examples not involving lens spaces were fruitless. Hence the following conjecture appeared:

**Fundamental Group Conjecture:** Two irreducible 3-manifolds with isomorphic fundamental group are homeomorphic except if they are lens spaces.

Since the work of Perelman, it is known that the Fundamental Group Conjecture is true. It is the successful conclusion of the work of many topologists. Here is a precise statement.

**Theorem 9** *Let $M^3$ and $N^3$ be two closed, connected 3-manifolds. If they are orientable assume that they are irreducible and not lens spaces. If they are non orientable assume that they are $P^2$-irreducible.*

*Let $\varphi : \pi_1 M \longrightarrow \pi_1 N$ be an isomorphism. Then there exists a homeomorphism $f : M \longrightarrow N$ which induces the isomorphism $\varphi$.*

Here is a **sketch of the proof**.

We consider first the orientable case. The first important step towards a proof was provided by Waldhausen [40] in his paper on sufficiently large 3-manifolds. The Corollary 6.5 says the following.

**Theorem 10** *Let $M$ and $N$ be two orientable and irreducible 3-manifolds. Suppose that $M$ is sufficiently large.*

*Let $\varphi : \pi_1 M \longrightarrow \pi_1 N$ be an isomorphism. Then there exists a homeomorphism $f : M \longrightarrow N$ which induces the isomorphism $\varphi$.*

For the proof of the Fundamental Group Conjecture, we are left now with two 3-manifolds none of them sufficiently large. For a long time (about 35 years!) the possibility existed for the existence of some unknown continent (similar to the long sought Austral Continent in the eighteenth century) inhabited by strange non sufficiently large 3-manifolds. Clearly William Thurston's Geometrization Conjecture implies that such a continent does not exist [54]. This is what Gregory Perelman's results are about. One rough way to state some consequences of his work which concern us here is the following.

**Theorem 11** *A non sufficiently large 3-manifold, irreducible and orientable, is either Seifert or hyperbolic.*

The "or" is exclusive since the center of the fundamental group of a Seifert manifold is non-trivial while it is trivial for a hyperbolic manifold.

The end of the proof of the Fundamental Group Conjecture splits then in two parts for orientable 3-manifolds. From now on, it does not matter in the arguments whether 3-manifolds are sufficiently large or not.

(1) Suppose that both 3-manifolds are hyperbolic. Then we are finished thanks to the Mostow Rigidity Theorem, which says:

**Theorem 12** *Let $G$ be the Lie group of isometries of a hyperbolic 3-space. Let $\mathcal{F}$ be the family of subgroups of $G$ which are isomorphic to the fundamental group of a hyperbolic closed 3-manifold. Then $\mathcal{F}$ is rigid.*

As a consequence hyperbolic 3-manifolds with isomorphic fundamental group are not only homeomorphic, they are isometric.

(2) Suppose that both 3-manifolds are Seifert. Then the proof is completed by the following theorem, which puts an end to years of efforts towards uniqueness results for Seifert manifolds. See the notes by Jankins and Neumann [56]. The whole last chapter is devoted to the proof of the following theorem. See also Orlik's book [46] p. 90.

**Theorem 13** *Let $M$ and $N$ be two closed oriented Seifert manifolds. Exclude lens spaces. Suppose that their fundamental groups are isomorphic. Then $M$ and $N$ are homeomorphic.*

There remains to consider the case of non-orientable 3-manifolds. Remarkably, this case was entirely solved by Heil in [41]. Here is the path followed by Heil.

**Proposition 7** *Let $M$ be a closed, connected, non-orientable 3-manifold. Then $M$ is sufficiently large.*

There are two steps in the proof.

**1st step.** $H^1(M; \mathbf{Z})$ is infinite. It is equivalent to say that the first Betti number $\beta_1$ is positive. Let us prove this assertion.

Poincaré duality with $\mathbf{Z}/2$ coefficients implies that the Euler characteristic $\chi(M)$ vanishes, since Poincaré duality is not affected by non-orientability if $\mathbf{Z}/2$ coefficients are used. If we consider Betti numbers over $\mathbf{Z}$ we have $0 = \chi(M) = \beta_0 - \beta_1 + \beta_2 - \beta_3$. But $\beta_3 = 0$ since $M$ is non-orientable. Hence $\beta_1 > 0$.

**2nd step.** Let $g \in H^1(M; \mathbf{Z})$ be an element of infinite order and indivisible. Since the circle $S^1$ is a $K\pi_1$ for the group $\mathbf{Z}$, the element $g$ can be represented by a map $f : M \longrightarrow S^1$. An argument which goes back to John Stallings (in his paper (1961) about 3-manifolds which fiber over the circle) shows that $f$ can be made transversal to the point $1 \in S^1$ in such a way that the surface $f^{-1}(1) \subset M$ is incompressible (after adequate surgeries). The important point here is that the surface obtained by transversality is 2-sided.

This being obtained, Heil shows how to adapt Waldhausen's proof of his Corollary 6.5 (i.e. Theorem 10 above) when $M$ is non-orientable. The condition of $P^2$-irreducibility is used here, as irreducibility was used by Waldhausen in the orientable case.

### 3.5 Several ways to classify 3-manifolds

*In this subsection 3-manifolds are assumed to be irreducible if orientable and $P^2$-irreducible if non-orientable.*

Let $M$ and $N$ be two 3-manifolds. Consider the following assertions.

(1) $M$ and $N$ have isomorphic fundamental groups.
(2) $M$ and $N$ have the same homotopy type.
(3) $M$ and $N$ are homeomorphic.

From what we said just above, the three assertions are equivalent for 3-manifolds except for lens spaces. Lens spaces are exceptional since $1^o$ does not imply $2^o$, the spaces $L(5, 1)$ and $L(5, 2)$ being the easiest counter-examples. $2^o$ does not imply $3^o$, the spaces $L(7, 1)$ and $L(7, 2)$ being the easiest counter-examples.

However there is the remarkable result.

**Theorem 14** *h-cobordant 3-manifolds are always homeomorphic.*

This statement is also true for lens spaces. The proof in this case is due to Atiyah and Bott [38]. See p. 479. In all other cases the theorem results from what we just said, since h-cobordant manifolds have the same homotopy type. Maybe it is good to recall that TOP = PL = DIFF for 3-manifolds.

# 4 Lens spaces as the boundary of cyclic quotient singularities

The novice reader will find in Henri Cartan's talk at the [27] IMU Congress a wonderful presentation of analytic spaces [21].

## 4.1 Normal complex surface singularities

In this section we consider (always complex) surface singularities, i.e. germs $(\Sigma, P)$ of complex 2-dimensional analytic spaces $\Sigma$ at a point $P \in \Sigma$. We assume the germs to be normal. A basic reference for surface singularities is Michael Artin's paper ref [35].

Here is a short summary about normal analytic surfaces.

$\Sigma$ can be smooth (also called regular) at $P$ or singular. The normality condition implies that $P$ is an irreducible isolated singularity. More precisely, there exists an open neighborhood $U$ of $P$ such that $U \backslash P$ is connected and consists of smooth points.

Conversely, if $P$ is an isolated singularity, $\Sigma$ is not necessarily normal, even if $\Sigma$ is irreducible. Here, irreducible in the sense used in analytic geometry, is equivalent to the fact that $U \backslash P$ is connected. In the case that the surface singularity is isolated and irreducible, the normalisation is a homeomorphism. In other words the topology is adequate, but the local algebra is not. The normalisation adds new functions to the original ones.

The main property one uses (often implicitely) of the normality condition is the following.

**Extension property:** Suppose that we have a continuous function $f : \Sigma \longrightarrow \mathbf{C}$ such that the restriction to the smooth part is analytic. Then $f$ is also analytic at $P$.

From a topological point of view, $\Sigma$ is locally a cone at $P$. More precisely there is a neighborhood of $P$ which is **homeomorphic** to a cone of vertex $P$, with basis a closed, connected and oriented 3-manifold $M$. Moreover, the oriented homeomorphism type of $M$ is well defined. I call $M$ **the boundary** of the singulartity. There are several ways to obtain this result. One is to adapt Alan Durfee's results (1983) about algebraic neighborhoods to the analytic category. Another is to use the existence of analytic triangulations and the Hauptvermutung of these triangulations (the reader is allowed to protest here against my disgraceful behaviour). It is pertinent to note that Heegaard was the first to introduce the boundary of a singularity. In more recent times, Mumford's theorem [31] was the starting point of the long story of the interplay between the topology and the analytic structure of singularities. Here it is.

**Theorem 15** *Let $(\Sigma, P)$ be a normal surface singularity. Suppose that its boundary is simply connected. Then $P$ is a smooth point. [In other words: smoothness (which is an analytic property) can be detected topologically].*

## 4.2 Quotient singularities

An excellent reference for this section is Brieskorn's [39]. I follow Brieskorn's paper, hoping not to betray him.

### *4.2.1 General facts about quotient singularities*

The starting point is provided by Cartan [23]. Let $X^n$ be an analytic $n$-dimensional manifold (hence $X^n$ is smooth by definition). Let $G$ be a group of analytic automorphisms of $X^n$, acting properly discontinuously. Cartan considers the quotient $X^n/G$ and provides it with a structure of analytic space. One of Cartan's results is the following.

**Theorem 16** *The space $X^n/G$ is a normal analytic space of dimension n. The projection $X^n \longrightarrow X^n/G$ is analytic, onto and with discrete fibers. Moreover, if G is finite the projection $X^n \longrightarrow X^n/G$ is a finite analytic morphism. [In particular it is a ramified covering in the analytic sense, and hence also in the topological sense say of Stein* [22], Fox [24], Montesinos [68]*].*

Following Brieskorn and Cartan we have the following definition.

**Definition 11** A **quotient singularity** is a singularity of a quotient space $X^n/G$.

**Theorem 17** (Cartan linearization theorem) *A quotient singularity is isomorphic to a quotient $\mathbf{C}^n/\Gamma$ where $\Gamma$ is a finite subgroup of $GL_n(\mathbf{C})$.*

*Remark* A finite subgroup of $GL_n(\mathbf{C})$ is not necessarily contained in the unitary group $U(n)$, but it is conjugate to a subgroup of $U(n)$. Hence in the literature we find the two points of view: subgroup of $GL_n(\mathbf{C})$ or subgroup of $U(n)$.

### *4.2.2 General facts about quotient surface singularities*

Up to now, we had no restriction on the dimension $n$. From now on, we consider the case $n = 2$, and we study **quotient surface singularities**.

We skip over David Prill's definition of a **small subgroup** of $GL_n(\mathbf{C})$. For $n = 2$ this means that no non-trivial element of the subgroup has the eigenvalue 1. This is equivalent to the fact that the subgroup acts freely outside the origin. From now on, we assume that the subgroups of $GL_2(\mathbf{C})$ by which we take quotients are small. The following theorem implies that it is enough to consider only small subgroups. It is also valid with no restriction on the dimension.

**Theorem 18** (Prill [37], also Gottschling [36]) *Every quotient surface singularity is isomorphic to a quotient $\mathbf{C}^2/G$ with G small.*

The Theorem 2.8 of Brieskorn's paper relates beautifully quotient surface singularities to their boundary:

**Theorem 19** *Let $(\Sigma, P)$ be a normal surface singularity. Then the two following assertions are equivalent:*

*(i) The singularity is isomorphic to a quotient surface singularity;*
*(ii) The boundary M of the singularity has a finite fundamental group.*

*Remark* The resolution of singularities in the case of surfaces implies that, if the fundamental group of the boundary is finite, the boundary is a Seifert manifold. This is a consequence of the fact that, from the resolution, we know that the boundary is a manifold obtained by "plumbing" in the sense introduced by Mumford. These manifolds are thoroughly studied by Neumann [53].

Therefore, by Theorem 7, $M$ is a spherical space form. So we have two Lie groups in competition: $SO(4)$ and $U(2)$. But the opposition is only apparent. After an obvious change of coordinates, we have $U(2) \subset SO(4) \subset GL_4(\mathbf{R})$. The following lemma seems to be well known. See du Val's book (1964) §41 see ref [33]. This follows also from Scott's analysis [57]. See Theorem 4.10 p. 455.

**Lemma 2** *Every finite subgroup of $SO(4)$ acting freely on $S^3$ is conjugate to a subgroup in $U(2)$.*

Now orientations come again into play. If we see a finite group in $U(2)$ instead of in $SO(4)$ the quotient receives an orientation from the complex structure. So this change of viewpoint chooses one of the two possible Seifert invariants for the quotient. It is the one with the Euler rational number $e_0 < 0$ in order that the intersection form on the plumbing graph be negative definite. See Neumann and Raymond's paper [49].

*We present now Brieskorn's proof of Theorem 19* That (i) implies (ii) is obvious. So let us prove that (ii) implies (i).

Let $N$ be a "good" neighborhood of $P$ with boundary the manifold $M$. Let $G$ be the fundamental group of $M$. Let $V \longrightarrow N \backslash P$ be the universal covering. By a theorem of Grauert and Remmert [28] this unramified covering can be completed to a ramified covering $\widetilde{f} : (\widetilde{V}, Q) \longrightarrow (N, P)$ with $\widetilde{V}$ normal. Since the fundamental group of $V$ is trivial, by Mumford's theorem $Q$ is in fact a smooth point in $\widetilde{V}$ and hence $\widetilde{V}$ is smooth.

Now, by construction, $\widetilde{f}$ identifies the quotient of $\widetilde{V}$ by the action of $G$, with $(N, P)$. By the linearization theorem of Cartan, the action of $G$ on $\widetilde{V}$ is equivalent to a linear one. □

### *4.2.3 Quotient surface singularities are taut*

Brieskorn also proves that quotient surface singulariries are taut (in German "starr"). I slightly modify Brieskorn's definition of taut, by taking orientations into account.

**Definition 12** A normal surface singularity is **taut** if its analytic type is determined by the oriented topology of its boundary.

**Theorem 20** *Let $\Sigma_1 = \mathbf{C}^2/G_1$ and $\Sigma_2 = \mathbf{C}^2/G_2$ be two quotient surface singularities. Then the following three assertions are equivalent.*

*(1)* $G_1$ *and* $G_2$ *are conjugate in* $GL_2(\mathbf{C})$.
*(2)* $\Sigma_1$ *and* $\Sigma_2$ *are analytically equivalent.*
*(3)* $\Sigma_1$ *and* $\Sigma_2$ *are topologically equivalent, by an orientation preserving homeomorphism.*

*Proof* (1) obviously implies (2). (2) implies (3) by the analytic invariance of the boundary. Let us prove that (3) implies (1). If the groups $G_i$ are not cyclic, this follows from the rigidity theorem of Threlfall-Seifert. If these groups are cyclic, this comes from the proof of Reidemeister's theorem, since the equality of the Reidemeister torsions implies that the eigenvalues coincide. □

### *4.2.4 The resolution graph of quotient surface singularities*

Let me first introduce the following notation. Fix two integers $1 \leq u < v$. We consider the continued fraction expansion:

$$\frac{v}{u} = b_1 - \frac{1}{b_2 - \dots}$$

with $2 \leq b_i$ for all $i$ between 1 and say $r$. We write $\frac{v}{u} = \text{Contfrac}(b_1, b_2, \dots, b_r)$.

Note that $\frac{v}{u'} = \text{Contfrac}(b_r, \dots, b_1)$ with $uu' \equiv 1 \bmod v$.

From the continued fraction expansion we construct a bamboo shaped plumbing graph (called straight line graph by Neumann [53], p. 317) weighted by $(e_1, \dots, e_r)$ with $e_i = -b_i$.

**The standard plumbing graph for the lens space** $L(n, q)$ is the bamboo associated to the continued fraction expansion given above of $\frac{n}{q}$. Hirzebruch [20] proved that this is the graph of the minimal resolution of the cyclic quotient singularity $\mathbf{C}^2/C_{n,q}$.

**Caution.** There are many Seifert foliations on a lens space. In fact there is an infinity of isotopy classes of Seifert foliations on a lens space. Hence an infinity of plumbing graphs, hence an infinity of weighted bamboos. The different bamboos correspond to different continued fraction expansions of the rational number $\frac{n}{q}$. The operations needed to move from one to another are described by Neumann [53].

For the quotient surface singularity which corresponds to the platonic triple $(\alpha_1, \alpha_2, \alpha_3)$ and some $(e; \beta_1, \beta_2, \beta_3)$ we choose the orientation of the Seifert manifold which has $e_0 < 0$. Then we construct a star shaped plumbing graph as follows. The center of the star has genus zero and Euler's weight $e - 3$. There are three branches attached to the center. The one which corresponds to $(\alpha_1, \beta_1; \alpha_2, \beta_2; \alpha_3, \beta_3)$ is obtained as follows. We expand $\frac{\alpha_i}{\alpha_i - \beta_i}$ in a continued fraction as above. We construct the corresponding bamboo. We attach the first vertex of this bamboo (the one with weight $e_1$) to the center with an edge. See Neumann's [53] Corollary 5.7, p. 327. Brieskorn proved that this graph is the resolution graph of the singularity.

#### *4.2.5 Klein singularities*

These singularities have two names: Klein or Kleinian singularities. See the Encyclopedia of Mathematics (Section Rational Singularities) or Wikipedia for the terminology. Moreover, they are also called "rational double points" (Durfee [50]) and also du Val singularities.

They are studied for instance in Durfee's paper [50] and the Springer Lecture Notes No. 777.

**Definition 13** A quotient surface singularity $\Sigma = \mathbf{C}^2/G$ is a **Klein singularity** if $G$ is conjugate to a subgroup of $SU(2)$.

The condition amounts to restrict our attention from $U(2)$ to $SU(2)$. It turns out that this is a very strong restriction. In fact few quotient surface singularities are Klein.

For lens spaces, among the $L(n, q)$, only $L(n, n-1)$ is Klein.

For the platonic triples, the Seifert manifolds with finite fundamental group which are Klein are those with the $\beta_i = 1$ and the Euler number (say $\beta$) also equal to 1.

As a result Klein singularities have a plumbing graph (i.e. a resolution graph) with all weights equal to $-2$. In terms of Lie algebras one gets:

- the graph $A_{n-1}$ for $L(n, n-1)$ with $n \geq 2$
- the graph $D_{n+2}$ for the triple $(2, 2, n)$ with $n \geq 2$
- the graph $E_6$ for the triple $(2, 3, 3)$
- the graph $E_7$ for the triple $(2, 3, 4)$
- the graph $E_8$ for the triple $(2, 3, 5)$

### 4.3 Cyclic quotient singularities

#### *4.3.1 Cyclic quotients and quasi-ordinary surface singularities*

**Definition 14** A normal surface singularity $\Sigma$ is called **cyclic quotient** if it is isomorphic to a singularity $\mathbf{C}^2/C_{n,q}$ for some cyclic group $C_{n,q} \subset U(2)$.

**Beware.** Do not forget "quotient" in the name. Cyclic singularities exist. They were defined and studied by Hirzebruch. See his Bourbaki Seminar (1970/71) [44]. The Section 4 is devoted to cyclic singularities and the resolution graph presented by Hirzebruch consists in a beautiful cycle. See also his paper published in 1973 p. 214 for the use of "cyclic" see ref [45]. These singulaties are also called cusp singularities.

The next proposition results immediately from what we said about quotient surface singularities.

**Proposition 8** *A normal surface singularity is a cyclic quotient if and only if one of the following equivalent conditions holds:*

*(1) its boundary is a lens space*
*(2) the fundamental group of its boundary is cyclic.*

I now say a word about **quasi-ordinary** singularities. But there is a catch here. I recall the standard definition.

**Definition 15** A hypersurface singularity $\Sigma \subset \mathbf{C}^3$ is called quasi-ordinary if there is a finite projection onto $\mathbf{C}^2$ such that the discriminant is contained in the two coordinate axes.

The catch is that the definition concerns hypersurfaces, in other words embedded, not abstract, singularities. These are rarely normal, contrary to the singularities we talk about in this paper. Therefore, their boundary is generally a pseudo-manifold. Its structure is described by Costa [58].

An important point in the study of these hypersurfaces is their parametrization (Joseph Lipman, Yih Nan Gau). It is nearly obvious that the normalization of an irreducible quasi-ordinary singularity is a cyclic quotient singularity. More precisely one has the next proposition.

**Proposition 9** *Let* $\pi : (\Sigma, P) \longrightarrow (\mathbf{C}^2, 0)$ *be a finite holomorphic mapping from the normal surface singularity* $(\Sigma, P)$ *ramified over the two coordinate axes. Then* $(\Sigma, P)$ *is a cyclic quotient surface singularity.*

*Proof* The boundary of the singularity is a ramified covering of the sphere $S^3$ ramified over the Hopf link. Hence it is a lens space. □

One can do somewhat better.

**Theorem 21** *Let* $\mathring{B}^4$ *be the unit open ball in* $\mathbf{C}^2$ *and let* $\breve{B}^4$ *be* $\mathring{B}^4$ *minus the two axes. Suppose that we have a finite unramifed covering* $\breve{\pi} : \breve{Y} \longrightarrow \breve{B}^4$*. Then this covering can be completed to a finite holomorphic mapping* $\pi : Y \longrightarrow \mathring{B}^4$ *ramified over the coordinate axes with* $Y$ *normal. This extension is essentially unique.*

Here is a short sketch of the (classical) proof. Each step is non trivial.

First step. The unramifed covering can be completed to a ramified covering in the topological sense. In this context this was done by Stein [22]. In the topological context this was done by Fox [24] and Montesinos [68].

Second step. The topological completion can be equipped with the structure of a normal analytic space. This is due to Grauert and Remmert [28].

Third step. Uniqueness is provided by the normality condition.

### 4.3.2 From the topology to the singularity

I come back to the last theorem. It is interesting to describe how the topology of the unramified covering determines the ramified covering. This amounts to describing coverings of the sphere $S^3$ ramified over the Hopf link. Now the complement of the Hopf link in $S^3$ deformation retracts onto $S^1 \times S^1$. We denote by $G_1 \times G_2$ the fundamental group of the torus, each $G_i$ denoting a copy of the integers $\mathbf{Z}$.

The unramified covering is determined by a subgroup $K \subset G_1 \times G_2$ of finite index. After ramification one obtains the lens space $L(n, q)$ and our aim is to determine the integers $n$ and $q$.

**Notations.** We denote by $e_i$ the canonical generator of $G_i$. We denote by $K_i$ the intersection $K_i = K \cap Ze_i$. We denote by $p_i : G_i \times G_2 \longrightarrow G_i$ the canonical projection on a factor. The image $p_i(K)$ is denoted by $H_i$. The canonical generator of $K_i$ is $k_i e_i$ with $k_i > 0$ and the canonical generator of $H_i$ is $h_i e_i$ with $h_i > 0$.

**Lemma 3** *We have canonical isomorphisms* $H_1/K_1 \overset{p_1}{\longleftarrow} K/(K_1 \times K_2) \overset{p_2}{\longrightarrow} H_2/K_2$.

*Proof of the lemma.* The onto projection $p_1 : K \longrightarrow H_1$ has kernel $K_2$. Hence we have an isomorphism $K/K_2 \overset{\approx}{\longrightarrow} H_1$.We then take the quotient of this isomorphism with $K_1$.

As a consequence the quotient $K/(K_1 \times K_2)$ is cyclic of order $k_1k_2/k$. Therefore the lens space $L(n, q)$ has $n = k_1k_2/k$. We also have the equalities $n = k_1/h_1 = k_2/h_2$. There remains to determine $q$. For this we propose a parenthesis.

**Parenthesis: How to determine the integers $n$ and $q$ from a genus one Heegaard decomposition.**

In the homology groups $H_1(\partial V_i)$, integer coefficients $\mathbf{Z}$ are understood. We consider a decomposition of a lens space say $L$ in two solid torii $V_1$ and $V_2$ with $V_1 \cap V_2 = \partial V_1 = \partial V_2 = T$. Each $H_1(\partial V_i)$ contains a meridian element $m_i$, which is canonical once orientations and signs are fixed. We assume that this is the case. The meridian is the first element of a basis of $H_1(\partial V_i)$. In order to obtain a basis for $H_1(\partial V_i)$ we have to choose adequately a second element $l_i$. For geographical reasons we call it a parallel (not a longitude!). The parallel is not canonical. We have the following equality

$$m_1 = nl_2 - qm_2$$

The minus sign is crucial. Since the parallel $l_2$ is not canonical, $q$ is only determined mod $n$. But this is exactly what we have from the beginning.

We also have the equality $m_2 = nl_1 - q'm_1$ with $qq' \equiv 1 \bmod n$. For details see Jankins–Neumann p. 30. **End of the parenthesis.**

We resume the determination of the integer $q$. We have in $G_1 \times G_2$:

$$m_2 = 0e_1 + k_2e_2 \quad l_2 = xe_1 + ye_2$$

We proceed to the computation of the integers $x$ and $y$. Since the index of $K$ in $G_1 \times G_2$ is equal to $k$ up to sign, the determinant of the coefficients must be equal to $k$. Hence $xk_2 = k$ and so $x = k/k_2$. Now $n = (k_1k_2/k)$. Hence $k/k_2 = k_1/n = h_1$. Hence $x = h_1$. Therefore $l_2 = h_1e_1 + ye_2$.

There remains to compute $y$. All integers $y$ such that $h_1e_1 + ye_2 \in K$ produce an element in $K$ which is a second element for a basis of $K$. Let us assume that we have two of them:

$$l_2 = h_1e_1 + ye_2 \quad \text{and} \quad l'_2 = h_1e_1 + y'e_2. \tag{1}$$

Hence $l_2 - l'_2 = (y - y')e_2 \in K$ and hence $\in K_2$. Therefore $(y - y')$ is divisible by $k_2$. Hence $y$ is well defined mod $k_2$.

On the other hand the projection of $l_2$ in $G_2$ is equal to $ye_2$. Hence $y$ is divisible by $h_2$. We write $y = uh_2$.

We claim that $u$ is equal to the $q$ which determines the lens space $L(n, q)$.

To begin with, $y$ is well defined $\bmod\ k_2$ and hence $u$ is well defined $\bmod\ k_2/h_2 = n$. This is exactly the indetermination of $q$. We started from $l_2 = xe_1 + ye_2$. After some computations we have now $l_2 = h_1e_1 + uh_2e_2$. Let us multiply this equality by $n = k_1/h_1 = k_2/h_2$. We get $nl_2 = nh_1e_1 + nuh_2e_2 = k_1e_1 + uk_2e_2 = m_1 + um_2$. Therefore

$$m_1 = nl_2 - um_2$$

**Conclusion.** To obtain the residue $q$ mod $n$, we look for elements in $K$ of the form $h_1e_1 + uh_2e_2$. Any integer $u$ such that this linear combination is in $K$ is a representative of the residue mod $n$ we are looking for.

**Comment.** The integers $h_1$ and $h_2$ can take any positive value. Hence there are infinitely many coverings which produce the "same" $L(n, q)$ and hence isomorphic cyclic quotient singularities. The simplest case is certainly $h_1 = 1 = h_2$. If this happens, the degree $k$ of the covering is equal to $n$.

### *4.3.3 The singularities $z^m = x^a y^b$*

These quasi-ordinary singularities were already present in Jung [3] and Hirzebruch [20]. This is a particular case of the situation we have studied above. But now we have at our disposal the homomorphism $\varphi : G_1 \times G_2 \longrightarrow \mathbf{Z}/m$ which determines the unramified covering. We have $\varphi(e_1) = a$ and $\varphi(e_2) = b$. The homomorphism $\varphi$ is onto if and only if the gcd of $(m, a, b)$ is equal to 1. This is equivalent to require that the singularity is irreducible (here "irreducible" is used for its meaning in the theory of analytic spaces). We assume that this condition is satisfied. The method presented above applied to this situation produces the following result.

**Theorem 22** *We consider the quasi-ordinary singularity $z^m = x^a y^b$, satisfying the irreducibility condition gcd $(m, a, b) = 1$. The normalisation of this singularity has the lens space $L(n, q)$ for boundary (this is equivalent to say that the normalisation is isomorphic to the cyclic quotient singularity $\mathbf{C}^2/C_{n,q}$) where the integers n and q are obtained as follows. Let $d_a$ = gcd (m , a) and $d_b$ = gcd (m , b). Also let $m_a = m/d_a$ and $m_b = m/d_b$. Then:*

*(1) the integer n is equal to $(m_a m_b)/m = m/(d_a d_b)$;*
*(2) the invariant q is equal* mod *n to the solutions of the unknown u in the equation $ubd_a \equiv -ad_b \bmod m$.*

This theorem is generalized by Popescu-Pampu [66] to arbitrary quasi-ordinary singularities.

We conclude with a few examples. The normalisation of the quasi-ordinary singularity with equation $z^n = xy^{n-q}$ is $\mathbf{C}^2/C_{n,q}$ with boundary $L(n, q)$. It is good to see that this computation agrees with Hirzebruch and Brieskorn [62], p. 25, obtained by a different method. For instance, for $q = n-1$ the quasi-ordinary singularity is already normal in $\mathbf{C}^3$. Its boundary is the lens space $L(n, n - 1)$ we already met in the subsection about Klein singularities.

Lens spaces are the only boundaries of quotient singularities which are also such a boundary if one reverses the orientation. This reversal on $L(n, n - 1)$ produces $L(n, 1)$ which is not Klein. See Neumann [53].

**Acknowledgements** I thank Françoise Michel for useful conversations. But the mistakes which remain are mine. I also thank the referee for helping me to be more clear in several instances.

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