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Property (k-β) of Musielak–Orlicz and Musielak–Orlicz–Cesàro spaces

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Abstract In this paper, we investigate the geometric property (k-β) for any fixed integer $k \geq 1$ of the space $l_\Phi((E_n))$ generated by a Musielak–Orlicz function Φ and a sequence (E_n) of finite dimensional spaces $E_n, n \in \mathbb{N}$, equipped with both the Luxemburg and the Amemiya norms. As a consequence, we obtain the property (k-β) of Musielak–Orlicz–Cesàro spaces ces_Φ using the approach recently considered by Saejung. Some applications to the Cesàro sequence spaces of order α and Cesàro difference sequence spaces of order m are also noted.

Keywords (β)-Property · Cesàro spaces · Difference spaces · Musielak–Orlicz spaces

Mathematics Subject Classification 46B20 · 46A45 · 46B45

1 Introduction

Geometric properties of Banach space such as the Kadec–Klee property [or (H)-property], Opial property, rotundity, nearly uniformly convexity property (NUC), (β)-property and their several generalizations play fundamental role for their various applications in the fixed point theory, optimization theory, differential and integral equations etc.

In the sequel, we shall initiate with basic notions of geometric properties of Banach spaces and Musielak–Orlicz spaces.

Let $(X, \|.\|)$ be a Banach space and l^0 be the space of all real sequences $x = (x(i))_{i=1}^{\infty}$. Let $S(X)$ and $B(X)$ be denote the unit sphere and closed unit ball of X, respectively.

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Recall that a sequence $(x_l) \subset X, x_l = (x_l(i))_{i=1}^{\infty}, l \in \mathbb{N}$, is said to be *$\varepsilon$-separated* sequence if the separation of sequence (x_l), $sep(x_l) = \inf\{\|x_l - x_m\| : l \neq m\} > \varepsilon$ for some $\varepsilon > 0$.

Define for any $x \notin B(X)$, the drop $D(x, B(X)) = conv(\{x\} \cup B(X))$, where $conv$ denotes the convex hull. Rolewicz [28] introduced the property (β) as follows:

For any subset C of X, the *Kuratowski measure of noncompactness* of C is defined as the infimum $\alpha(C)$ of those $\epsilon > 0$ for which there is a covering of C by a finite number of sets of diameter less than ϵ. Then a Banach space X is said to have the *property* (β) if, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|x\| \in (1, 1+\delta)$ implies

$$\alpha(D(x, B(X)) \backslash B(X)) < \epsilon.$$

A very useful characterization of property (β) was given by Kutzarova [23] in the following way:

Banach space X has the property (β) if and only if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for each element $x \in B(X)$ and each sequence $(x_n) \subset B(X)$ with $sep(x_n) \geq \epsilon$, there is an index k such that

$$\left\| \frac{x + x_k}{2} \right\| \leq 1 - \delta.$$

Rolewicz [28] proved that uniform convexity of X implies the property (β) and the property (β) implies *nearly uniformly convex (or nearly uniform convexity)* (NUC). Therefore, we have

$$\begin{array}{ccc} \text{Banach-Saks Property} \Leftarrow \text{Property } (\beta) \Rightarrow & (NUC) \Rightarrow (D) \Rightarrow \text{Reflexivity} \\ & \Downarrow \\ & (UKK) \Rightarrow \text{Property } (H). \end{array}$$

We refer to the reader [11] for definitions and above implications of the property (β). Let $k \geq 1$ be an integer. A Banach space is said to have the $(k\text{-}\beta)$ property (see [22]) if for each $\epsilon > 0$ there exists a δ, $0 < \delta < 1$ such that for every element $x \in B(X)$ and any sequence $(x_l) \subset B(X)$ with $sep(x_l) > \epsilon$ there are indices $l_1, l_2, \ldots, l_k \in \mathbb{N}$ for which

$$\left\| \frac{x + x_{l_1} + x_{l_2} + \ldots + x_{l_k}}{k+1} \right\| \leq 1 - \delta \text{ holds.}$$

Note that $(1\text{-}\beta)$ property coincides with (β) property. Let $k \geq 2$ be an integer. A Banach space is said to be the k-nearly uniformly convex property $(k\text{-}NUC)$ (see [22]) if for any $\epsilon > 0$, there exists a $\delta > 0$ such that for every sequence $(x_l) \subset B(X)$ with $sep(x_l) > \epsilon$ there are indices $l_1, l_2, \ldots, l_k \in \mathbb{N}$ for which

$$\left\| \frac{x_{l_1} + x_{l_2} + \ldots + x_{l_k}}{k} \right\| \leq 1 - \delta \text{ holds.}$$

Kutzarova [22] has shown that if a Banach space X is $(k\text{-}NUC)$ for some integer $k \geq 2$, then X is (NUC). But the converse is not true in general, for example, the *Baernstein* space B is (NUC) (see [2]) but it is not $(k\text{-}NUC)$ for any integer $k \geq 2$. Further, it is proved that for any Banach space X, $(k\text{-}\beta) \Rightarrow ((k+1)\text{-}NUC)$ for every $k \geq 1$ and $(k\text{-}NUC) \Rightarrow (k\text{-}\beta)$ for every $k \geq 2$. Indeed, $(k\text{-}\beta) \Rightarrow ((k+1)\text{-}\beta)$ for $k \geq 1$ and hence if X is $(k\text{-}\beta)$ for $k \geq 1$ then X is $(k\text{-}NUC)$ for $k \geq 2$. But $(k\text{-}\beta)$ spaces and hence $((k+1)\text{-}NUC)$ spaces for $k \geq 1$ need not be $(1\text{-}\beta)$ spaces (i.e., spaces with (β)-property). For example, *Schachermayer's* space is $(8\text{-}NUC)$ that is $(8\text{-}\beta)$ but not $(1\text{-}\beta)$ (see [22] for details).

A map $\varphi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function (see [5]) if it is an even, convex, continuous at 0, left continuous on $\mathbb{R}_+$, $\varphi(0) = 0$ and $\varphi(u) \to \infty$ as $u \to \infty$. A sequence $\Phi = (\varphi_n)_{n=1}^{\infty}$ of Orlicz functions φ_n is called Musielak–Orlicz function. We say that a Musielak–Orlicz function Φ satisfies condition (∞_1) if

$$\lim_{u\to+\infty} \frac{\varphi_n(u)}{u} = +\infty \quad \text{for each } n \in \mathbb{N}. \qquad (\infty_1)$$

For a Musielak–Orlicz function Φ, its complementary function $\Psi = (\psi_n)_{n=1}^{\infty}$ of Φ in the sense of Young is defined as below:

$$\psi_n(u) = \sup_{v\geq 0}\{\, |u|v - \varphi_n(v)\} \quad \text{for all } u \in \mathbb{R} \quad \text{and} \quad n \in \mathbb{N}.$$

Let $(E_n, \|\cdot\|_n)$ be finite dimensional real Banach spaces for each $n \in \mathbb{N}$. For a given sequence (E_n), we consider elements from the cartesian product $\prod_{n=1}^{\infty} E_n$, namely sequences $x = (x(n))_{n=1}^{\infty}$ such that $x(n) \in E_n$ for each $n \in \mathbb{N}$. For a given Musielak–Orlicz function Φ, on $\prod_{n=1}^{\infty} E_n$ a convex modular $\sigma_\Phi(x)$ is defined as follows:

$$\sigma_\Phi(x) = \sum_{n=1}^{\infty} \varphi_n(\|x(n)\|_n)$$

and the linear space

$$l_\Phi((E_n)) = \left\{x \in \prod_{n=1}^{\infty} E_n : \sigma_\Phi(rx) < \infty \text{ for some } \ r > 0\right\}$$

is called Musielak–Orlicz sequence space generated by a Musielak–Orlicz function Φ and a sequence (E_n) of finite dimensional spaces. Throughout this paper, $\|\cdot\|_n$ will be written by $\|\cdot\|$ in order to avoid ambiguity. We consider $l_\Phi((E_n))$ induced by the Luxemburg norm $\|\cdot\|_\Phi^L$ and the Amemiya norm $\|\cdot\|_\Phi^A$ are defined below:

$$\|x\|_\Phi^L = \inf\left\{r > 0 : \sigma_\Phi\left(\frac{x}{r}\right) \leq 1\right\}$$

$$\text{and } \|x\|_\Phi^A = \inf_{k>0}\left\{\frac{1}{k}(1 + \sigma_\Phi(kx))\right\}.$$

Indeed, these two norms are equivalent as evident from the inequality $\|x\|_\Phi^L \leq \|x\|_\Phi^A \leq 2\|x\|_\Phi^L$ (see [5]). The Musielak–Orlicz sequence space $l_\Phi((E_n))$ equipped with the norms $\|\cdot\|_\Phi^L$ as well as $\|\cdot\|_\Phi^A$ forms a Banach space denoted by $l_\Phi^L((E_n))$ and $l_\Phi^A((E_n))$, respectively. It is to be pointed out here that for any $x \in l_\Phi^A((E_n))$ there exists a $k > 0$ such that $\|x\|_\Phi^A = \frac{1}{k}(1 + \sigma_\Phi(kx))$ whenever for each $n \in \mathbb{N}$, $\frac{\varphi_n(u)}{u} \to \infty$ as $u \to \infty$. If $\varphi(t) = |t|^p$ for $1 \leq p < \infty$ and $\varphi_n(t) = |t|^{p_n}$ for $1 \leq \hat{p} < \infty$, $\hat{p} = (p_n)$, $n \in \mathbb{N}$, then $l_\Phi((E_n))$ reduces to $l_p((E_n))$ and $l_{\hat{p}}((E_n))$, respectively. Basic properties of Orlicz function and deep results on the geometry of Orlicz spaces have been found in the dissertation of Chen (see [5]).

A Musielak–Orlicz function $\Phi = (\varphi_n)_{n=1}^{\infty}$ satisfies the δ_2^0-condition, denoted by $\Phi \in \delta_2^0$, if there are positive constants a, K, a natural number m and a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers such that $(c_n)_{n=m}^{\infty} \in l_1$ and the inequality

$$\varphi_n(2u) \leq K\varphi_n(u) + c_n \qquad (1)$$

holds for every $n \in \mathbb{N}$ and $u \in \mathbb{R}$ whenever $\varphi_n(u) \leq a$. If a Musielak–Orlicz function Φ satisfies the δ_2^0-condition with $m = 1$, then Φ is said to be satisfying the δ_2-condition (see

[17]). A Musielak–Orlicz function $\Phi = (\varphi_n)_{n=1}^{\infty}$ satisfies the condition $(*)$ (see [18]) if for any $\varepsilon \in (0, 1)$ there is a $\delta > 0$ such that

$$\varphi_n(u) < 1 - \varepsilon \text{ implies } \varphi_n((1+\delta)u) \leq 1, \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad u \geq 0. \tag{2}$$

A Musielak–Orlicz function Φ is to said to be vanishing only at zero, which is denoted by $\Phi > 0$, if $\varphi_n(u) > 0$ for any $n \in \mathbb{N}$ and $u > 0$.
The Musielak–Orlicz–Cesàro space ces_Φ was introduced by Wangkeeree in [31] and it was defined by

$$ces_\Phi = \left\{ x \in l^0 : \varsigma_\Phi(rx) = \sum_{n=1}^{\infty} \varphi_n \left(\frac{r}{n} \sum_{k=1}^{n} |x(k)| \right) < \infty \text{ for some } r > 0 \right\}.$$

The sequence space ces_Φ is a normed linear space equipped with both the Luxemburg norm $\| \cdot \|_\Phi^L$ and the Amemiya norm $\| \cdot \|_\Phi^A$ defined similarly for the convex modular $\varsigma_\Phi(x)$. When a Musielak–Orlicz function Φ is replaced by an Orlicz function φ only, then ces_Φ reduces to the Orlicz–Cesàro space ces_φ studied by Cui et al. [10]. Several geometric properties for the spaces ces_φ, Cesàro function spaces and ces_Φ are considered in [10,16,19,21] and [31], respectively.

In recent years, a quite attention is given to the study of certain geometric properties such as the Kadec–Klee property (H)-property, uniform Kadec–Klee property, uniform Opial property, (β)-property, rotundity, locally uniform rotundity, nearly uniform convexity (NUC), k-nearly uniform convexity $(k\text{-}NUC)$, $k \geq 2$ etc. for Cesàro spaces, Cesàro–Orlicz sequence spaces, Musielak–Orlicz sequence spaces and others. For instance, from a geometric point of view, the property (β) is extensively studied in many research articles, for example in [6], and [7]. The property (β) is one of the most significant geometric properties of Banach space because if a Banach space X has the (β)-property then it implies that X is reflexive, both X and its dual X^* have the fixed point property, X is (NUC), has the (H) property and *drop property*. On the other hand, property $(k\text{-}NUC)$, $k \geq 2$ is studied for Cesàro spaces in [8], for Orlicz spaces in [9], for generalized Cesàro spaces in [30] and for Cesàro–Musielak–Orlicz sequence spaces in [31].

Saejung [29] studied the geometry of Cesàro sequence spaces ces_p for $1 < p < \infty$ using an alternative approach. Indeed, he proved that ces_p for $1 < p < \infty$ are isometrically embedded in the infinite l_p-sum $l_p((\mathbb{R}^n))$ of finite dimensional spaces $\mathbb{R}^n$ and studied the property (β) and the uniform Opial property of $l_p((\mathbb{R}^n))$. As these properties are inherited by isometric subspaces so ces_p possess these properties too. In the direction of Saejung [29], we shall first establish that ces_Φ is linearly isometric with a closed subspace of the space $l_\Phi((E_n))$ generated by an Musielak–Orlicz function Φ and a sequence (E_n) of finite dimensional spaces E_n, $n \in \mathbb{N}$. Similarly, we then show that the space $l_\Phi((E_n))$ has the property $(k\text{-}\beta)$ for fixed integer $k \geq 1$ induced by both the Luxemburg and the Amemiya norms. Since the property $(k\text{-}\beta)$ is inherited by subspaces, consequently ces_Φ will have the same property. As a consequence, we obtain parallel results related to the property (β), $(k\text{-}NUC)$ property for fixed integer $k \geq 2$ of the spaces such as Cesàro in [8], Orlicz in [9], generalized Cesàro in [30], Cesàro–Musielak–Orlicz in [31], Cesàro–Orlicz in [10]. Further, in the last section of this paper we shall give some applications to $ces(\alpha, p)$, the Cesàro spaces of order α and $O_{\hat{p}}^{(m)}$, the Cesàro difference spaces of order m.

2 Main results

First, we present the following lemma.

Lemma 1 *The Musielak–Orlicz–Cesàro space ces_Φ is linearly isometric with a closed subspace of $l_\Phi((\mathbb{R}^n))$, where $\mathbb{R}^n$ is the Euclidean space endowed with the following norm: $\|(\alpha_1, \ldots, \alpha_n)\| = \sum_{i=1}^n |\alpha_i|$ for $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$.*

Proof For all $x = (x(i)) \in ces_\Phi$, the following linear isometric map $T : ces_\Phi \to l_\Phi((\mathbb{R}^n))$ is defined:

$$T(x(i)) = \left(x(1), \left(\frac{x(1)}{2}, \frac{x(2)}{2}\right), \ldots, \left(\frac{x(1)}{n}, \frac{x(2)}{n}, \ldots, \frac{x(n)}{n}\right), \ldots\right).$$

Then

$$\begin{aligned}
&\|T((x(i)))\|_{l_\Phi((\mathbb{R}^n))} \\
&= \|T(x(1), x(2), \ldots, x(i), \ldots)\|_{l_\Phi((\mathbb{R}^n))} \\
&= \left\|\left(x(1), \left(\frac{x(1)}{2}, \frac{x(2)}{2}\right), \ldots, \left(\frac{x(1)}{n}, \frac{x(2)}{n}, \ldots, \frac{x(n)}{n}\right), \ldots\right)\right\|_{l_\Phi((\mathbb{R}^n))} \\
&= \inf\left\{r > 0 : \sum_{n=1}^{\infty} \varphi_n\left(\frac{1}{rn}\sum_{k=1}^{n}|x(k)|\right) \le 1\right\} \\
&= \inf\left\{r > 0 : \varsigma_\Phi\left(\frac{x}{r}\right) \le 1\right\} \\
&= \|(x(i))\|_{ces_\Phi}.
\end{aligned}$$

Hence the lemma is proved. □

Note Instead of studying geometric properties of ces_Φ it is enough to study geometric properties of $l_\Phi((\mathbb{R}^n))$ and if such geometric properties are inherited by subspaces then ces_Φ will have the same properties. Thanks to Lemma 1 conditions that are sufficient for some geometric properties of $l_\Phi((\mathbb{R}^n))$ are also sufficient for these geometric properties of ces_Φ.

To establish our results, the following important lemma will be needed.

Lemma 2 *Let Ψ be a complementary function to Φ. Then $\Psi \in \delta_2$ if and only if there exist constants $\theta \in (0, 1)$, $\beta \in (0, 1)$, $u_0 > 0$ and a sequence (c_n) of non negative real numbers such that $(c_n) \in l_1$ and*

$$\varphi_n(\beta u) \le (1 - \theta)\beta\varphi_n(u) + c_n$$

holds for every $u \in \mathbb{R}$ satisfying $\varphi_n(u) \le u_0$ for each $n \in \mathbb{N}$. The result also holds when $u_0 = 1$.

Proof The proof of this lemma is a combination of Lemma 2.5. and Remark 2.0.1. presented in [26]. So it is omitted. □

Define $h_\Phi((E_n))$ is a subspace of $l_\Phi((E_n))$ as

$$h_\Phi((E_n)) = \{x \in l^0 : \sigma_\Phi(rx) < \infty, \text{ for all } r > 0\},$$

equipped with both the Luxemburg norm $\|\cdot\|_\Phi^L$ and the Amemiya norm $\|\cdot\|_\Phi^A$ and denoted by $h_\Phi^L((E_n))$ and $h_\Phi^A((E_n))$, respectively.

We assume that the Musielak–Orlicz function $\Phi = (\varphi_n)_{n=1}^\infty$ is finite. In the sequel, the following known lemmas are used:

Lemma 3 *Let* $x \in h_\Phi((E_n))$ *be an arbitrary element. Then* $\|x\|_\Phi^L = 1$ *if and only if* $\sigma_\Phi(x) = 1$.

Proof The proof will run on the parallel lines of the proof of Lemma 2.1 given in [10]. □

Lemma 4 *Suppose* $\Phi \in \delta_2$ *and* $\Phi > 0$. *Then for any* $(x_l) \subset l_\Phi((E_n))$, $\|x_l\|_\Phi^L \to 0$ $(\|x_l\|_\Phi^A \to 0)$ *if and only if* $\sigma_\Phi(x_l) \to 0$.

Proof For the proof of this lemma, the work of Kamińska (see [18]) is referred to the reader. □

Lemma 5 *If* $\Phi \in \delta_2$, *i.e., inequality (1) holds, then for any* $x \in l_\Phi((E_n))$,

$$\|x\|_\Phi^L = 1 \quad \textit{if and only if} \quad \sigma_\Phi(x) = 1.$$

Proof Since $\Phi \in \delta_2$ implies that $l_\Phi((E_n)) = h_\Phi((E_n))$, the proof follows from Lemma 3. □

Lemma 6 *Suppose* $\Phi \in \delta_2$, *i.e., inequality* (1) *holds and* Φ *satisfies the condition* $(*)$, *i.e., inequality* (2) *holds. Then for any* $x \in l_\Phi((E_n))$ *and every* $\epsilon \in (0, 1)$ *there exists* $\delta(\epsilon) \in (0, 1)$ *such that* $\sigma_\Phi(x) \leq 1 - \epsilon$ *implies* $\|x\|_\Phi^L \leq 1 - \delta$.

Proof The proof of this lemma can be given in a similar way as the proof of Lemma 9 in [18]. □

Lemma 7 *Let* $\Phi \in \delta_2$, *i.e., inequality* (1) *holds,* $\Phi > 0$ *and satisfies the condition* $(*)$, *i.e., inequality* (2) *holds. Then for each* $d \in (0, 1)$ *and* $\epsilon > 0$ *there exists* $\delta = \delta(d, \epsilon) > 0$ *such that* $\sigma_\Phi(u) \leq d$, $\sigma_\Phi(v) \leq \delta$ *imply*

$$|\sigma_\Phi(u + v) - \sigma_\Phi(u)| < \epsilon \quad \textit{for any} \ \ u, v \in l_\Phi((E_n)). \tag{3}$$

Proof For the proof of this lemma, the authors refer any one of the references [10,18,25] to the reader. □

Now, we are in a position to state our first result. The result is as follows:

Theorem 1 *Let* $\Phi = (\varphi_n)_{n=1}^\infty$ *be a Musielak–Orlicz function vanishing only at zero and* $\Psi = (\psi_n)_{n=1}^\infty$ *be the complementary function to* Φ. *If* $\Phi \in \delta_2, \Psi \in \delta_2$ *and* Φ *satisfies the condition* $(*)$, *i.e., inequality (2) holds. Then* $l_\Phi^L((E_n))$ *has the* $(k$-$\beta)$*-property for any fixed integer* $k \geq 1$.

Proof Let $k \geq 1$, $\epsilon > 0$ be arbitrary. Choose $x \in Bl_\Phi^L((E_n))$, $(x_l)_{l=1}^\infty \subset Bl_\Phi^L((E_n))$, $x_l = (x_l(i))_{i=1}^\infty$, $l \in \mathbb{N}$, be such that $sep(x_l) > \epsilon$. For each $m \in \mathbb{N}$ denote

$$x_l^m = (0, 0, \ldots, 0, x_l(m), x_l(m+1), \ldots).$$

Since for each $i \in \mathbb{N}$, the sequence $(x_l(i))_{l=1}^\infty$ is bounded, so by Bolzano–Weierstrass theorem $(x_l(i))_{l=1}^\infty$ has convergent subsequence for each $i \in \mathbb{N}$. By Cantor's diagonal method one can find a subsequence $(x_{l_k})_{k=1}^\infty$ of $(x_l)_{l=1}^\infty$ such that for each $i \in \mathbb{N}$, $(x_{l_k}(i))_{k=1}^\infty$ converges, i.e., the sequence $(x_{l_k}(i))_{k=1}^\infty$ converges pointwise and so one can make the coordinates $x_{l_k}(1), x_{l_k}(2), \ldots, x_{l_k}(m-1)$ differ by as a little as one want for k sufficiently large. Since $(x_{l_k})_{k=1}^\infty$ is a subsequence of $(x_l)_{l=1}^\infty$, so one gets $\epsilon < sep(x_l) \leq sep(x_{l_k})$. Therefore, for every $m \in \mathbb{N}$, there exists a $k_m \in \mathbb{N}$ such that $sep(x_{l_k}^m) \geq \epsilon$ for all $k \geq k_m$. Hence by definition of the separation of sequence, for each $m \in \mathbb{N}$, there exists $l_m \in \mathbb{N}$ such that

$$\|x_{l_m}^m\|_\Phi^L \geq \frac{\min(\epsilon, 1)}{2} \text{ holds.} \tag{4}$$

By Lemma 4, there exists $\eta \in (0,1)$ such that $\sigma_\Phi(x) \geq \eta$ whenever $\|x\|_\Phi^L \geq \frac{\min(\epsilon,1)}{2}$, $x \in l_\Phi^L((E_n))$. Defining $\epsilon_1 = \frac{\eta\theta}{4k(k+1)}$, where $\theta = \theta(k)$ is the constant from Lemma 2. Since Φ vanishes only at zero and satisfy the conditions δ_2 and $(*)$, so by Lemma 7 there exists a $\delta = \delta(1,\epsilon_1) > 0$ such that $\sigma_\Phi(u) \leq 1$ and $\sigma_\Phi(v) \leq \delta$ imply

$$|\sigma_\Phi(u+v) - \sigma_\Phi(u)| < \epsilon_1. \tag{5}$$

Without loss of generality, one can assume that $\delta = \delta(1,\epsilon_1) \leq \frac{\eta}{2}$. Setting the constants m_1, $m_2, m_3, \ldots, m_{k-1} \in \mathbb{N}$ such that $m_1 < m_2 < m_3 < m_4 < \ldots < m_{k-1}$ for $x_1 = x_{l_1}$, $x_2 = x_{l_2}, \ldots, x_{k-1} = x_{l_{k-1}}$. Since $x, x_j \in B(l_\Phi^L((E_n)))$, so for every $\delta > 0$ there exist natural constants $m_1, m_2, m_3, \ldots, m_{k-1}$, one obtain $\sigma_\Phi(x^{m_1}) \leq \delta$, and $\sigma_\Phi(x_j^{m_j}) \leq \delta$ for all $j = 1, 2, \ldots, k-1$. Since $(c_n)_{n=1}^\infty \in l_1$ as in Lemma 2, so it can be assumed that $\sum_{n=m_k+1}^\infty c_n \leq \frac{\eta\theta}{2(k+1)}$. By inequality (4) there exists $l_k \in \mathbb{N}$ such that $\|x_{l_k}^{m_k+1}\|_\Phi^L \geq \frac{\min(\epsilon,1)}{2}$. Consequently, one gets $\sigma_\Phi(x_{l_k}^{m_k+1}) \geq \eta$. Since Φ is convex, so applying Lemma 2 and using the inequality (5), one obtains

$$\begin{aligned}
\sigma_\Phi\Big(\frac{x + x_{l_1} + x_{l_2} + \ldots + x_{l_k}}{k+1}\Big) &= \sum_{n=1}^{m_1} \varphi_n\Big(\Big\|\frac{x(n) + x_{l_1}(n) + x_{l_2}(n) + \ldots + x_{l_k}(n)}{k+1}\Big\|\Big) \\
&\quad + \sum_{n=m_1+1}^{\infty} \varphi_n\Big(\Big\|\frac{x(n) + x_{l_1}(n) + x_{l_2}(n) + \ldots + x_{l_k}(n)}{k+1}\Big\|\Big) \\
&\leq \frac{1}{k+1}\sum_{n=1}^{m_1}\Big\{\varphi_n(\|x(n)\|) + \sum_{i=1}^{k}\varphi_n(\|x_{l_i}(n)\|)\Big\} + \sum_{n=m_1+1}^{\infty}\varphi_n\Big(\Big\|\frac{x_{l_1}(n) + \ldots + x_{l_k}(n)}{k+1}\Big\|\Big) \\
&\quad + \epsilon_1 \qquad \text{[Using inequality (5) and the fact that each } \varphi_n \text{ is convex]} \\
&= \frac{1}{k+1}\sum_{n=1}^{m_1}\varphi_n(\|x(n)\|) + \frac{1}{k+1}\sum_{n=1}^{m_1}\sum_{i=1}^{k}\varphi_n(\|x_{l_i}(n)\|) \\
&\quad + \sum_{n=m_1+1}^{m_2}\varphi_n\Big(\Big\|\frac{x_{l_1}(n) + \ldots + x_{l_k}(n)}{k+1}\Big\|\Big) + \sum_{n=m_2+1}^{\infty}\varphi_n\Big(\Big\|\frac{x_{l_1}(n) + \ldots + x_{l_k}(n)}{k+1}\Big\|\Big) + \epsilon_1 \\
&\leq \frac{1}{k+1}\sigma_\Phi(x) + \frac{1}{k+1}\sum_{n=1}^{m_1}\sum_{i=1}^{k}\varphi_n(\|x_{l_i}(n)\|) + \sum_{n=m_1+1}^{m_2}\varphi_n\Big(\Big\|\frac{x_{l_1}(n) + \ldots + x_{l_k}(n)}{k+1}\Big\|\Big) \\
&\quad + \sum_{n=m_2+1}^{\infty}\varphi_n\Big(\Big\|\frac{x_{l_2}(n) + \ldots + x_{l_k}(n)}{k+1}\Big\|\Big) + 2\epsilon_1.
\end{aligned}$$

Now repeating the same process using inequality (5) k times and since $\sigma_\Phi(x) = 1$, one gets

$$\begin{aligned}
&\sigma_\Phi\Big(\frac{x + x_{l_1} + x_{l_2} + \ldots + x_{l_k}}{k+1}\Big) \\
&\quad \leq \frac{1}{k+1} + \frac{1}{k+1}\sum_{n=1}^{m_1}\sum_{i=1}^{k}\varphi_n(\|x_{l_i}(n)\|) + \frac{1}{k+1}\sum_{n=m_1+1}^{m_2}\sum_{i=1}^{k}\varphi_n(\|x_{l_i}(n)\|)
\end{aligned}$$

$$+\frac{1}{k+1}\sum_{n=m_2+1}^{m_3}\sum_{i=1}^{k}\varphi_n(\|x_{l_i}(n)\|)+\ldots+\frac{1}{k+1}\sum_{n=m_{k-1}+1}^{m_k}\sum_{i=1}^{k}\varphi_n(\|x_{l_i}(n)\|)$$
$$+\sum_{n=m_k+1}^{\infty}\varphi_n\Big(\Big\|\frac{x_{l_k}(n)}{k+1}\Big\|\Big)+k\epsilon_1.$$

Now from each term, we separate $i = k$th term and the rest of $(k-1)$ terms are taken together and since $\sigma_\Phi(x_{l_i}) = 1$ for $i = 1, 2, \ldots, k-1$, so from the right hand side of the last inequality, one obtains

$$\frac{1}{k+1}+\frac{1}{k+1}\sum_{i=1}^{k-1}\Big\{\sum_{n=1}^{m_1}\varphi_n(\|x_{l_i}(n)\|)+\sum_{n=m_1+1}^{m_2}\varphi_n(\|x_{l_i}(n)\|)+\cdots$$
$$+\sum_{n=m_{k-1}+1}^{m_k}\varphi_n(\|x_{l_i}(n)\|)\Big\}+\frac{1}{k+1}\Big\{\sum_{n=1}^{m_1}\varphi_n(\|x_{l_k}(n)\|)$$
$$+\sum_{n=m_1+1}^{m_2}\varphi_n(\|x_{l_k}(n)\|)+\cdots+\sum_{n=m_{k-1}+1}^{m_k}\varphi_n(\|x_{l_k}(n)\|)\Big\}$$
$$+\sum_{n=m_k+1}^{\infty}\varphi_n\Big(\Big\|\frac{x_{l_k}(n)}{k+1}\Big\|\Big)+k\epsilon_1$$
$$\le\frac{1}{k+1}+\frac{1}{k+1}\sum_{i=1}^{k-1}\sigma_\Phi(x_{l_i})+\frac{1}{k+1}\sum_{n=1}^{m_k}\varphi_n(\|x_{l_k}(n)\|)$$
$$+\sum_{n=m_k+1}^{\infty}\varphi_n\Big(\Big\|\frac{x_{l_k}(n)}{k+1}\Big\|\Big)+k\epsilon_1\le\frac{k}{k+1}+\frac{1}{k+1}\sum_{n=1}^{m_k}\varphi_n(\|x_{l_k}(n)\|)$$
$$+\frac{1-\theta}{k+1}\sum_{n=m_k+1}^{\infty}\varphi_n(\|x_{l_k}(n)\|)+\sum_{n=m_k+1}^{\infty}c_n+k\epsilon_1$$
$$\le\frac{k}{k+1}+\frac{1}{k+1}\sum_{n=1}^{\infty}\varphi_n(\|x_{l_k}(n)\|)-\frac{\theta}{k+1}\sum_{n=m_k+1}^{\infty}\varphi_n(\|x_{l_k}(n)\|)+\frac{\eta\theta}{2(k+1)}+k\epsilon_1$$
$$\le 1-\frac{\theta}{k+1}.\eta+\frac{\eta\theta}{2(k+1)}+k\epsilon_1=1-\frac{\eta\theta}{(k+1)}$$
$$+\frac{\eta\theta}{2(k+1)}+\frac{\eta\theta}{4(k+1)}=1-\frac{\eta\theta}{4(k+1)},$$

where $\epsilon_1 = \frac{\eta\theta}{4k(k+1)}$.

Hence, by Lemma 6, there exists $\tau(=\frac{\eta\theta}{4(k+1)}) > 0$ such that $\Big\|\frac{x+x_{l_1}+x_{l_2}+\ldots+x_{l_k}}{k+1}\Big\|_\Phi^L < 1-\tau$. Thus the space $l_\Phi^L((E_n))$ has the property $(k\text{-}\beta)$ for integer $k \ge 1$. □

Corollary 1 *If $E_n = \mathbb{R}^1$ for any $n \in \mathbb{N}$, then l_Φ^L has the property $(k\text{-}\beta)$ for $k \ge 1$. In particular, when $\varphi_n = \varphi$ for all $n \in \mathbb{N}$, $k = 1$ then l_φ^L has the property (β) as obtained by Cui et al.* [6] *and Cui and Thompson* [12]. *In addition, l_φ^L is $(k\text{-}NUC)$ for any $k \ge 2$ as obtained by Cui et al.* [9].

Corollary 2 *If $\varphi_n(u) = |u|^{p_n}$ with $p_n = p$ for all $n \in \mathbb{N}$, $E_n = \mathbb{R}^n$, $n \in \mathbb{N}$ and $1 < p < \infty$, then $l_p^L((E_n))$ possesses the property (k-β) for $k \geq 1$. Hence by Lemma 1, we have ces_p^L possesses the property (k-β) for $k \geq 1$ too and therefore sequence space ces_p^L has the property (β) as obtained by Cui and Meng [7] and is (k-NUC) for any $k \geq 2$ as obtained by Cui and Hudzik [8].*

Corollary 3 *If $\varphi_n(u) = |u|^{p_n}$ with $\liminf_{n\to\infty} p_n > 1$ and $E_n = \mathbb{R}^n$, $n \in \mathbb{N}$, then the Nakano sequence space $l_{\hat{p}}((E_n))$ has the property (k-β) for $k \geq 1$ and hence by Lemma 1 the space $ces^L(p)$ is (k-NUC) for any $k \geq 2$ established by Sanhan and Suantai [30]. When $E_n = \mathbb{R}^1$ for any $n \in \mathbb{N}$, $k = 1$, then $l_{\hat{p}}$ has the property (β) as obtained by Dhompongsa [13].*

Corollary 4 *Suppose $E_n = \mathbb{R}^n$, $n \in \mathbb{N}$. Since $l_\Phi^L((E_n))$ has the property (k-β) for integer $k \geq 1$, so by Lemma 1, the sequence space ces_Φ^L is (k-NUC) for any $k \geq 2$ as studied by Wangkeeree [31].*

Theorem 2 *Let Φ be a Musielak–Orlicz function satisfying the condition (∞_1), vanishing only at zero and $\Psi = (\psi_n)_{n=1}^\infty$ be the complementary function to Φ. If $\Phi \in \delta_2$, $\Psi \in \delta_2$ and Φ satisfies the condition ($*$), i.e., inequality (2) holds, then $l_\Phi^A((E_n))$ has the (k-β)-property for any fixed integer $k \geq 1$.*

Proof Let $k \geq 1$, $\epsilon > 0$ be arbitrary. Take $x \in B(l_\Phi^A((E_n)))$, $x_l \in B(l_\Phi^A((E_n)))$, $x_l = (x_l(i))_{i=1}^\infty$, $l \in \mathbb{N}$ be such that $sep(x_l) > \epsilon$. As in the previous theorem, for each $m \in \mathbb{N}$, we denote $x_l^m = (0, 0, \ldots, 0, x_l(m), x_l(m+1), \ldots)$. Consider the subsequence $(x_{l_k})_{k=1}^\infty$ of $(x_l)_{l=1}^\infty$. Hence proceeds in a similar way as first part of the Theorem 1, for every $m \in \mathbb{N}$ there exists a $l_m \in \mathbb{N}$, one obtains

$$\|x_{l_m}^m\|_\Phi^L \geq \frac{\min(\epsilon, 1)}{2}.$$

This inequality and Lemma 4 together imply that, there exist $\eta \in (0, 1)$, $l_k, m_1 \in \mathbb{N}$ such that

$$\sigma_\Phi(x_{l_k}^{m_1+1}) \geq \eta \text{ holds.} \tag{6}$$

Choose $k_0,\ k_n \geq 1, n \in \mathbb{N}$ such that

$$\|x_0\|_\Phi^A = \frac{1}{k_0}(1 + \sigma_\Phi(k_0 x_n)) \ \& \ \|x_n\|_\Phi^A = \frac{1}{k_n}(1 + \sigma_\Phi(k_n x_n)) \text{ holds.}$$

Since $\Psi \in \delta_2$, so the sequence (k_n) is bounded (see [5]). Take $\sup\{k_n :\ n \in \mathbb{N}\} = M$. It is obvious that M is finite. We consider fixed natural numbers $l_1, l_2, \ldots, l_k$ such that $l_1 < l_2 < \cdots < l_k$.

Denote (here the following symbol $\prod$ indicates the product of real numbers)

$$G = k_0 \prod_{i=1}^{k} k_{l_i},\ \ g_0 = k_{l_1} k_{l_2} \ldots k_{l_k}, g_j = k_0 \prod_{i \neq j} k_{l_i} \ \text{ for } \ 1 \leq j \leq k \ \text{ and } \ g = \frac{G}{\sum_{i=0}^{k} g_i}.$$

Note that $g_0 k_0 = G$, $g_i k_{l_i} = G$ for integer $1 \leq i \leq k$ and $\frac{g_k}{\sum_{i=0}^k g_i} = \frac{g_k}{g_k + \sum_{i=0}^{k-1} g_i} \leq \frac{M^k}{M^k+1} =: \mu$.

Choose $\varepsilon_1 = \frac{\eta\theta}{4(k+1)M^k}$, where $\theta = \theta(k)$ is a constant from Lemma 2. Since Φ vanishes only at zero and satisfy the conditions δ_2 and ($*$), so by Lemma 7 there exists a δ, $0 < \delta = \delta(1, \epsilon_1) < \frac{\eta}{2}$ such that $\sigma_\Phi(u) \leq M$ and $\sigma_\Phi(v) \leq \delta$ imply

$$|\sigma_\Phi(u + v) - \sigma_\Phi(u)| < \epsilon_1 \quad \text{for any } \ u, v \in l_\Phi^A((E_n)). \tag{7}$$

Since $\sigma_\Phi(x_0) < \infty$, $\sigma_\Phi(x_{l_i}) < \infty$ for $i = 1, 2, \ldots, (k-1)$ and $\sigma_\Phi\left(\frac{x_0+x_{l_1}+\ldots+x_{l_{k-1}}}{k+1}\right) < \infty$, so there exists a constant $m_1 \in \mathbb{N}$ such that

$$\sum_{n=m_1+1}^{\infty} \varphi_n\left(\left\|\frac{x_0(n)+x_{l_1}(n)+\ldots+x_{l_{k-1}}(n)}{k+1}\right\|\right) \leq \delta,$$

$$\sum_{n=m_1+1}^{\infty} \varphi_n\left(\frac{g_k}{\sum_{i=0}^{k} g_i} k_{l_k}\|x_{l_k}(n)\|\right) \leq \frac{M^k}{M^k+1}\sum_{n=m_1+1}^{\infty} \varphi_n(M\|x_{l_k}(n)\|) \leq M.$$

It is to be noted that for $i = 1, 2, \ldots, (k-1)$ there exists $\eta \in (0,1)$ for which

$$\sigma_\Phi(x_{l_i}^{m_1+1}) = \sum_{n=m_1+1}^{\infty} \varphi_n\left(\|x_{l_i}(n)\|\right) \leq \delta < \eta \text{ holds.}$$

Since $\Psi \in \delta_2$, so by Lemma 2, there exists $\theta \in (0,1)$, $\mu \in (0,1)$ and a sequence (c_n) of non negative real numbers such that $(c_n) \in l_1$ and

$$\varphi_n(\mu u) \leq (1-\theta)\mu\varphi_n(u) + c_n$$

holds whenever $\varphi_n(\frac{u}{M}) \leq 1$ and for each $n \in \mathbb{N}$.

The convexity of Musielak–Orlicz function Φ implies that for any $\vartheta \in [0, \mu]$ such that

$$\varphi_n(\vartheta u) \leq (1-\theta)\vartheta\varphi_n(u) + c_n$$

holds whenever $\varphi_n(\frac{u}{M}) \leq 1$ and sequence $(c_n) \in l_1, n \in \mathbb{N}$. Thus for $\varphi_n(\frac{u}{M}) \leq 1$, one obtains

$$\varphi_n\left(\frac{g_k}{\sum\limits_{i=0}^{k} g_i} u\right) \leq (1-\theta)\frac{g_k}{\sum_{i=0}^{k} g_i}\varphi_n(u) + c_n. \tag{8}$$

Since $(c_n) \in l_1$, so there exists a $m_1 \in \mathbb{N}$ such that $\sum_{n=m_1+1}^{\infty} c_n \leq \frac{\eta\theta}{2(k+1)M^k}$. With the help of inequalities (6), (7) and (8), the definition of norm $\|\cdot\|_\Phi^A$ gives

$$\begin{aligned}
&\|x_0 + x_{l_1} + \cdots + x_{l_k}\|_\Phi^A \\
&= \frac{\sum_{i=0}^{k} g_i}{G}\left[1 + \sigma_\Phi\left(\frac{G}{\sum_{i=0}^{k} g_i}\left(x_0 + x_{l_1} + \cdots + x_{l_k}\right)\right)\right]\left(\text{where } G = k_0\prod_{i=1}^{k} k_{l_i}\right) \\
&= \frac{\sum_{i=0}^{k} g_i}{G}\left[1 + \sum_{n=1}^{m_1}\varphi_n\left(\frac{G}{\sum_{i=0}^{k} g_i}\left(\left\|x_0(n) + x_{l_1}(n) + \cdots + x_{l_k}(n)\right\|\right)\right)\right. \\
&\quad \left. + \sum_{n=m_1+1}^{\infty}\varphi_n\left(\frac{G}{\sum_{i=0}^{k} g_i}\left(\left\|x_0(n) + x_{l_1}(n) + \cdots + x_{l_k}(n)\right\|\right)\right)\right] \\
&\leq \frac{\sum_{i=0}^{k} g_i}{G}\left[1 + \sum_{n=1}^{m_1}\varphi_n\left(\frac{g_0}{\sum_{i=0}^{k} g_i}k_0\|x_0(n)\|\right.\right. \\
&\quad \left. + \frac{g_1}{\sum_{i=0}^{k} g_i}k_{l_1}\|x_{l_1}(n)\| + \cdots + \frac{g_k}{\sum_{i=0}^{k} g_i}k_{l_k}\|x_{l_k}(n)\|\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=m_1+1}^{\infty}\varphi_n\left(\frac{G}{\sum_{i=0}^k g_i}\left(\|x_0(n)+x_{l_1}(n)+\cdots+x_{l_{k-1}}(n)\|\right)+\frac{G}{\sum_{i=0}^k g_i}\|x_{l_k}(n)\|\right)\Bigg]\\
&\le\frac{\sum_{i=0}^k g_i}{G}\left[1+\sum\nolimits_{n=1}^{m_1}\left(\frac{g_0}{\sum\limits_{i=0}^k g_i}\varphi_n(k_0\|x_0(n)\|)+\frac{g_1}{\sum\limits_{i=0}^k g_i}\varphi_n(k_{l_1}\|x_{l_1}(n)\|)+\cdots\right.\right.\\
&\left.\left.\cdots+\frac{g_k}{\sum_{i=0}^k g_i}\varphi_n(k_{l_k}\|x_{l_k}(n)\|)\right)+\sum\nolimits_{n=m_1+1}^{\infty}\varphi_n\left(\frac{g_k}{\sum\limits_{i=0}^k g_i}k_{l_k}\|x_{l_k}(n)\|\right)+\varepsilon_1\right]\\
&=\frac{1}{k_0}+\sum_{n=1}^{m_1}\frac{1}{k_0}\varphi_n(k_0\|x_0(n)\|)+\sum_{j=1}^{k}\left\{\frac{1}{k_{l_j}}+\sum_{n=1}^{m_1}\frac{1}{k_{l_j}}\varphi_n(k_{l_j}\|x_{l_j}(n)\|)\right\}\\
&\quad+\frac{\sum_{i=0}^k g_i}{G}\left[\sum_{n=m_1+1}^{\infty}\varphi_n\left(\frac{g_k}{\sum_{i=0}^k g_i}k_{l_k}\|x_{l_k}(n)\|\right)+\varepsilon_1\right]\\
&\le\frac{1}{k_0}+\sum_{n=1}^{m_1}\frac{1}{k_0}\varphi_n(k_0\|x_0(n)\|)+\sum_{j=1}^{k}\left\{\frac{1}{k_{l_j}}+\sum_{n=1}^{m_1}\frac{1}{k_{l_j}}\varphi_n(k_{l_j}\|x_{l_j}(n)\|)\right\}\\
&\quad+\frac{\sum_{i=0}^k g_i}{G}\left[(1-\theta)\frac{g_k}{\sum_{i=0}^k g_i}\sum_{n=m_1+1}^{\infty}\varphi_n(k_{l_k}\|x_{l_k}(n)\|)+\sum_{n=m_1+1}^{\infty}c_n+\varepsilon_1\right]\\
&\le\|x_0\|_\Phi^A+\|x_{l_1}\|_\Phi^A+\cdots+\|x_{l_k}\|_\Phi^A-\theta\frac{g_k}{G}k_{l_k}\sum_{n=m_1+1}^{\infty}\varphi_n(\|x_{l_k}(n)\|)\\
&\quad+\frac{\sum\limits_{i=0}^k g_i}{G}\left(\sum_{n=m_1+1}^{\infty}c_n+\varepsilon_1\right)\le(k+1)-\eta\theta+(k+1)M^k\frac{\eta\theta}{2(k+1)M^k}\\
&\quad+(k+1)M^k\frac{\eta\theta}{4(k+1)M^k}=(k+1)\left(1-\frac{\eta\theta}{4(k+1)}\right).
\end{aligned}
$$

Therefore

$$
\left\|\frac{x_0+x_{l_1}+\cdots+x_{l_k}}{k+1}\right\|_\Phi^A\le 1-\frac{\eta\theta}{4(k+1)}.
$$

Thus for any fixed integer $k\ge 1$, the space $l_\Phi^A((E_n))$ possesses the property (k-β). □

Corollary 5 *If $E_n=\mathbb{R}^1$ for any $n\in\mathbb{N}$, then l_Φ^A has the property (k-β) for $k\ge 1$. In particular, when $\varphi_n=\varphi$ for all $n\in\mathbb{N}$, $k=1$ then l_φ^A has the property (β) as obtained by Cui et al.* [6]. *Further l_φ^A is (k-NUC) for any $k\ge 2$ as studied by Cui and Hudzik* [9].

Corollary 6 *Suppose $E_n=\mathbb{R}^n$, $n\in\mathbb{N}$. Since $l_\Phi^A((E_n))$ has the property (k-β) for $k\ge 1$, so by Lemma 1, we have that ces_Φ^A possesses the property (k-β) for each fixed integer $k\ge 1$.*

3 Some applications

In this section, the results related to the Cesàro sequence spaces of order $\alpha(\geq 1)$ and Cesàro difference sequence spaces of order m are discussed.

3.1 Cesàro sequence spaces of order α

First, we begin with the definition of Cesàro sequence spaces of order $\alpha(\geq 1)$. Let $p > 1$. Then Cesàro sequence spaces of order $\alpha(\geq 1)$ is denoted by $ces(\alpha, p)$ and defined by

$$ces(\alpha, p)=\left\{x \in l^0 : \sum_{n=0}^{\infty}\left(\frac{1}{\binom{n+\alpha}{n}}\sum_{k=0}^{n}\binom{n-k+\alpha-1}{n-k}|x(k)|\right)^p < \infty\right\} \text{ (see [3], pp. 113).}$$

Note that $ces(1, p)$ coincides with ces_p and the spaces $ces(\alpha, p)$ contain all l_p. Further, the spaces $ces(\alpha, p)$ do not depend on α for $\alpha \geq 1$. The $ces(\alpha, p)$ are Banach spaces with respect to the norm

$$\|x\|_{ces(\alpha,p)} = \left(\sum_{n=0}^{\infty}\left(\frac{1}{\binom{n+\alpha}{n}}\sum_{k=0}^{n}\binom{n-k+\alpha-1}{n-k}|x(k)|\right)^p\right)^{\frac{1}{p}}.$$

Recently, Braha [4] studied geometric properties such as (β)-property, $(k\text{-}NUC)$ property for integer $k \geq 2$ and uniform Opial property of the second order Cesàro space $ces(2, p)$. As these properties are inherited by subspaces so the results can be concluded immediately with the help of following Lemma 8 and Theorem 1. Now, we establish the following lemma.

Lemma 8 *The sequence space $ces(\alpha, p)$ is linearly isometric to a closed subspace in the infinite l_p-sum of finite dimensional spaces $l_p((E_{n+1}))$, where $E_{n+1} = \mathbb{R}^{n+1}$, $n \in \mathbb{N}_0$ is the $(n+1)$-dimensional Euclidean space equipped with the norm defined below:*

$$\|(\alpha_0, \alpha_1, \ldots, \alpha_n)\| = \sum_{i=0}^{n}|\alpha_i| \quad \text{for } (\alpha_0, \alpha_1, \ldots, \alpha_n) \in E_{n+1}. \tag{9}$$

Proof For all $x = (x(i)) \in ces(\alpha, p)$, the following linear isometric map $T : ces(\alpha, p) \to l_p((E_{n+1}))$ is defined:

$$T((x(i))) = \left(x(0), \left(\frac{\binom{\alpha}{1}}{\binom{\alpha+1}{1}}x(0), \frac{\binom{\alpha-1}{0}}{\binom{\alpha+1}{1}}x(1)\right), \ldots, \left(\frac{\binom{\alpha+n-1}{n}}{\binom{\alpha+n}{n}}x(0), \ldots, \frac{\binom{\alpha-1}{0}}{\binom{\alpha+n}{n}}x(n)\right), \ldots\right).$$

Then

$$\begin{aligned}\|T((x(i)))\|_{l_p((E_{n+1}))} &= \|T(x(0), x(1), \ldots, x(i), \ldots)\|_{l_p((E_{n+1}))}\\ &= \left\|\left(x(0), \left(\frac{\binom{\alpha}{1}}{\binom{\alpha+1}{1}}x(0), \frac{\binom{\alpha-1}{0}}{\binom{\alpha+1}{1}}x(1)\right), \ldots, \left(\frac{\binom{\alpha+n-1}{n}}{\binom{\alpha+n}{n}}x(0), \ldots, \frac{\binom{\alpha-1}{0}}{\binom{\alpha+n}{n}}x(n)\right), \ldots\right)\right\|_{l_p((E_{n+1}))}\\ &= \left(\sum_{n=0}^{\infty}\left(\frac{1}{\binom{n+\alpha}{n}}\sum_{k=0}^{n}\binom{n-k+\alpha-1}{n-k}|x(k)|\right)^p\right)^{\frac{1}{p}}\\ &= \|(x(i))\|_{ces(\alpha,p)}.\end{aligned}$$

Thus the lemma is proved. □

Corollary 7 *Consider the $(n+1)$-dimensional Euclidean space $E_{n+1} = \mathbb{R}^{n+1}$, $n \in \mathbb{N}_0$ and if $\varphi(t) = |t|^p$, $1 \le p < \infty$. Then by Theorems 1 and 2, it is known that $l_p((E_{n+1}))$ has the property $(k$-$\beta)$ for $k \ge 1$ and hence property $(k - NUC)$ for $k \ge 2$ induced by both the Luxemburg and the Amemiya norms. Since the property $(k$-$\beta)$ for $k \ge 1$ is inherited by subspaces so the spaces $ces(\alpha, p)$ for $\alpha \ge 1$ possess this property too.*

3.2 Cesàro difference sequence spaces of order *m*

For an arbitrary positive integer m, the results of Kizmaz who introduced the difference sequence spaces in [20] are generalized to mth order difference sequence spaces by Malkowsky and Parashar in [24]. In 1983, Orhan first introduced and studied the Cesàro difference sequence spaces $O_p^{(1)}$, $1 \le p < \infty$ as defined below:

$$O_p^{(1)} = \left\{ x \in l^0 : \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |\Delta^{(1)} x(k)| \right)^p < \infty \right\} \quad \text{for } 1 \le p < \infty,$$

where $(\Delta^{(1)} x)_k = (\Delta^{(1)} x(k)) = (x(k) - x(k-1))$, $k \in \mathbb{N}_0$ with the assumption that all negative subscripts of x are equal to zero. Orhan in [27] has also proved that the strict inclusion $ces_p \subset O_p^{(1)}$ holds for $1 \le p < \infty$ (see [27], p. 59). Using mth order difference operator $\Delta^{(m)}$, the author Et in [15] considered generalized difference sequence spaces $O_p^{(m)}$ as defined below:

$$O_p^{(m)} = \left\{ x \in l^0 : \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |\Delta^{(m)} x(k)| \right)^p < \infty \right\} \quad \text{for } 1 \le p < \infty,$$

where the sequence $(\Delta^{(m)} x)_k = (\Delta^{(m)} x(k))$ is defined as $\Delta^{(m)} x(k) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x(k-i)$, $k \in \mathbb{N}_0$ with the convention that all negative subscripts of x are equal to zero. For $1 \le p < \infty$, $O_p^{(m)}$ (in particular $O_p^{(1)}$, when $m = 1$) is a complete normed linear space induced by the following norm:

$$\|x\|_p = \left(\sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |\Delta^{(m)} x(k)| \right)^p \right)^{\frac{1}{p}}.$$

Quite recently, in 2014, another generalization $O_{\hat{p}}^{(m)}$ of the sequence space $O_p^{(m)}$ was presented by Et et al. [14]. They considered a positive bounded sequence $\hat{p} = (p_n)$ of real numbers with $p_n \ge 1$ instead of a fixed $p \ge 1$. Indeed, if we define the convex modular $\zeta(x) = \sum_{n=0}^{\infty} (\frac{1}{n+1} \sum_{k=0}^{n} |\Delta^{(m)} x(k)|)^{p_n}$, then $O_{\hat{p}}^{(m)}$ is defined as follows:

$$O_{\hat{p}}^{(m)} = \{x \in l^0 : \zeta(rx) < \infty \quad \text{for some } r > 0\}.$$

The authors also studied the property (H), uniform Opial property and Banach Saks property of type p for the space $O_{\hat{p}}^{(m)}$. Altay [1] introduced p-summable difference sequence spaces $l_p^{\Delta^{(m)}}$ and studied several topological properties and matrix transformations. The space $l_p^{\Delta^{(m)}}$, which is introduced as follows:

$$l_p^{\Delta^{(m)}} = \left\{ x \in l^0 : \sum_{n=0}^{\infty} |\Delta^{(m)} x(n)|^p < \infty \right\} \quad \text{for } 1 \le p < \infty,$$

is a Banach space equipped with the norm $\|x\|_p = \|(\Delta^{(m)} x)_n\|_p$.

Let $(E_{n+1}, \|.\|_{n+1})$ be finite dimensional real Banach spaces for each $n \in \mathbb{N}_0$. Consider the cartesian product $\prod_{n=0}^{\infty} E_{n+1}$, which consist of all sequences $x = (x(n))_{n=0}^{\infty}$ such that $x(n) \in E_{n+1}$ for each $n \in \mathbb{N}_0$. Then the Nakano difference sequence spaces $l_{\hat{p}}^{\Delta^{(m)}}((E_{n+1}))$ is defined as

$$l_{\hat{p}}^{\Delta^{(m)}}((E_{n+1})) = \left\{ x \in \prod_{n=0}^{\infty} E_{n+1} : \rho(rx) < \infty \quad \text{for some } r > 0 \right\},$$

where on $\prod_{n=0}^{\infty} E_{n+1}$, a convex modular ρ is defined as $\rho(x) = \sum_{n=0}^{\infty} \|\Delta^{(m)} x(n)\|_{n+1}^{p_n}$. It is easy to establish that the pair $(l_{\hat{p}}^{\Delta^{(m)}}((E_{n+1})), \|\cdot\|_{\hat{p}}^{\Delta^{(m)}})$ is a Banach space, where

$$\|x\|_{\hat{p}}^{\Delta^{(m)}} = \inf \left\{ r > 0 : \rho\left(\frac{x}{r}\right) \leq 1 \right\}.$$

We now prove the following lemma.

Lemma 9 *The sequence space $O_{\hat{p}}^{(m)}$ is linearly isometric to a closed subspace of Nakano difference sequence spaces $l_{\hat{p}}^{\Delta^{(m)}}((E_{n+1}))$, where $E_{n+1} = \mathbb{R}^{n+1}$, $n \in \mathbb{N}_0$ is the $(n+1)$-dimensional Euclidean space equipped with the norm given by equality (9).*

Proof For all $x = (x(i)) \in O_{\hat{p}}^{(m)}$, the following linear isometric map $T : O_{\hat{p}}^{(m)} \to l_{\hat{p}}^{\Delta^{(m)}}((E_{n+1}))$ is defined:

$$\begin{aligned} &T((x(i))) \\ &= \left(\Delta^{(m)} x(0), \frac{1}{2}(\Delta^{(m)} x(0), \Delta^{(m)} x(1)), \ldots, \frac{1}{n+1}(\Delta^{(m)} x(0), \ldots, \Delta^{(m)} x(n)), \ldots \right). \end{aligned}$$

Indeed the following is holds:

$$\begin{aligned} &\|T((x(i)))\|_{l_{\hat{p}}^{\Delta^{(m)}}((E_{n+1}))} \\ &= \|T(x(0), x(1), \ldots, x(i), \ldots)\|_{l_{\hat{p}}^{\Delta^{(m)}}((E_{n+1}))} \\ &= \Big\| \Big(\Delta^{(m)} x(0), \frac{1}{2}(\Delta^{(m)} x(0), \Delta^{(m)} x(1)), \ldots, \frac{1}{n+1}(\Delta^{(m)} x(0), \ldots \\ &\ldots, \Delta^{(m)} x(n)), \ldots \Big) \Big\|_{l_{\hat{p}}^{\Delta^{(m)}}((E_{n+1}))} \\ &= \inf \left\{ r > 0 : \sum_{n=0}^{\infty} \left(\frac{1}{r(n+1)} \sum_{k=0}^{n} |\Delta^{(m)} x(k)| \right)^{p_n} \leq 1 \right\} \\ &= \inf \left\{ r > 0 : \zeta\left(\frac{x}{r}\right) \leq 1 \right\} \\ &= \|(x(i))\|_{O_{\hat{p}}^{(m)}}. \end{aligned}$$

Thus the lemma is proved. □

Corollary 8 *Proceeding in a similar approach considered in Theorems 1 and 2 it can be easy to establish that the space $l_{\hat{p}}^{\Delta^{(m)}}((E_{n+1}))$ possesses the property $(k\text{-}\beta)$ for $k \geq 1$ equipped with both the Luxemburg and the Amemiya norms. Therefore, by Lemma 9, the space $O_{\hat{p}}^{(m)}$ possesses the property $(k\text{-}\beta)$ for any fixed integer $k \geq 1$.*

4 Conclusions

The geometric properties (k-β), $k \geq 1$ for the space $l_{\Phi}((E_n))$ generated by a Musielak–Orlicz function Φ and a sequence (E_n) of n-dimensional spaces E_n, $n \in \mathbb{N}$ are studied and derived the similar properties for the Musielak–Orlicz–Cesàro space ces_{Φ} equipped with both the Luxemburg and the Amemiya norms. The work provides several new results and strengthen many earlier known results. The uniform Opial property for $l_{\Phi}((E_n))$ can be easily established by applying the same techniques which are developed ourselves in our earlier work [25].

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