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ORIGINAL PAPER

Existence and uniqueness of solutions for a first-order discrete fractional boundary value problem

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Abstract In this paper we use topological degree to establish the existence of nontrivial solutions for a first-order discrete fractional boundary value problem with a sign-changing nonlinearity. Also using the monotone iterative technique we discuss problems with a non-negative nonlinearity and establish the uniqueness of positive solutions.

Keywords Discrete fractional boundary value problems · Nontrivial (positive) solutions · Topological degree · Monotone iterative technique

Mathematics Subject Classification 34B10 · 34B18 · 34A34 · 45G15 · 45M20

1 Introduction

In this paper we study the existence and uniqueness of solutions for the first-order discrete fractional boundary value problem

$$\begin{cases} \Delta_{\nu-1}^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)), & t \in [0, T]_{\mathbb{Z}}, \\ y(\nu-1) = y(\nu+T), \end{cases} \tag{1.1}$$

where ν is a real number with $0 < \nu < 1$ and Δ^{ν} is a discrete fractional operator. For the nonlinear term f, we assume that

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(H1) $f(t, y) \in C([\nu - 1, \nu + T - 1]_{\mathbb{Z}_{\nu-1}} \times \mathbb{R}, \mathbb{R})$.

Note that, in this paper we use $[a, b]_E$ to stand for $[a, b] \cap E$ for some set E.

In [1–4], Atici and Eloe developed the fundamental theory of both discrete delta and nabla fractional calculus. For more recent results we refer the reader to [5–9] and the references cited therein. In [5], the authors considered a three-point fractional sum boundary value problem, where the nonlinearity f is either superlinear or sublinear:

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u} = 0 \quad \text{or} \quad \infty, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u} = \infty \text{ or } 0; \tag{1.2}$$

here $f_0 = 0$, $f_\infty = \infty$ is the superlinear case, and $f_0 = \infty$, $f_\infty = 0$ is the sublinear case. In [6–8], the authors adopted the growth conditions in (1.2) to study many types of discrete fractional boundary value problems with nonnegative and semipositone nonlinearities. For more details in this direction, we refer the reader to the recent book [9], which summarizes some results on discrete fractional equations.

Motivated by the above, in this paper we use topological degree to study the existence of nontrivial solutions for the discrete fractional boundary value problem (1.1). The novelty is threefold: (1) with the aid of some inequalities of Green's function, our growth condition on the nonlinearity improves that in (1.2) (see conditions (H3) and (H4) in the following section); (2) nontrivial solutions are obtained using topological degree with a semipositone nonlinearity (this is seldom considered in discrete fractional equations); (3) with a sublinear growth condition, a unique positive solution is obtained from the monotone iterative technique with a nonnegative nonlinearity (in addition the iterative sequences are given).

2 Preliminaries

We first introduce some background materials from discrete fractional calculus, and for more details, we refer the reader to [1–4,9].

Definition 2.1 We define $t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ for any $t, \nu \in \mathbb{R}$ for which the right-hand side is well-defined. We use the convention that if $t + 1 - \nu$ is a pole of the Gamma function and $t + 1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2.2 For $\nu > 0$, the $\nu-$th fractional sum of a function f is $\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - s - 1)^{\underline{\nu-1}} f(s)$, for $t \in \mathbb{N}_{a+N-\nu}$. We also define the $\nu-$th fractional difference for $\nu > 0$ by $\Delta_a^\nu f(t) = \Delta^N \Delta_a^{\nu-N} f(t)$, for $t \in \mathbb{N}_{a+N-\nu}$, where $N \in \mathbb{N}$ with $0 \leq N - 1 < \nu \leq N$.

From [7], we know (1.1) is equivalent to

$$y(t) = \sum_{s=0}^{T} \frac{G(t, s)}{\Gamma(\nu)} f(s + \nu - 1, y(s + \nu - 1)), \quad \text{for } t \in [\nu - 1, \nu + T - 1]_{\mathbb{Z}_{\nu-1}}, \tag{2.1}$$

where

$$G(t, s) = \begin{cases} \frac{(\nu+T-s-1)^{\underline{\nu-1}} t^{\underline{\nu-1}}}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}} + (t - s - 1)^{\underline{\nu-1}}, & 0 \leq s \leq t - \nu \leq T, \\ \frac{(\nu+T-s-1)^{\underline{\nu-1}} t^{\underline{\nu-1}}}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}}, & t - \nu < s \leq T. \end{cases} \tag{2.2}$$

Let $C^* = 1 + \frac{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}}{(\nu+T-1)^{\underline{\nu-1}}}$, for all $(t,s) \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}} \times [0,T]_{\mathbb{Z}}$, and note the inequality

$$\begin{aligned} 0 &< \frac{(\nu+T)^{\underline{\nu-1}}}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}}(\nu+T-s-1)^{\underline{\nu-1}} \\ &\le G(t,s) \le \frac{C^*\Gamma(\nu)}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}}(\nu+T-s-1)^{\underline{\nu-1}}. \end{aligned} \tag{2.3}$$

Let $\phi(t+\nu-1) = (\nu+T-t-1)^{\underline{\nu-1}}$, $t \in [0,T]_{\mathbb{Z}}$ and $\phi^*(t) = (2\nu+T-t-2)^{\underline{\nu-1}}$, $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. Then, for all $s \in [0,T]_{\mathbb{Z}}$, we have

$$\sum_{t=\nu-1}^{\nu+T-1} \frac{G(t,s)}{\Gamma(\nu)}\phi^*(t) = \sum_{t=0}^{T} \frac{G(t+\nu-1,s)}{\Gamma(\nu)}\phi(t+\nu-1). \tag{2.4}$$

Consequently, for all $s \in [0,T]_{\mathbb{Z}}$, from (2.3) and (2.4) we have

$$\begin{aligned} \sum_{t=0}^{T} \frac{(\nu+T)^{\underline{\nu-1}}\phi(t+\nu-1)}{\Gamma(\nu)(\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}})}\cdot\phi(s+\nu-1) &\le \sum_{t=\nu-1}^{\nu+T-1} \frac{G(t,s)}{\Gamma(\nu)}\phi^*(t) \\ &\le \sum_{t=0}^{T} \frac{C^*\phi(t+\nu-1)}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}}\cdot\phi(s+\nu-1). \end{aligned} \tag{2.5}$$

For convenience, we let

$$\begin{aligned} \kappa_1 &= \sum_{t=0}^{T} \frac{(\nu+T)^{\underline{\nu-1}}}{\Gamma(\nu)(\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}})}\phi(t+\nu-1) \quad \text{and} \\ \kappa_2 &= \sum_{t=0}^{T} \frac{C^*}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}}\phi(t+\nu-1). \end{aligned}$$

Let E be the collection of all maps from $[\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$ to $\mathbb{R}$, equipped with the max norm, $\|\cdot\|$. Then E is a Banach space. Let $P \subset E$ be $P = \{y \in E : y(t) \ge 0,\ t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}\}$. Then P is a cone in E. Define a linear operator L by $(Ly)(t) = \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} y(s+\nu-1)$, for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. Then we easily have $L(P) \subset P$, i.e., L is a positive operator.

Lemma 2.3 *Let $r(L)$ be the spectral radius of L. Then $\kappa_1 \le r(L) \le \kappa_2$.*

Proof For all $n \in \mathbb{N}^+$, we have

$$\|L^n\| = \max_{t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}} \sum_{s_1=0}^{T}\sum_{s_2=0}^{T}\cdots\sum_{s_n=0}^{T} \frac{G(t,s_1)}{\Gamma(\nu)}\frac{G(s_1,s_2)}{\Gamma(\nu)}\cdots\frac{G(s_{n-1},s_n)}{\Gamma(\nu)}.$$

From (2.3), for all $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$ we have

$$\begin{aligned} \kappa_1 &= \sum_{s=0}^{T} \frac{(\nu+T)^{\underline{\nu-1}}(\nu+T-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)(\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}})} \le \sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)} \\ &\le \sum_{s=0}^{T} \frac{C^*(\nu+T-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}} = \kappa_2. \end{aligned} \tag{2.6}$$

Hence, we have $\kappa_1^n \leq \|L^n\| \leq \kappa_2^n$. From Gelfand's theorem, we have $r(L) = \lim_{n\to\infty} \sqrt[n]{\|L^n\|} \in [\kappa_1, \kappa_2]$. This completes the proof. □

Lemma 2.4 *Let* $P_0 = \left\{ y \in P : \sum_{t=0}^{T} \phi(t+\nu-1)y(t+\nu-1) \geq \frac{\kappa_1(\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}})}{C^*}\|y\| \right\}$. *Then* P_0 *is a cone in* E, *and* $L(P) \subset P_0$.

Proof Note that $\sum_{t=0}^{T} \phi(t+\nu-1)y(t+\nu-1) = \sum_{t=\nu-1}^{\nu+T-1} \phi^*(t)y(t)$. Then we have

$$\begin{aligned}\sum_{t=\nu-1}^{\nu+T-1} \phi^*(t)(Ly)(t) &= \sum_{t=\nu-1}^{\nu+T-1} \phi^*(t) \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} y(s+\nu-1)\\ &\geq \kappa_1 \sum_{s=0}^{T} \phi(s+\nu-1)y(s+\nu-1)\\ &\geq \frac{\kappa_1(\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}})}{C^*} \sum_{s=0}^{T} \frac{G(\tau,s)}{\Gamma(\nu)} y(s+\nu-1),\end{aligned}$$

for all $\tau \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. This implies that $\sum_{t=\nu-1}^{\nu+T-1} \phi^*(t)(Ly)(t) \geq \frac{\kappa_1(\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}})}{C^*}\|Ly\|$. This completes the proof. □

We now recall the following three well known results (see [10]).

Lemma 2.5 *Let* E *be a Banach space and* Ω *a bounded open set in* E. *Suppose* $A : \overline{\Omega} \to E$ *is a continuous compact operator. If there exists a* $\omega_0 \neq 0$ *such that* $\omega - A\omega \neq \lambda\omega_0, \forall \omega \in \partial\Omega, \lambda \geq 0$, *then the topological degree* $\deg(I - A, \Omega, 0) = 0$.

Lemma 2.6 *Let* E *be a Banach space and* Ω *a bounded open set in* E *with* $0 \in \Omega$. *Suppose* $A : \overline{\Omega} \to E$ *is a continuous compact operator. If* $\omega - \lambda A\omega \neq 0, \forall \omega \in \partial\Omega, \lambda \in [0, 1]$, *then* $\deg(I - A, \Omega, 0) = 1$.

Lemma 2.7 *Let* E *be a partially ordered Banach space, and* $x_0, y_0 \in E$ *with* $x_0 \leq y_0$, $D = [x_0, y_0]$. *Suppose that* $A : D \to E$ *satisfies the following conditions*

1. *A is an increasing operator;*
2. $x_0 \leq Ax_0$, $y_0 \geq Ay_0$, *i.e.,* x_0 *and* y_0 *is a subsolution and a supersolution of A;*
3. *A is a continuous compact operator.*

Then A has the smallest fixed point x^* *and the largest fixed point* y^* *in* $[x_0, y_0]$, *respectively. Moreover,* $x^* = \lim_{n\to\infty} A^n x_0$ *and* $y^* = \lim_{n\to\infty} A^n y_0$.

3 Nontrivial solutions for (1.1)

Let $(Ay)(t) = \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} f(s+\nu-1, y(s+\nu-1))$, for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$, where G is defined in (2.2). Then we note, from the Arzelà–Ascoli theorem, that A is a completely continuous operator on E. Moreover, y is a solution of (1.1) if and only if y is a fixed point of the operator A.

Let $\lambda_1 = \kappa_1^{-1}$, $\lambda_2 = \kappa_2^{-1}$, and $B_\varrho := \{x \in E : \|x\| < \varrho\}$ for $\varrho > 0$. For convenience, we use $c_1, c_2, \ldots$ to stand for different positive constants. Now we list our assumptions on f.

(H2) There exists $M \geq 0$ such that $f(t, y) \geq -M$ for all $y \in \mathbb{R}$ and $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$.

(H3) $\liminf_{y\to+\infty}\frac{f(t,y)}{y} > \lambda_1$, $\limsup_{y\to 0}\left|\frac{f(t,y)}{y}\right| < \lambda_2$ uniformly on $t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}$.

(H4) $\liminf_{y\to 0}\frac{f(t,y)}{|y|} > \lambda_1$, $\limsup_{y\to+\infty}\frac{f(t,y)}{y} < \lambda_2$ uniformly on $t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}$.

Theorem 3.1 *Suppose that (H1)–(H3) hold. Then* (1.1) *has at least one nontrivial solution.*

Proof Let

$$\overline{y}(t) = M\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)} \quad \text{for } t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}.$$

From the first limit in (H3), there exist $\varepsilon > 0$ and $c_1 > 0$ such that

$$f(t,y) \geq (\lambda_1+\varepsilon)y - c_1, \quad \forall y\in\mathbb{R}, \quad t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}. \tag{3.1}$$

We now prove that

$$y \neq Ay + \lambda\phi^{**}, \quad \forall y\in E, \quad \|y\| = R, \quad \lambda\geq 0, \tag{3.2}$$

where $\phi^{**}\in P_0$ is an arbitrarily fixed element and $R > \|\overline{y}\| + \frac{C^*(\varepsilon M\kappa_1\kappa_2+c_1\kappa_1+M\kappa_2-M\kappa_1)}{\varepsilon\kappa_1^2(\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}})}\sum_{s=0}^{T}\phi(s+\nu-1)$.

Suppose there exist $y_0\in E$, $\|y_0\| = R$ and $\lambda_0\geq 0$ such that

$$\begin{aligned} y_0(t) = (Ay_0)(t) + \lambda_0\phi^{**}(t) = \sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}f(s+\nu-1, y_0(s+\nu-1)) \\ + \lambda_0\phi^{**}(t), \quad \forall t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}. \end{aligned} \tag{3.3}$$

Then we have

$$\begin{aligned} y_0(t) + \overline{y}(t) = \sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}[f(s+\nu-1, y_0(s+\nu-1)) + M] \\ + \lambda_0\phi^{**}(t), \quad \forall t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}. \end{aligned}$$

Note $\phi^{**}\in P_0$ and Lemma 2.4, so $y_0+\overline{y}\in P_0$. From (3.1), (2.4) and (2.5), we have

$$\begin{aligned} &\sum_{t=\nu-1}^{\nu+T-1}(Ay_0)(t)\phi^*(t) - \sum_{t=\nu-1}^{\nu+T-1}y_0(t)\phi^*(t) \\ &= \sum_{t=\nu-1}^{\nu+T-1}\phi^*(t)\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}f(s+\nu-1, y_0(s+\nu-1)) - \sum_{t=\nu-1}^{\nu+T-1}y_0(t)\phi^*(t) \\ &= \sum_{t=\nu-1}^{\nu+T-1}\phi^*(t)\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}[f(s+\nu-1, y_0(s+\nu-1)) + M] \\ &\quad -M\sum_{t=\nu-1}^{\nu+T-1}\phi^*(t)\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)} - \sum_{t=\nu-1}^{\nu+T-1}y_0(t)\phi^*(t) \\ &\geq \sum_{s=0}^{T}\kappa_1\phi(s+\nu-1)[(\lambda_1+\varepsilon)y_0(s+\nu-1) - c_1 + M] \end{aligned}$$

$$
\begin{aligned}
&-M\kappa_2\sum_{s=0}^{T}\phi(s+\nu-1)-\sum_{t=\nu-1}^{\nu+T-1}y_0(t)\phi^*(t)\\
&=\kappa_1\lambda_1\sum_{s=0}^{T}\phi(s+\nu-1)y_0(s+\nu-1)\\
&\quad-\sum_{t=\nu-1}^{\nu+T-1}y_0(t)\phi^*(t)+\varepsilon\kappa_1\sum_{s=0}^{T}\phi(s+\nu-1)y_0(s+\nu-1)\\
&\quad-(c_1\kappa_1+M\kappa_2-M\kappa_1)\sum_{s=0}^{T}\phi(s+\nu-1)\\
&=\varepsilon\kappa_1\sum_{t=\nu-1}^{\nu+T-1}y_0(t)\phi^*(t)-(c_1\kappa_1+M\kappa_2-M\kappa_1)\sum_{s=0}^{T}\phi(s+\nu-1)\\
&=\varepsilon\kappa_1\sum_{t=\nu-1}^{\nu+T-1}[y_0(t)+\overline{y}(t)]\phi^*(t)-\varepsilon\kappa_1\sum_{t=\nu-1}^{\nu+T-1}\phi^*(t)M\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}\\
&\quad-(c_1\kappa_1+M\kappa_2-M\kappa_1)\sum_{s=0}^{T}\phi(s+\nu-1)\\
&\geq\varepsilon\kappa_1\frac{\kappa_1(\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}})}{C^*}\|y_0+\overline{y}\|\\
&\quad-(\varepsilon M\kappa_1\kappa_2+c_1\kappa_1+M\kappa_2-M\kappa_1)\sum_{s=0}^{T}\phi(s+\nu-1)\\
&\geq\varepsilon\kappa_1\frac{\kappa_1(\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}})}{C^*}\|y_0\|-\varepsilon\kappa_1\frac{\kappa_1(\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}})}{C^*}\|\overline{y}\|\\
&\quad-(\varepsilon M\kappa_1\kappa_2+c_1\kappa_1+M\kappa_2-M\kappa_1)\sum_{s=0}^{T}\phi(s+\nu-1)\\
&>0.
\end{aligned}
\tag{3.4}
$$

Note $\phi^{**}\in P_0$ and $\lambda_0\geq 0$, so (3.3) enables us to obtain

$$
\sum_{t=\nu-1}^{\nu+T-1}y_0(t)\phi^*(t)-\sum_{t=\nu-1}^{\nu+T-1}(Ay_0)(t)\phi^*(t)=\sum_{t=\nu-1}^{\nu+T-1}\lambda_0\phi^{**}(t)\phi^*(t)\geq 0.
$$

That is a contradiction with (3.4). As a result, (3.2) holds true. Now Lemma 2.5 implies that

$$
\deg(I-A,B_R,0)=0. \tag{3.5}
$$

From the second limit in (H3), there exist $r\in(0,R)$ and $\varepsilon\in(0,\lambda_2)$ such that $|f(t,y)|\leq(\lambda_2-\varepsilon)|y|$, for all $|y|\leq r$, $t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}$. Next we claim

$$
Ay\neq\lambda y,\quad\forall y\in\partial B_r,\lambda\geq 1. \tag{3.6}
$$

If this is false, then there exists a $y_1\in\partial B_r$ and a $\overline{\lambda}_0\geq 1$ such that $Ay_1=\overline{\lambda}_0y_1$. We may assume that $\overline{\lambda}_0>1$ (otherwise we have a fixed point y_1). Hence, we have $|Ay_1|=|\overline{\lambda}_0y_1|\geq|y_1|$ and

$$
\begin{aligned}
|y_1(t)| &\le \left| \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} f(s+\nu-1, y_1(s+\nu-1)) \right| \\
&\le \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} |f(s+\nu-1, y_1(s+\nu-1))| \\
&\le \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} (\lambda_2-\varepsilon) |y_1(s+\nu-1)|.
\end{aligned}
$$

Multiply both sides of the above inequality by $\phi^*(t)$ and sum from $\nu-1$ to $\nu+T-1$, and from (2.4) and (2.5) we obtain

$$
\begin{aligned}
\sum_{t=\nu-1}^{\nu+T-1} |y_1(t)|\phi^*(t) &\le \sum_{t=\nu-1}^{\nu+T-1} \phi^*(t) \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} (\lambda_2-\varepsilon)|y_1(s+\nu-1)| \\
&= \sum_{s=0}^{T} \left(\sum_{t=\nu-1}^{\nu+T-1} \frac{G(t,s)}{\Gamma(\nu)} \phi^*(t) \right) (\lambda_2-\varepsilon)|y_1(s+\nu-1)| \\
&\le \sum_{s=0}^{T} \kappa_2 \phi(s+\nu-1)(\lambda_2-\varepsilon)|y_1(s+\nu-1)| \\
&\le \frac{\lambda_2-\varepsilon}{\lambda_2} \sum_{t=\nu-1}^{\nu+T-1} |y_1(t)|\phi^*(t).
\end{aligned}
$$

This implies that $\sum_{t=\nu-1}^{\nu+T-1} |y_1(t)|\phi^*(t) = 0$ and then $|y_1(t)| \equiv 0$ for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. This contradicts $y_1 \in \partial B_r$. Consequently, (3.6) is true. Now Lemma 2.6 implies that

$$
\deg(I-A, B_r, 0) = 1. \tag{3.7}
$$

From (3.5) and (3.7), we have that $\deg(I-A, B_R\backslash\overline{B}_r, 0) = \deg(I-A, B_R, 0) - \deg(I-A, B_r, 0) = -1$, which implies that $I-A$ has at least a zero point in $B_R\backslash\overline{B}_r$, i.e., A has at least one fixed point in $B_R\backslash\overline{B}_r$. Thus (1.1) has at least one nontrivial solution. This completes the proof. □

Theorem 3.2 *Suppose that (H1), (H2) and (H4) hold. Then* (1.1) *has at least one nontrivial solution.*

Proof From the first limit in (H4), there exist $\varepsilon > 0$ and $r_1 > 0$ such that

$$
f(t,y) \ge (\lambda_1+\varepsilon)|y|, \quad \forall |y| \le r_1, \quad t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}. \tag{3.8}
$$

Therefore, for each $y \in \overline{B}_{r_1}$, we obtain $(Ay)(t) \ge (\lambda_1+\varepsilon)\sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)}|y(s+\nu-1)|$ for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. Then $A(\overline{B}_{r_1}) \subset P$. For every $y \in \partial B_{r_1} \cap P$, we claim

$$
y \ne Ay + \lambda\phi^{***}, \quad \lambda \ge 0, \tag{3.9}
$$

where $\phi^{***} \in P$ is an arbitrarily fixed element. Suppose there exist $y_0 \in \partial B_{r_1} \cap P$ and $\lambda_0 \ge 0$ such that $y_0(t) = (Ay_0)(t) + \lambda_0\phi^{***}(t)$ for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. This implies that

$$
y_0(t) \ge (Ay_0)(t) \ge (\lambda_1+\varepsilon) \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} y_0(s+\nu-1).
$$

Multiply both sides of the above inequality by $\phi^*(t)$ and sum from $\nu-1$ to $\nu+T-1$, and from (2.4) and (2.5) we obtain

$$\begin{aligned}\sum_{t=\nu-1}^{\nu+T-1} y_0(t)\phi^*(t) &\geq \sum_{t=\nu-1}^{\nu+T-1}\phi^*(t)\sum_{s=0}^{T}(\lambda_1+\varepsilon)\frac{G(t,s)}{\Gamma(\nu)}y_0(s+\nu-1)\\ &=(\lambda_1+\varepsilon)\sum_{s=0}^{T}\left(\sum_{t=\nu-1}^{\nu+T-1}\frac{G(t,s)}{\Gamma(\nu)}\phi^*(t)\right)y_0(s+\nu-1)\\ &\geq \kappa_1(\lambda_1+\varepsilon)\sum_{s=0}^{T}y_0(s+\nu-1)\phi(s+\nu-1)\\ &\geq \frac{\lambda_1+\varepsilon}{\lambda_1}\sum_{t=\nu-1}^{\nu+T-1}y_0(t)\phi^*(t),\end{aligned}$$

and then $y_0(t)\equiv 0$ for $t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}$. This contradicts $y_0\in\partial B_{r_1}\cap P$. Hence, (3.9) is satisfied. Note that $A(\overline{B}_{r_1})\subset P$. Then from the permanence property of the fixed point index and Lemma 2.5, we obtain

$$\deg(I-A,B_{r_1},0)=i(A,B_{r_1}\cap P,P)=0, \tag{3.10}$$

where i denotes the fixed point index on P.

Define an operator $\overline{A}y=A(y-\overline{y})+\overline{y}$. Then $\overline{A}:E\to P$ is a completely continuous operator. From the second limit in (H4), there exist $r_2\in(r_1+\|\overline{y}\|,+\infty)$ and $\sigma\in(0,1)$ such that

$$f(t,y)\leq\sigma\lambda_2 y,\quad \forall y\geq r_2, t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}. \tag{3.11}$$

Let $L_1y=\sigma\lambda_2Ly$. Then $L_1:E\to E$ is a bounded linear operator and $L_1(P)\subset P$. Moreover, note $\sigma\in(0,1)$, and Lemma 2.3 enables us to find $r(L_1)=\sigma\lambda_2r(L)\leq\sigma\lambda_2\kappa_2=\sigma<1$. Therefore, $(I-L_1)$ has an inverse operator, denoted by $(I-L_1)^{-1}$.

Let $W=\{y\in P: y=\lambda\overline{A}y, 0\leq\lambda\leq 1\}$. We show that W is bounded. For $y\in W$, we note that if $y(s+\nu-1)-\overline{y}(s+\nu-1)<0$, then we have

$$y(s+\nu-1)-\overline{y}(s+\nu-1)\geq y(s+\nu-1)-\|\overline{y}\|\geq-\|\overline{y}\|>-r_2.$$

Consequently, if $y(s+\nu-1)-\overline{y}(s+\nu-1)<r_2$, we obtain $\|y-\overline{y}\|\leq r_2$ and then by the continuity of f, there exists $M_1>0$ such that

$$|f(s+\nu-1,y(s+\nu-1)-\overline{y}(s+\nu-1))|\leq M_1,\quad \text{for } s\in[0,T]_{\mathbb{Z}}.$$

Therefore, for all $y\in W$, from (3.11) and (2.3) we have

$$\begin{aligned}y(t)&\leq(\overline{A}y)(t)=(A(y-\overline{y}))(t)+\overline{y}(t)\\ &=\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}f(s+\nu-1,y(s+\nu-1)-\overline{y}(s+\nu-1))+M\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}\\ &=\sum_{\{s\in[0,T]_{\mathbb{Z}}:y(s+\nu-1)-\overline{y}(s+\nu-1)\geq r_2\}}\frac{G(t,s)}{\Gamma(\nu)}f(s+\nu-1,y(s+\nu-1)-\overline{y}(s+\nu-1))\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\{s\in[0,T]_{\mathbb{Z}}:y(s+\nu-1)-\overline{y}(s+\nu-1)<r_2\}}\frac{G(t,s)}{\Gamma(\nu)}f(s+\nu-1,y(s+\nu-1)-\overline{y}(s+\nu-1))\\
&+M\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}\\
\le{}&\sigma\lambda_2\sum_{\{s\in[0,T]_{\mathbb{Z}}:y(s+\nu-1)-\overline{y}(s+\nu-1)\ge r_2\}}\frac{G(t,s)}{\Gamma(\nu)}(y(s+\nu-1)-\overline{y}(s+\nu-1))\\
&+M_1\sum_{\{s\in[0,T]_{\mathbb{Z}}:y(s+\nu-1)-\overline{y}(s+\nu-1)<r_2\}}\frac{G(t,s)}{\Gamma(\nu)}+M\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}\\
\le{}&\sigma\lambda_2\sum_{\{s\in[0,T]_{\mathbb{Z}}:y(s+\nu-1)-\overline{y}(s+\nu-1)\ge r_2\}}\frac{G(t,s)}{\Gamma(\nu)}y(s+\nu-1)+M_1\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}\\
&+M\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}\\
\le{}&\sigma\lambda_2\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}y(s+\nu-1)+(M_1+M)\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}\\
\le{}&\sigma\lambda_2\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}y(s+\nu-1)+\frac{(M_1+M)C^*}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}}\sum_{s=0}^{T}(\nu+T-s-1)^{\underline{\nu-1}}\\
={}&(L_1y)(t)+\frac{(M_1+M)C^*}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}}\sum_{s=0}^{T}(\nu+T-s-1)^{\underline{\nu-1}}.
\end{aligned}
$$

Then $((I-L_1)y)(t)\le\frac{(M_1+M)C^*}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}}\sum_{s=0}^{T}(\nu+T-s-1)^{\underline{\nu-1}}$ for $t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}$. Hence,

$$
y(t)\le(I-L_1)^{-1}\frac{(M_1+M)C^*}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}}\sum_{s=0}^{T}(\nu+T-s-1)^{\underline{\nu-1}}
$$

for $t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}$, and so W is bounded.

Let $r_3>\max\{r_2,\sup W+\|\overline{y}\|\}$. Then $\overline{A}$ has no fixed points on ∂B_{r_3}. If this is false, then there is a $y_0\in\partial B_{r_3}$ such that $\overline{A}y_0=y_0$, and thus $y_0\in W$, $\|y_0\|=r_3>\sup W$, and this is a contradiction. As a result, from the permanence property and the homotopy invariance property of the fixed point index, we obtain $\deg(I-\overline{A},B_{r_3},0)=i(\overline{A},B_{r_3}\cap P,P)=1$. Next we claim that

$$
\deg(I-A,B_{r_3},0)=\deg(I-\overline{A},B_{r_3},0)=1. \tag{3.12}
$$

Let $H(l,y)=A(y-l\overline{y})+l\overline{y}$ for $(l,y)\in[0,1]\times\partial B_{r_3}$. Then $H(0,y)=Ay$, $H(1,y)=\overline{A}y$. Suppose that there exists $(l_0,y_1)\in[0,1]\times\partial B_{r_3}$ such that $H(l_0,y_1)=y_1$. Then $A(y_1-l_0\overline{y})+l_0\overline{y}=y_1$, which implies that $A(y_1-l_0\overline{y})=y_1-l_0\overline{y}$, and $\overline{A}(y_1-l_0\overline{y}+\overline{y})=y_1-l_0\overline{y}+\overline{y}$. Therefore, $y_1-l_0\overline{y}+\overline{y}\in W$, and $\|y_1-l_0\overline{y}+\overline{y}\|\ge\|y_1\|-(1-l_0)\|\overline{y}\|=r_3-(1-l_0)\|\overline{y}\|>\sup W$. This is a contradiction. From the homotopy invariance of the topological degree, we see that (3.12) holds. From (3.10) and (3.12), we have $\deg(I-A,B_{r_3}\backslash\overline{B}_{r_1},0)=\deg(I-A,B_{r_3},0)-\deg(I-A,B_{r_1},0)=1$, which implies that $I-A$ has at least a zero point in $B_{r_3}\backslash\overline{B}_{r_1}$, i.e., A has at least one fixed point in $B_{r_3}\backslash\overline{B}_{r_1}$. Thus (1.1) has at least one nontrivial solution. This completes the proof. □

4 Positive solutions for (1.1)

In this section we have the following assumptions on f.

(H1') $f(t, y) \in C([\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}} \times \mathbb{R}^+, \mathbb{R}^+)$, and $f(t, y) > 0$ for all $(t, y) \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}} \times \mathbb{R}^+$.

(H5) There exits $\mu \in (0, 1)$ such that $f(t, ky) \ge k^\mu f(t, y)$ for $k \in (0, 1)$.

(H6) $f(t, y)$ is increasing in y, i.e., $f(t, y_1) \le f(t, y_2)$ holds for $y_1 \le y_2$.

Theorem 4.1 *Suppose that (H1)′ and (H5)–(H6) hold. Then there exist $y_0^*, y_1^* \in P\backslash\{0\}$ such that $y_0^* \le Ay_0^*$, $y_1^* \ge Ay_1^*$.*

Proof Let $w(t) = \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)}$ for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. Then from (2.6), $w(t) \in [\kappa_1, \kappa_2]$. Therefore, from (H1)′, there exist $a_w, b_w > 0$ such that $0 < a_w \le f(t, w) \le b_w$. This implies that

$$a_w w(t) \le \sum_{s=0}^{T} \frac{G(t, s)}{\Gamma(\nu)} f(s+\nu-1, w(s+\nu-1)) := w_0(t) \le b_w w(t), \quad \forall t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}. \tag{4.1}$$

Let $y_0(t) = \delta w_0(t)$ with $0 < \delta < \min\{1/b_w, a_w^{\mu/(1-\mu)}\}$. Then choosing $\varepsilon \in (0, \min\{1, a_w\})$, we have

$$\begin{aligned}(A\varepsilon y_0)(t) &= \sum_{s=0}^{T} \frac{G(t, s)}{\Gamma(\nu)} f(s+\nu-1, \varepsilon y_0(s+\nu-1)) \\ &= \sum_{s=0}^{T} \frac{G(t, s)}{\Gamma(\nu)} f\left(s+\nu-1, \frac{\varepsilon y_0(s+\nu-1)}{w(s+\nu-1)} w(s+\nu-1)\right) \\ &\ge \varepsilon^\mu (\delta a_w)^\mu w_0(t) \ge \varepsilon\delta w_0(t) = \varepsilon y_0(t).\end{aligned}$$

Now, let $y_0^* = \varepsilon y_0$. Then $y_0^* \le Ay_0^*$.

Let $y_1 = \xi w_0(t)$ with $\xi > \max\{1/a_w, b_w^{\mu/(1-\mu)}\}$. Taking $\overline{\varepsilon} > \max\{1, b_w\}$, we obtain

$$\begin{aligned}\overline{\varepsilon} y_1(t) \ge \overline{\varepsilon}^\mu \xi w_0(t) &= \overline{\varepsilon}^\mu \xi \sum_{s=0}^{T} \frac{G(t, s)}{\Gamma(\nu)} f(s+\nu-1, w(s+\nu-1)) \\ &\ge \overline{\varepsilon}^\mu \xi \sum_{s=0}^{T} \frac{G(t, s)}{\Gamma(\nu)} f\left(s+\nu-1, \frac{w(s+\nu-1)}{\overline{\varepsilon} y_1(s+\nu-1)} \overline{\varepsilon} y_1(s+\nu-1)\right) \\ &\ge \overline{\varepsilon}^\mu \xi \overline{\varepsilon}^{-\mu} (\xi b_w)^{-\mu} \sum_{s=0}^{T} \frac{G(t, s)}{\Gamma(\nu)} f\left(s+\nu-1, \overline{\varepsilon} y_1(s+\nu-1)\right) \\ &\ge \sum_{s=0}^{T} \frac{G(t, s)}{\Gamma(\nu)} f\left(s+\nu-1, \overline{\varepsilon} y_1(s+\nu-1)\right) \\ &= (A\overline{\varepsilon} y_1)(t).\end{aligned}$$

Let $y_1^* = \overline{\varepsilon} y_1$. Then $Ay_1^* \le y_1^*$. This completes the proof. □

Theorem 4.2 *Let y_0^*, y_1^* be defined in Theorem 4.1. Then (1.1) has a unique positive solution $y^* \in [y_0^*, y_1^*]$. Moreover, for any $y_0 \in [y_0^*, y_1^*]$, the sequence $y_n = A^n y_0 \to y^*$ $(n \to \infty)$ uniformly in $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$.*

Proof From (H6), A is an increasing operator. Then from Section 3 and Theorem 4.1, we note that all conditions of Lemma 2.7 are satisfied. Then A has the smallest fixed point y_0^{**} and the largest fixed point y_1^{**} in $[y_0^*, y_1^*]$, respectively. Moreover, $y_0^{**} = \lim_{n\to\infty} A^n y_0^*$ and $y_1^{**} = \lim_{n\to\infty} A^n y_1^*$. Next we claim that $y_0^{**}(t) \equiv y_1^{**}(t)$ for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. Indeed, from (4.1), there exist $b_i \geq a_i > 0 (i = 1, 2)$ such that

$$a_1 w(t) \leq y_0^{**}(t) \leq b_1 w(t),\ a_2 w(t) \leq y_1^{**}(t) \leq b_2 w(t), \quad \forall t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}},$$

from the fact that y_0^{**} and y_1^{**} are fixed points of A with $y_0^{**}, y_1^{**} \in [y_0^*, y_1^*]$. Therefore, $y_0^{**} \geq \frac{a_1}{b_2} y_1^{**}$. Let $k_0 = \sup\{k > 0 : y_0^{**} \geq k y_1^{**}\}$. Then $k_0 > 0$ and $y_0^{**} \geq k_0 y_1^{**}$. We now show $k_0 \geq 1$. Suppose the contrary. Then $k_0 < 1$ and

$$\begin{aligned} y_0^{**}(t) &= \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} f(s+\nu-1, y_0^{**}(s+\nu-1)) \\ &\geq \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} f(s+\nu-1, k_0 y_1^{**}(s+\nu-1)) \\ &\geq \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} k_0^{\mu} f(s+\nu-1, y_1^{**}(s+\nu-1)). \end{aligned}$$

Let $g(s+\nu-1) = f(s+\nu-1, k_0 y_1^{**}(s+\nu-1)) - k_0^{\mu} f(s+\nu-1, y_1^{**}(s+\nu-1))$ for $s \in [0, T]_{\mathbb{Z}}$. Then $g \in P\backslash\{0\}$, and from (4.1) there exist $b_3 \geq a_3 > 0$ such that

$$a_3 w(t) \leq \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} g(s+\nu-1) \leq b_3 w(t).$$

Consequently, we have

$$\begin{aligned} y_0^{**}(t) &\geq \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} g(s+\nu-1) + \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} k_0^{\mu} f(s+\nu-1, y_1^{**}(s+\nu-1)) \\ &\geq \frac{a_3}{b_2} y_1^{**}(t) + k_0^{\mu} y_1^{**}(t) > k_0 y_1^{**}(t), \end{aligned}$$

contradicting the definition of k_0. As a result, $k_0 \geq 1$, and $y_0^{**} \geq y_1^{**}$. This implies that $y_0^{**}(t) \equiv y_1^{**}(t)$ for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. Hence, (1.1) has a unique positive solution $y^* \in [y_0^*, y_1^*]$. Moreover, for any $y_0 \in [y_0^*, y_1^*]$, then we have $y_0^* \leq y_0 \leq y_1^*$ and $y_0^* \leq Ay_0^* \leq Ay_0 \leq Ay_1^* \leq y_1^*$. For $n \in \mathbb{N}^+$ large enough, we have

$$y_0^* \leq Ay_0^* \leq \cdots \leq A^n y_0^* \leq A^n y_0 \leq A^n y_1^* \leq \cdots \leq Ay_1^* \leq y_1^*.$$

Let $n \to \infty$. Then $y_n = A^n y_0 \to y^*$ from the fact that $\lim_{n\to\infty} A^n y_0^* = \lim_{n\to\infty} A^n y_1^* = y^*$. This completes the proof. □

Remark 4.3 In [11], the authors established the existence of a unique positive solution to the fractional q-difference boundary value problem

$$\begin{cases} (D_q^{\alpha} y)(x) = -f(x, y(x)), & 0 < x < 1, \quad 2 < \alpha \leq 3, \\ y(0) = (D_q y)(0) = 0, & (D_q y)(1) = 0, \end{cases} \tag{4.2}$$

where f is a nonnegative continuous function and one of their assumptions is

$(\mathrm{H})_{\mathrm{Yang}}$ for any $x \in [0, 1]$, $y \in \mathbb{R}^+$, there exist two constants m, n with $m \le 0 \le n < 1$ such that

$$c^n f(x, y) \le f(x, cy) \le c^m f(x, y), \quad \text{for } 0 < c \le 1. \tag{4.3}$$

In our argument above we only need the left inequality of (4.3) (see (H5)) to establish a unique positive solution for (1.1).

Compliance with ethical standards

Conflict of interest The authors declare that they have no competing interests.

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