

An extension of Jensen's operator inequality and its application to Young inequality

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Received: 9 October 2017 / Accepted: 9 January 2018 / Published online: 22 January 2018
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Abstract Jensen's operator inequality for convexifiable functions is obtained. This result contains classical Jensen's operator inequality as a particular case. As a consequence, a new refinement and a reverse of Young's inequality are given.

Keywords Convexifiable functions · Jensen's inequality · Young inequality · Operator inequality

Mathematics Subject Classification Primary 47A63, 26A51; Secondary 26D15, 47A64, 46L05

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1 Introduction and preliminaries

In this article, \mathcal{H} will denote a Hilbert space, and the term ‘‘operator’’ we shall mean endomorphism of \mathcal{H} . The following result that provides an operator version for the Jensen inequality is due to Mond and Pečarić [12]:

Theorem 1.1 (Jensen’s operator inequality for convex functions) *Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If $f(t)$ is a convex function on $[m, M]$, then*

$$f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle, \tag{1.1}$$

for every unit vector $x \in \mathcal{H}$.

Over the years, various extensions and generalizations of (1.1) have been obtained in the literature, e.g., [6, 7, 13]. For this background we refer to any expository text such as [5].

The aim of this paper is to find an inequality which contains (1.1) as a special case. Our result also allows to obtain a refinement and a reverse for the scalar Young inequality. More precisely, it will be shown that for two non-negative numbers a, b we have

$$\begin{aligned} K^r(h, 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{r}{4}\right)\left(\frac{a-b}{D}\right)^2\right) \\ \leq \frac{a \nabla_v b}{a \sharp_v b} \\ \leq K^R(h, 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{R}{4}\right)\left(\frac{a-b}{D}\right)^2\right), \end{aligned}$$

where $r = \min\{v, 1 - v\}$, $R = \max\{v, 1 - v\}$, $D = \max\{a, b\}$ and $K(h, 2) = \frac{(h+1)^2}{4h}$ is the Kantorovich constant with $h = \frac{b}{a}$.

To make the text more self-contained we give a brief overview of convexifiable functions. Given a continuous $f : I \rightarrow \mathbb{R}$ defined on the compact interval $I \subset \mathbb{R}$, consider a function $\varphi : I \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $\varphi(x, \alpha) = f(x) - \frac{1}{2}\alpha x^2$. If $\varphi(x, \alpha)$ is a convex function on I for some $\alpha = \alpha^*$, then $\varphi(x, \alpha)$ is called a convexification of f and α^* a convexifier on I . A function f is convexifiable if it has a convexification. It is noted in [17, Corollary 2.9] that if the continuously differentiable function f has Lipschitz derivative (i.e., $|f'(x) - f'(y)| \leq L|x - y|$ for any $x, y \in I$ and some constant L), then $\alpha = -L$ is a convexifier of f .

The following fact concerning convexifiable functions plays an important role in our discussion (see [17, Corollary 2.8]):

If f is twice continuously differentiable, then $\alpha = \min_{t \in I} f''(t)$ is a convexifier of f . (P)

The reader may consult [16] for additional information about this topic. For all other notions used in the paper, we refer the reader to the monograph [5].

2 Main results

After the above preparation, we are ready to prove the analogue of (1.1) for non-convex functions.

Theorem 2.1 (Jensen’s operator inequality for non-convex functions) *Let f be a continuous convexifiable function on the interval I and α a convexifier of f . Then*

$$f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle - \frac{1}{2}\alpha(\langle A^2x, x \rangle - \langle Ax, x \rangle^2), \tag{2.1}$$

for every self-adjoint operator A with $Sp(A) \subseteq I$ and every unit vector $x \in \mathcal{H}$.

Proof The idea of proof evolves from the approach in [15]. Let $g_\alpha : I \rightarrow \mathbb{R}$ with $g_\alpha(x) = \varphi(x, \alpha)$. According to the assumption, $g_\alpha(x)$ is convex. Therefore

$$g_\alpha(\langle Ax, x \rangle) \leq \langle g_\alpha(A)x, x \rangle,$$

for every unit vector $x \in \mathcal{H}$. This expression is equivalent to the desired inequality (2.1). \square

A few remarks concerning Theorem 2.1 are in order.

Remark 2.1 (a) Using the fact that for a convex function f one can choose the convexifier $\alpha = 0$, one recovers the inequality (1.1).

(b) For continuously differentiable function f with Lipschitz derivative and Lipschitz constant L , we have

$$f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle + \frac{1}{2}L(\langle A^2x, x \rangle - \langle Ax, x \rangle^2).$$

An important special case of Theorem 2.1, which refines inequality (1.1) can be explicitly stated using the property (P).

Remark 2.2 Let $f : I \rightarrow \mathbb{R}$ be a twice continuously differentiable strictly convex function and $\alpha = \min_{t \in I} f''(t)$. Then

$$f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle - \frac{1}{2}\alpha(\langle A^2x, x \rangle - \langle Ax, x \rangle^2), \tag{2.2}$$

for every positive operator A with $Sp(A) \subseteq I$ and every unit vector $x \in \mathcal{H}$.

The inequality (2.2) is obtained in the paper [13, Theorem 3.3] (where this result was derived for the strongly convex functions) with a different technique (see also [4]).

The proof of the following corollary is adapted from the one of [5, Theorem 1.3], but we put a sketch of the proof for the reader.

Corollary 2.1 *Let f be a continuous convexifiable function on the interval I and α a convexifier. Let A_1, \dots, A_n be self-adjoint operators on \mathcal{H} with $Sp(A_i) \subseteq I$ for $1 \leq i \leq n$ and $x_1, \dots, x_n \in \mathcal{H}$ be such that $\sum_{i=1}^n \|x_i\|^2 = 1$. Then*

$$f\left(\sum_{i=1}^n \langle A_i x_i, x_i \rangle\right) \leq \sum_{i=1}^n \langle f(A_i)x_i, x_i \rangle - \frac{1}{2}\alpha\left(\sum_{i=1}^n \langle A_i^2 x_i, x_i \rangle - \left(\sum_{i=1}^n \langle A_i x_i, x_i \rangle\right)^2\right). \tag{2.3}$$

Proof In fact, $x := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is a unit vector in the Hilbert space \mathcal{H}^n . If we introduce the “diagonal” operator on \mathcal{H}^n

$$A := \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix},$$

then, obviously, $Sp(A) \subseteq I$, $\|x\| = 1$, $\langle f(A)x, x \rangle = \sum_{i=1}^n \langle f(A_i)x_i, x_i \rangle$, $\langle Ax, x \rangle = \sum_{i=1}^n \langle A_i x_i, x_i \rangle$, $\langle A^2 x, x \rangle = \sum_{i=1}^n \langle A_i^2 x_i, x_i \rangle$. Hence, to complete the proof, it is enough to apply Theorem 2.1 for A and x . \square

Corollary 2.1 leads us to the following result. The argument depends on an idea of [1, Corollary 1].

Corollary 2.2 *Let f be a continuous convexifiable function on the interval I and α a convexifier. Let A_1, \dots, A_n be self-adjoint operators on \mathcal{H} with $Sp(A_i) \subseteq I$ for $1 \leq i \leq n$ and let p_1, \dots, p_n be positive scalars such that $\sum_{i=1}^n p_i = 1$. Then*

$$f\left(\sum_{i=1}^n \langle p_i A_i x, x \rangle\right) \leq \sum_{i=1}^n \langle p_i f(A_i)x, x \rangle - \frac{1}{2}\alpha\left(\sum_{i=1}^n \langle p_i A_i^2 x, x \rangle - \left(\sum_{i=1}^n \langle p_i A_i x, x \rangle\right)^2\right), \tag{2.4}$$

for every unit vector $x \in \mathcal{H}$.

Proof Suppose that $x \in \mathcal{H}$ is a unit vector. Putting $x_i = \sqrt{p_i}x \in \mathcal{H}$ so that $\sum_{i=1}^n \|x_i\|^2 = 1$ and applying Corollary 2.1 we obtain the desired result (2.4). \square

The clear advantage of our approach over the Jensen operator inequality is shown in the following example. Before proceeding we recall the following multiple operator version of Jensen’s inequality [1, Corollary 1]: Let $f : [m, M] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and A_i be self-adjoint operators with $Sp(A_i) \subseteq [m, M]$, $i = 1, \dots, n$ for some scalars $m < M$. If $p_i \geq 0$, $i = 1, \dots, n$ with $\sum_{i=1}^n p_i = 1$, then

$$f\left(\sum_{i=1}^n \langle p_i A_i x, x \rangle\right) \leq \sum_{i=1}^n \langle p_i f(A_i)x, x \rangle, \tag{2.5}$$

for every $x \in \mathcal{H}$ with $\|x\| = 1$.

Example 2.1 We use the same idea from [15, Illustration 1]. Let $f(t) = \sin t$ ($0 \leq t \leq 2\pi$), $\alpha = \min_{0 \leq t \leq 2\pi} f''(t) = -1$, $n = 2$, $p_1 = p$, $p_2 = 1 - p$, $\mathcal{H} = \mathbb{R}^2$, $A_1 = \begin{pmatrix} 2\pi & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2\pi \end{pmatrix}$ and $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. After simple calculations (thanks to the continuous functional calculus), from (2.4) we infer that

$$\sin(2\pi(1-p)) \leq 2\pi^2 p(1-p), \quad 0 \leq p \leq 1 \tag{2.6}$$

and (2.5) implies

$$\sin(2\pi(1-p)) \leq 0, \quad 0 \leq p \leq 1. \tag{2.7}$$

Not so surprisingly, the inequality (2.7) can break down when $\frac{1}{2} \leq p \leq 1$ (i.e., (2.5) is not applicable here). However, the new upper bound in (2.6) holds.

The weighted version of [15, Theorem 3] follows from Corollary 2.2, i.e.,

$$f\left(\sum_{i=1}^n p_i t_i\right) \leq \sum_{i=1}^n p_i f(t_i) - \frac{1}{2}\alpha\left(\sum_{i=1}^n p_i t_i^2 - \left(\sum_{i=1}^n p_i t_i\right)^2\right), \tag{2.8}$$

where $t_i \in I$ and $\sum_{i=1}^n p_i = 1$. For the case $n = 2$, the inequality (2.8) reduces to

$$f((1 - v)t_1 + vt_2) \leq (1 - v)f(t_1) + vf(t_2) - \frac{v(1 - v)}{2}\alpha(t_1 - t_2)^2, \tag{2.9}$$

where $0 \leq v \leq 1$. In particular

$$f\left(\frac{t_1 + t_2}{2}\right) \leq \frac{f(t_1) + f(t_2)}{2} - \frac{1}{8}\alpha(t_1 - t_2)^2. \tag{2.10}$$

It is notable that Theorem 2.1 is equivalent to the inequality (2.8). The following provides a refinement of the arithmetic-geometric mean inequality.

Proposition 2.1 For each $a, b > 0$ and $0 \leq v \leq 1$, we have

$$\sqrt{ab} \leq H_v(a, b) - \frac{d}{8}\left((1 - 2v)\left(\log \frac{a}{b}\right)\right)^2 \leq \frac{a + b}{2} - \frac{d}{8}\left(\log \frac{a}{b}\right)^2 \leq \frac{a + b}{2}, \tag{2.11}$$

where $d = \min\{a, b\}$ and $H_v(a, b) = \frac{a^{1-v}b^v + b^{1-v}a^v}{2}$ is the Heinz mean.

Proof Assume that f is a twice differentiable convex function such that $\alpha \leq f''$ where $\alpha \in \mathbb{R}$. Under these conditions, it follows that

$$\begin{aligned} f\left(\frac{a + b}{2}\right) &= f\left(\frac{(1 - v)a + vb + (1 - v)b + va}{2}\right) \\ &\leq \frac{f((1 - v)a + vb) + f((1 - v)b + va)}{2} - \frac{1}{8}\alpha((a - b)(1 - 2v))^2 \quad (\text{by (2.10)}) \\ &\leq \frac{f(a) + f(b)}{2} - \frac{1}{8}\alpha(a - b)^2 \quad (\text{by (2.9)}) \\ &\leq \frac{f(a) + f(b)}{2}, \end{aligned}$$

for $\alpha \geq 0$. Now taking $f(t) = e^t$ with $t \in I = [a, b]$ in the above inequalities, we deduce the desired inequality (2.11). □

Remark 2.3 As Bhatia pointed out in [2], the Heinz means interpolate between the geometric mean and the arithmetic mean, i.e.,

$$\sqrt{ab} \leq H_v(a, b) \leq \frac{a + b}{2}. \tag{2.12}$$

Of course, the first inequality in (2.11) yields an improvement of (2.12). The inequalities in (2.11) also sharpen up the following inequality which is due to Dragomir (see [3, Remark 1]):

$$\frac{d}{8}\left(\log \frac{a}{b}\right)^2 \leq \frac{a + b}{2} - \sqrt{ab}.$$

Studying about the arithmetic-geometric mean inequality, we cannot avoid mentioning its cousin, the Young inequality. The following inequalities provides a multiplicative type refinement and reverse of the Young’s inequality:

$$K^r(h, 2) \leq \frac{(1 - v)a + vb}{a^{1-v}b^v} \leq K^R(h, 2), \tag{2.13}$$

where $0 \leq v \leq 1$, $r = \min\{v, 1 - v\}$, $R = \max\{v, 1 - v\}$ and $K(h, 2) = \frac{(h+1)^2}{4h}$ with $h = \frac{b}{a}$. The first one was proved by Zuo et al. [18, Corollary 3], while the second one was given by Liao et al. [9, Corollary 2.2].

Our aim in the following is to establish a refinement for the inequalities in (2.13). The crucial role for our purposes will play the following facts:

If f is a convex function on the fixed closed interval I , then

$$n\lambda \left\{ \sum_{i=1}^n \frac{1}{n} f(x_i) - f\left(\sum_{i=1}^n \frac{1}{n} x_i\right) \right\} \leq \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right), \tag{2.14}$$

$$\sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \leq n\mu \left\{ \sum_{i=1}^n \frac{1}{n} f(x_i) - f\left(\sum_{i=1}^n \frac{1}{n} x_i\right) \right\}, \tag{2.15}$$

where $p_1, \dots, p_n \geq 0$ with $\sum_{i=1}^n p_i = 1$, $\lambda = \min\{p_1, \dots, p_n\}$, $\mu = \max\{p_1, \dots, p_n\}$. Notice that the first inequality goes back to Pečarić et al. [10, Theorem 1, p. 717], while the second one was obtained by Mitroi in [11, Corollary 3.1].

Now we come to the announced theorem. In order to simplify the notations, we put $a\#_v b = a^{1-v}b^v$ and $a\nabla_v b = (1-v)a + vb$.

Theorem 2.2 *Let $a, b > 0$ and $0 \leq v \leq 1$. Then*

$$\begin{aligned} &K^r(h, 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{r}{4}\right)\left(\frac{a-b}{D}\right)^2\right) \\ &\leq \frac{a\nabla_v b}{a\#_v b} \\ &\leq K^R(h, 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{R}{4}\right)\left(\frac{a-b}{D}\right)^2\right), \end{aligned} \tag{2.16}$$

where $r = \min\{v, 1-v\}$, $R = \max\{v, 1-v\}$, $D = \max\{a, b\}$ and $K(h, 2) = \frac{(h+1)^2}{4h}$ with $h = \frac{b}{a}$.

Proof Employing the inequality (2.14) for the twice differentiable convex function f with $\alpha \leq f''$, we have

$$\begin{aligned} &n\lambda \left\{ \frac{1}{n} \sum_{i=1}^n f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \right\} - \sum_{i=1}^n p_i f(x_i) + f\left(\sum_{i=1}^n p_i x_i\right) \\ &\leq \frac{\alpha}{2} \left\{ n\lambda \left[\frac{1}{n} \sum_{i=1}^n x_i^2 - \left(\frac{1}{n} \sum_{i=1}^n x_i\right)^2 \right] - \left(\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i\right)^2 \right) \right\}. \end{aligned}$$

Here we set $n = 2$, $x_1 = a, x_2 = b, p_1 = 1-v, p_2 = v, \lambda = r$ and $f(x) = -\log x$ with $I = [a, b]$ (so $\alpha = \min_{x \in I} f''(x) = \frac{1}{D^2}$). Thus we deduce the first inequality in (2.16). The second inequality in (2.16) is also obtained similarly by using the inequality (2.15). \square

Remark 2.4 (a) Since $\frac{v(1-v)}{2} - \frac{r}{4} \geq 0$ for each $0 \leq v \leq 1$, we have $\exp\left(\left(\frac{v(1-v)}{2} - \frac{r}{4}\right)\left(\frac{a-b}{D}\right)^2\right) \geq 1$. Therefore the first inequality in (2.16) provides an improvement for the first inequality in (2.13).

(b) Since $\frac{v(1-v)}{2} - \frac{R}{4} \leq 0$ for each $0 \leq v \leq 1$, we get $\exp\left(\left(\frac{v(1-v)}{2} - \frac{R}{4}\right)\left(\frac{a-b}{D}\right)^2\right) \leq 1$. Therefore the second inequality in (2.16) provides an improvement for the second inequality in (2.13).

Proposition 2.2 *Under the same assumptions in Theorem 2.2, we have*

$$\frac{(h + 1)^2}{4h} \geq \exp\left(\frac{1}{4} \left(\frac{a - b}{D}\right)^2\right).$$

Proof We prove the case $a \leq b$, then $h \geq 1$. We set $f_1(h) \equiv 2 \log(h + 1) - \log h - 2 \log 2 - \frac{1}{4} \frac{(h-1)^2}{h^2}$. It is quite easy to see that $f_1'(h) = \frac{(2h+1)(h-1)^2}{2h^3(h+1)} \geq 0$, so that $f_1(h) \geq f_1(1) = 0$. For the case $a \geq b$, (then $0 < h \leq 1$), we also set $f_2(h) \equiv 2 \log(h+1) - \log h - 2 \log 2 - \frac{1}{4}(h-1)^2$. By direct calculation $f_2'(h) = -\frac{(h-1)^2(h+2)}{2h(h+1)} \leq 0$, so that $f_2(h) \geq f_2(1) = 0$. Thus the statement follows. \square

Remark 2.5 Dragomir obtained a refinement and reverse of Young’s inequality in [3, Theorem 3] as:

$$\exp\left(\frac{v(1-v)}{2} \left(\frac{a-b}{D}\right)^2\right) \leq \frac{a \nabla_v b}{a \sharp_v b} \leq \exp\left(\frac{v(1-v)}{2} \left(\frac{a-b}{d}\right)^2\right), \tag{2.17}$$

where $d = \min\{a, b\}$. From the following facts (a) and (b), we claim that our inequalities are non-trivial results.

- (a) From Proposition 2.2, our lower bound in (2.16) is tighter than the one in (2.17).
- (b) Numerical computations show that there is no ordering between the right hand side in (2.16) and the one in the second inequality of (2.17) shown in [3, Theorem 3]. For example, if we take $a = 2, b = 1$ and $v = 0.1$, then

$$K^{R'}(h, 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{R}{4}\right) \left(\frac{a-b}{D}\right)^2\right) - \exp\left(\frac{v(1-v)}{2} \left(\frac{a-b}{d}\right)^2\right) \simeq 0.0168761,$$

whereas it approximately equals -0.0436069 when $a = 2, b = 1$ and $v = 0.3$.

We give a further remark in relation to comparisons with other inequalities.

Remark 2.6 The following refined Young inequality and its reverse are known

$$K^{r'}(\sqrt{t}, 2)t^v + r(1 - \sqrt{t})^2 \leq (1 - v) + vt \leq K^{R'}(\sqrt{t}, 2)t^v + r(1 - \sqrt{t})^2, \tag{2.18}$$

where $t > 0, r' = \min\{2r, 1 - 2r\}$ and $R' = \max\{2r, 1 - 2r\}$. The first and second inequality were given in [14, Lemma 2.1] and in [9, Theorem 2.1], respectively.

Numerical computations show that there is no ordering between our inequalities (2.16) and the above ones. Actually, if we take $v = 0.45$ and $t = 0.1$ (we set $t = \frac{b}{a}$ with $a \geq b$ in (2.16)), then

$$K^{R'}(\sqrt{t}, 2)t^v + r(1 - \sqrt{t})^2 - t^v K^R(h, 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{R}{4}\right) (1-t)^2\right) \simeq 0.0363059,$$

while it equals approximately -0.0860004 when $v = 0.9$ and $t = 0.1$.

Similarly, when $v = 0.45$ and $t = 0.1$ we get

$$K^{r'}(\sqrt{t}, 2)t^v + r(1 - \sqrt{t})^2 - t^v K^r(h, 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{r}{4}\right) (1-t)^2\right) \simeq -0.0126828,$$

while it equals approximately 0.037896 when $v = 0.9$ and $t = 0.1$.

Obviously, in the inequality (2.13), we cannot replace $K^r(h, 2)$ by $K^R(h, 2)$, or vice versa. In this regard, we have the following theorem. The proof is almost the same as that of Theorem 2.2 (it is enough to use the convexity of the function $g_\beta(x) = \frac{\beta}{2}x^2 - f(x)$ where $\beta = \min_{x \in I} f''(x)$).

Theorem 2.3 *Let all the assumptions of Theorem 2.2 hold except that $d = \min\{a, b\}$. Then*

$$\begin{aligned}
 K^R(h, 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{R}{4}\right)\left(\frac{a-b}{d}\right)^2\right) &\leq \frac{A \nabla_v B}{A \sharp_v B} \\
 &\leq K^r(h, 2) \\
 &\exp\left(\left(\frac{v(1-v)}{2} - \frac{r}{4}\right)\left(\frac{a-b}{d}\right)^2\right).
 \end{aligned}$$

We end this paper by presenting the operator inequalities based on Theorems 2.2 and 2.3, thanks to the Kubo-Ando theory [8].

Corollary 2.3 *Let A, B be two positive invertible operators and positive real numbers m, m', M, M' that satisfy one of the following conditions:*

- (i) $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$.
- (ii) $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$.

Then

$$\begin{aligned}
 &K^r(h, 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{r}{4}\right)\left(\frac{1-h}{h}\right)^2\right) A \sharp_v B \\
 &\leq A \nabla_v B \\
 &\leq K^R(h', 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{R}{4}\right)\left(\frac{1-h'}{h'}\right)^2\right) A \sharp_v B
 \end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
 &K^R(h, 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{R}{4}\right)\left(\frac{1-h'}{h'}\right)^2\right) A \sharp_v B \\
 &\leq A \nabla_v B \\
 &\leq K^r(h', 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{r}{4}\right)\left(\frac{1-h}{h}\right)^2\right) A \sharp_v B,
 \end{aligned}$$

where $r = \min\{v, 1-v\}$, $R = \max\{v, 1-v\}$ and $K(h, 2) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$.

Proof On account of (2.16), we have

$$\begin{aligned}
 &\min_{h \leq x \leq h'} \left\{ K^r(x, 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{r}{4}\right)\left(\frac{1-x}{\max\{1, x\}}\right)^2\right) \right\} T^v \\
 &\leq (1-v)I + vT \\
 &\leq \max_{h \leq x \leq h'} \left\{ K^R(x, 2) \exp\left(\left(\frac{v(1-v)}{2} - \frac{R}{4}\right)\left(\frac{1-x}{\max\{1, x\}}\right)^2\right) \right\} T^v,
 \end{aligned}$$

for the positive operator T such that $hI \leq T \leq h'I$. Setting $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$.

In the first case we have $I < hI = \frac{M}{m}I \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \frac{M'}{m'}I = h'I$, which implies that

$$\begin{aligned} & \min_{1 \leq h \leq x \leq h'} \left\{ K^r(x, 2) \exp \left(\left(\frac{v(1-v)}{2} - \frac{r}{4} \right) \left(\frac{1-x}{x} \right)^2 \right) \right\} \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right)^v \\ & \leq (1-v)I + vA^{-\frac{1}{2}}BA^{-\frac{1}{2}} \\ & \leq \max_{1 \leq h \leq x \leq h'} \left\{ K^R(x, 2) \exp \left(\left(\frac{v(1-v)}{2} - \frac{R}{4} \right) \left(\frac{1-x}{x} \right)^2 \right) \right\} \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right)^v. \end{aligned} \quad (2.20)$$

We can write (2.20) in the form

$$\begin{aligned} & K^r(h, 2) \exp \left(\left(\frac{v(1-v)}{2} - \frac{r}{4} \right) \left(\frac{1-h}{h} \right)^2 \right) \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right)^v \\ & \leq (1-v)I + vA^{-\frac{1}{2}}BA^{-\frac{1}{2}} \\ & \leq K^R(h', 2) \exp \left(\left(\frac{v(1-v)}{2} - \frac{R}{4} \right) \left(\frac{1-h'}{h'} \right)^2 \right) \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right)^v. \end{aligned}$$

Finally, multiplying both sides of the previous inequality by $A^{\frac{1}{2}}$ we get the desired result (2.19).

The proof of other cases is similar, we omit the details. \square

Acknowledgements The authors thank anonymous referees for giving valuable comments and suggestions to improve our manuscript. The author (S.F.) was partially supported by JSPS KAKENHI Grant Number 16K05257.

References

1. Agarwal, R.P., Dragomir, S.S.: A survey of Jensen type inequalities for functions of selfadjoint operators in Hilbert spaces. *Comput. Math. Appl.* **59**(12), 3785–3812 (2010)
2. Bhatia, R.: Interpolating the arithmetic-geometric mean inequality and its operator version. *Linear Algebra Appl.* **413**, 355–363 (2006)
3. Dragomir, S.S.: On new refinements and reverse of Young's operator inequality. [arXiv:1510.01314v1](https://arxiv.org/abs/1510.01314v1)
4. Dragomir, S.S.: Some Jensen's type inequalities for twice differentiable functions of selfadjoint operators in Hilbert spaces. *Filomat* **23**(3), 211–222 (2009)
5. Furuta, T., Mičić Hot, J., Pečarić, J. Seo, Y.: *Mond-Pečarić Method in Operator Inequalities*. Monographs in Inequalities 1, Element, Zagreb (2005)
6. Horváth, L., Khan, K.A., Pečarić, J.: Cyclic refinements of the different versions of operator Jensen's inequality. *Electron. J. Linear Algebra* **31**(1), 125–133 (2016)
7. Kian, M.: Operator Jensen inequality for superquadratic functions. *Linear Algebra Appl.* **456**, 82–87 (2014)
8. Kubo, F., Ando, T.: Means of positive linear operators. *Math. Ann.* **246**(3), 205–224 (1980)
9. Liao, W., Wu, J., Zhao, J.: New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant. *Taiwanese J. Math.* **19**(2), 467–479 (2015)
10. Mitrinović, D.S., Pečarić, J., Fink, A.M.: *Classical and New Inequalities in Analysis*. Kluwer Academic Publishers, Dordrecht (1993)
11. Mitroi, F.C.: About the precision in Jensen–Steffensen inequality. *An. Univ. Craiova Ser. Mat. Inform.* **37**(4), 73–84 (2010)
12. Mond, B., Pečarić, J.: Convex inequalities in Hilbert space. *Houston J. Math.* **19**(3), 405–420 (1993)

13. Moradi, H.R., Omidvar, M.E., Adil Khan, M., Nikodem, K.: Around Jensen's inequality for strongly convex functions. *Aequationes Math.* (2017). <https://doi.org/10.1007/s00010-017-0496-5>
14. Wu, J., Zhao, J.: Operator inequalities and reverse inequalities related to the Kittaneh–Manasrah inequalities. *Linear Multilinear Algebra* **62**, 884–894 (2014)
15. Zlobec, S.: Jensen's inequality for nonconvex functions. *Math. Commun.* **9**(2), 119–124 (2004)
16. Zlobec, S.: Convexifiable functions in integral calculus. *Glasnik Matematički* **40**(2), 241–247 (2005)
17. Zlobec, S.: Characterization of convexifiable functions. *Optimization* **55**(3), 251–261 (2006)
18. Zuo, H., Shi, G., Fujii, M.: Refined Young inequality with Kantorovich constant. *J. Math. Inequal.* **5**(4), 551–556 (2011)