RACSAM (2018) 112:425–435
https://doi.org/10.1007/s13398-017-0388-5

ORIGINAL PAPER

Stability of general A-cubic functional equations in modular spaces

G. Zamani Eskandani[1] · John Michael Rassias[2]

Received: 5 October 2016 / Accepted: 24 February 2017 / Published online: 21 March 2017

Abstract In this paper, by using fixed point theory, we investigate the generalized Hyers–Ulam stability of an α-cubic functional equation in modular spaces.

Keywords Fixed point · Modular space · Generalized Hyers–Ulam stability

Mathematics Subject Classification Primary 39B52; Secondary 39B72 · 47H09

1 Introduction and preliminaries

In 1940, Ulam [23] asked the first question on the stability problem. In 1941, Hyers [9] solved the problem of Ulam. This result was generalized by Aoki [1] for additive mappings and by Rassias [20] for linear mappings by considering an *unbounded Cauchy difference*. In 1994, a further generalization was obtained by Găvruta [8]. Rassias [16–19] generalized Hyers result. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [2,5–7,11,15,16,21]). We also refer the readers to the books: Czerwik [3] and Hyers, Isac and Rassias [10].

The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano [12] and were intensively developed by Amemiya, Koshi, Shimogaki, Yamamuro [14,25] and others.

✉ G. Zamani Eskandani
zamani@tabrizu.ac.ir

John Michael Rassias
jrassias@primedu.uoa.gr; jrass@otenet.gr

1 Faculty of Mathematical Science, University of Tabriz, Tabriz, Iran

2 Pedagogical Department, National and Capodistrian University of Athens, 4 Agamemnonos Street, Aghia Paraskevi, Athens 15342, Greece

Definition 1.1 Let X be a vector space over a field K ($\mathbb{R}$ or $\mathbb{C}$). A generalized functional $\rho : X \longrightarrow [0, \infty]$ is called a modular if for arbitrary $x, y \in X$, ρ satisfies:

(a) $\rho(x) = 0$ if and only if $x = 0$,
(b) $\rho(ax) = \rho(x)$ for every scalar a with $|a| = 1$,
(c) $\rho(ax + by) \leq \rho(x) + \rho(y)$, whenever $a, b \geq 0$ and $a + b = 1$.

If we replace (c) by

(c') $\rho(ax+by) \leq a\rho(x)+b\rho(y)$, whenever $a, b \geq 0$ and $a+b = 1$, then the modular ρ is called convex. A modular ρ defines a corresponding modular space, i.e., the vector space X_ρ given by:

$$X_\rho = \{x \in X |\, \rho(\lambda x) \longrightarrow 0 \; as \; \lambda \longrightarrow 0\}.$$

Definition 1.2 Let $\{x_n\}$ and x be in X_ρ. Then

(i) The sequence $\{x_n\}$, with $x_n \in X_\rho$, is ρ-convergent to x if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(ii) The sequence $\{x_n\}$, with $x_n \in X_\rho$, is called ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(iii) A subset S of X_ρ is called ρ-complete if and only if any ρ-Cauchy sequence is ρ-convergent to an element of S.

Fatou property The modular ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ-convergent to x. A function modular is said to satisfy the Δ_α-condition ($\alpha \in \mathbb{N}, \alpha \geq 2$) if there exists $\kappa > 0$ such that $\rho(\alpha x) \leq \kappa\rho(x)$, for all $x \in X_\rho$.

Remark Δ_α-condition implies Δ_2-condition.

Definition 1.3 Let X_ρ be a modular space and C be a nonempty subset of X_ρ. The self-map $T : C \to C$ is said to be quasicontraction if there exists $k < 1$ such that

$$\rho\big(Tx - Ty\big) \leq k \max \Big\{\rho(x - y), \rho(x - Ty), \rho(y - Tx), \rho(x - Tx), \rho(y - Ty)\Big\},$$

for any $x, y \in C$.

Definition 1.4 Given a modular space X_ρ, a nonempty subset $C \subseteq X_\rho$, and a mapping $T : C \to C$, the orbit of T around a point x is the set

$$O(T) := \{x, Tx, T^2x, \ldots\},$$

the quantity

$$\delta_\rho(T) := \sup\{\rho(u - v)|u, v \in O(T)\},$$

is then associated to T and is called the orbital diameter of T at x. In particular, if $\delta_\rho(T) < \infty$, one says that T has a bounded orbit at x.

Theorem 1.5 ([13]) *Let X_ρ be a modular space such that ρ satisfies the Fatou property and let $C \subseteq X_\rho$ be a ρ-complete subset. If $T : C \to C$ is a quasicontraction and T has a bounded orbit at x_0, then the sequence $\{T^n x_0\}$ is ρ-convergent to a point $\omega \in C$.*

Stability of quadratic and generalized Jensen functional equation in modular spaces have been investigated in [22] and [24].

In this paper, we investigate the generalized Hyers–Ulam stability of the α- cubic functional equation

$$\begin{aligned} f(\alpha x + y) + f(\alpha x - y) + f(x + \alpha y) - f(x - \alpha y) \\ = 2\alpha f(x + y) + 2\alpha(\alpha^2 - 1)[f(x) + f(y)], \end{aligned} \tag{1.1}$$

with $\alpha \in \mathbb{N}, \alpha \neq 1$ via the extensive studies of fixed point theory in modular spaces.

2 Stability of α-cubic functional equation (1.1)

Throughout this section, we assume that ρ is a convex modular on ρ-complete modular space X_ρ with the Fatou property such that satisfies the Δ_α-condition with $0 < \kappa \leq \alpha$. In addition, let V be a linear space. For convenience, we use the following abbreviation for a given function $f : V \longrightarrow X_\rho$:

$$D_\alpha f(x, y) := f(\alpha x + y) + f(\alpha x - y) + f(x + \alpha y) - f(x - \alpha y)$$
$$-2\alpha f(x + y) - 2\alpha(\alpha^2 - 1)[f(x) + f(y)]$$

with $\alpha \in \mathbb{N}, \alpha \neq 1$ and for all $x, y \in V$. We shall need the following lemmas:

Lemma 2.1 *If a mapping $f : X \to Y$ satisfies the functional equation*

$$f(x + \alpha y) - f(x - \alpha y) = \alpha[f(x + y) - f(x - y)] + 2\alpha(\alpha^2 - 1)f(y), \tag{2.1}$$

with $\alpha \in \mathbb{N}, \alpha \neq 1$ and for all $x, y \in X$, then f is cubic.

Proof Replacing (x, y) with $(0, 0)$ in (2.1), we get $2\alpha(\alpha^2 - 1)f(0) = 0$ with $\alpha \in \mathbb{N}, \alpha \neq 1$. Therefore $f(0) = 0$. Replacing (x, y) with $(0, x)$ and $(0, -x)$ in (2.1), we get, respectively, equations:

$$f(\alpha x) - f(-\alpha x) = \alpha[(2\alpha^2 - 1)f(x) - f(-x)],$$
$$f(-\alpha x) - f(\alpha x) = \alpha[(2\alpha^2 - 1)f(-x) - f(x)]. \tag{2.2}$$

By adding these two equations, one can obtain $f(-x) = -f(x)$. By using (2.2) and $f(-x) = -f(x)$, we get $f(\alpha x) = \alpha^3 f(x)$ with $\alpha \in \mathbb{N}, \alpha \neq 1$ and for all $x \in X$ (See [4]). □

Lemma 2.2 *If a mapping $f : X \to Y$ satisfies (1.1) for all $x, y \in X$, then f is cubic.*

Proof Replacing (x, y) with $(0, 0)$ in (1.1), we get $f(0) = 0$. Replacing (x, y) with $(x, 0)$ in (1.1), we get,

$$f(\alpha x) = \alpha^3 f(x), \tag{2.3}$$

for all $x \in X$. By setting $x = 0$ and using (2.3), we get $f(-y) = -f(y)$ for all $y \in X$, that is f is odd. Replacing (x, y) with $(x, -y)$ in (1.1) and using oddness of f, we get,

$$f(\alpha x - y) + f(\alpha x + y) + f(x - \alpha y) - f(x + \alpha y)$$
$$= 2\alpha f(x - y) + 2\alpha(\alpha^2 - 1)[f(x) - f(y)] \tag{2.4}$$

for all $x, y \in X$. It follows from (1.1) and (2.4) that

$$f(x + \alpha y) - f(x - \alpha y) = \alpha[f(x + y) - f(x - y)] + 2\alpha(\alpha^2 - 1)f(y), \tag{2.5}$$

for all $x, y \in X$. It follows from Lemma 2.1 that f is cubic. □

Theorem 2.3 *Let $\varphi : V^2 \longrightarrow [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{\alpha^{3n}} \varphi(\alpha^n x, \alpha^n y) = 0, \tag{2.6}$$

and

$$\varphi(\alpha x, \alpha y) \leq \alpha^3 L\varphi(x, y), \tag{2.7}$$

for all $x, y \in V$ *with* $L < 1$*. Suppose that* $f : V \longrightarrow X_\rho$ *satisfies the condition*

$$\rho\big(D_\alpha f(x, y)\big) \le \varphi(x, y), \tag{2.8}$$

for all $x, y \in V$ *and* $f(0) = 0$*. Then there exists a unique cubic mapping* $C_\alpha : V \longrightarrow X_\rho$ *such that*

$$\rho\big(C_\alpha(x) - f(x)\big) \le \frac{1}{\alpha^3(1-L)}\varphi(x, 0), \tag{2.9}$$

for all $x \in V$*.*

Proof We consider the set

$$M = \{g : V \to X_\rho\}$$

and define the function $\overline{\rho}$ on M as follows,

$$\overline{\rho}(g) =: \inf\{c > 0 : \ \rho(g(x)) \le c\varphi(x, 0), \ \ \forall x \in V\}.$$

We show that $\overline{\rho}$ is a convex modular on M. It is also easy to verify that $\overline{\rho}$ satisfies the axioms (a) and (b) of a modular. We will next show that $\overline{\rho}$ is convex, and hence (c') is satisfied. Let $\epsilon > 0$ be given. Then there exist real constants $c_1 > 0$ and $c_2 > 0$ such that

$$\overline{\rho}(g) \le c_1 \le \overline{\rho}(g) + \epsilon, \ \ \overline{\rho}(h) \le c_2 \le \overline{\rho}(h) + \epsilon.$$

Also

$$\rho(g(x)) \le c_1\varphi(x, 0), \ \ \rho(h(x)) \le c_2\varphi(x, 0).$$

for all $x \in V$. If $a + b = 1$ and $a, b \ge 0$, then we get

$$\begin{aligned}\rho(ag(x) + bh(x)) &\le a\rho(g(x)) + b\rho(h(x))\\ &\le (c_1 a + c_2 b)\varphi(x, 0),\end{aligned}$$

so we get

$$\overline{\rho}(ag + bh) \le a\overline{\rho}(g) + b\overline{\rho}(h) + (a + b)\epsilon,$$

This concludes that $\overline{\rho}$ is a convex modular on M. Now we show that $M_{\overline{\rho}}$ is $\overline{\rho}$-complete. Let $\{g_n\}$ be a $\overline{\rho}$-Cauchy sequence in $M_{\overline{\rho}}$ and let $\epsilon > 0$ be given. There exists a positive integer $n_0 \in \mathbb{N}$ such that

$$\overline{\rho}(g_n - g_m) < \epsilon, \tag{2.10}$$

for all $n, m \ge n_0$. We have

$$\rho(g_n(x) - g_m(x)) \le \epsilon\varphi(x, 0) \tag{2.11}$$

for all $x \in V$ and $n, m \ge n_0$. Therefore if x is any given point of V, $\{g_n(x)\}$ is a ρ-Cauchy sequence in X_ρ. Since X_ρ is ρ-complete, so $\{g_n(x)\}$ is convergent in X_ρ, for each $x \in V$. Hence, we can define a function $g : V \to X_\rho$ by:

$$g(x) := \lim_{n\to\infty} g_n(x), \tag{2.12}$$

for all $x \in V$. Since ρ satisfies the Fatou property, it follows from (2.11) that

$$\rho(g_n(x) - g(x)) \le \liminf_{m\to\infty} \rho(g_n(x) - g_m(x)) \le \epsilon\varphi(x, 0), \tag{2.13}$$

so

$$\overline{\rho}(g_n - g) \leq \epsilon, \tag{2.14}$$

for all $n \geq n_0$. Thus, $\{g_n\}$ is $\overline{\rho}$-converges, so that $M_{\overline{\rho}}$ is $\overline{\rho}$-complete.

Now we show that $\overline{\rho}$ satisfies the Fatou property. Suppose that $\{g_n\}$ is a sequence in $M_{\overline{\rho}}$ which is $\overline{\rho}$- convergent to an element $g \in M_{\overline{\rho}}$. Let $\epsilon > 0$ be given. For each $n \in \mathbb{N}$, let c_n be a constant such that

$$\overline{\rho}(g_n) \leq c_n \leq \overline{\rho}(g_n) + \epsilon. \tag{2.15}$$

so

$$\rho(g_n(x)) \leq c_n \varphi(x, 0), \tag{2.16}$$

for all $x \in V$. Since ρ satisfies the Fatou property, we have

$$\begin{aligned} \rho(g(x)) &\leq \liminf_{n\to\infty} \rho(g_n(x)) \\ &\leq \liminf_{n\to\infty} c_n \varphi(x, 0) \\ &\leq \left[\liminf_{n\to\infty} \overline{\rho}(g_n) + \epsilon\right] \varphi(x, 0) \end{aligned}$$

Thus, we have

$$\overline{\rho}(g) \leq \liminf_{n\to\infty} \overline{\rho}(g_n) + \epsilon.$$

So $\overline{\rho}$ satisfies the Fatou property. We consider the function $\tau : M_{\overline{\rho}} \to M_{\overline{\rho}}$ defined by:

$$\tau g(x) = \frac{1}{\alpha^3} g(\alpha x),$$

for all $x \in V$ and $g \in M_{\overline{\rho}}$. Let $g, h \in M_{\overline{\rho}}$ and let $c \in [0, 1]$ be an arbitrary constant with $\overline{\rho}(g - h) < c$. From the definition of $\overline{\rho}$, we have $\rho(g(x) - h(x)) \leq c\varphi(x, 0)$ for all $x \in V$. By (2.7) and the last inequality, we get

$$\begin{aligned} \rho\Big(\frac{g(\alpha x)}{\alpha^3} - \frac{h(\alpha x)}{\alpha^3}\Big) &\leq \frac{1}{\alpha^3} \rho(g(\alpha x) - h(\alpha x)) \\ &\leq \frac{1}{\alpha^3} c\varphi(\alpha x, 0) \\ &\leq cL\varphi(x, 0), \end{aligned}$$

for all $x \in V$. Hence, $\overline{\rho}(\tau g - \tau h) \leq L\overline{\rho}(g - h)$, for all $g, h \in M_{\overline{\rho}}$, that is, τ is a $\overline{\rho}$-contraction. Next, we show that τ has a bounded orbit at f. Letting $y = 0$ in (2.8), we get

$$\rho\left(\frac{f(\alpha x)}{\alpha^3} - f(x)\right) \leq \frac{1}{\alpha^3} \varphi(x, 0), \tag{2.17}$$

for all $x \in V$. Replacing x with αx in (2.17), we get

$$\rho\left(\frac{f(\alpha^2 x)}{\alpha^3} - f(\alpha x)\right) \leq \frac{1}{\alpha^3} \varphi(\alpha x, 0), \tag{2.18}$$

By using (2.17) and (2.18), we get

$$\rho\left(\frac{f(\alpha^2x)}{\alpha^6}-f(x)\right)\leq\rho(\frac{f(\alpha^2x)}{\alpha^6}-\frac{f(\alpha x)}{\alpha^3})+\rho(\frac{f(\alpha x)}{\alpha^3}-f(x))$$
$$\leq\frac{1}{\alpha^6}\varphi(\alpha x,0)+\frac{1}{\alpha^3}\varphi(x,0), \tag{2.19}$$

for all $x\in V$. By induction, we can easily see that

$$\rho(\frac{f(\alpha^nx)}{\alpha^{3n}}-f(x))\leq\sum_{i=1}^{n}\frac{1}{\alpha^{3i}}\varphi(\alpha^{i-1}x,0)$$
$$\leq\frac{1}{L\alpha^3}\varphi(x,0)\sum_{i=1}^{n}L^i$$
$$\leq\frac{1}{\alpha^3(1-L)}\varphi(x,0), \tag{2.20}$$

for all $x\in V$. It follows from inequality (2.20) that

$$\rho\left(\frac{f(\alpha^nx)}{\alpha^{3n}}-\frac{f(\alpha^kx)}{\alpha^{3k}}\right)\leq\frac{1}{2}\rho\left(2\frac{f(\alpha^nx)}{\alpha^{3n}}-2f(x)\right)+\frac{1}{2}\rho\left(2\frac{f(\alpha^kx)}{\alpha^{3k}}-2f(x)\right)$$
$$\leq\frac{\kappa}{2}\rho\left(\frac{f(\alpha^nx)}{\alpha^{3n}}-f(x)\right)+\frac{\kappa}{2}\rho\left(\frac{f(\alpha^kx)}{\alpha^{3k}}-f(x)\right)$$
$$\leq\frac{\kappa}{\alpha^3(1-L)}\varphi(x,0),$$

for every $x\in V$ and $n,k\in\mathbb{N}$, By the definition of $\overline{\rho}$, we conclude that

$$\overline{\rho}(\tau^nf-\tau^kf)\leq\frac{\kappa}{\alpha^3(1-L)},$$

which implies the boundedness of an orbit of τ at f. It follows from Theorem 1.5 that, the sequence $\{\tau^nf\}$ $\overline{\rho}$-converges to $C_\alpha\in M_{\overline{\rho}}$. Now, by the $\overline{\rho}$-contractivity of τ, we have

$$\overline{\rho}(\tau^nf-\tau C_\alpha)\leq L\overline{\rho}(\tau^{n-1}f-C_\alpha).$$

Passing to the limit $n\to\infty$ and applying the Fatou property of $\overline{\rho}$, we obtain that

$$\overline{\rho}(\tau C_\alpha-C_\alpha)\leq\liminf_{n\to\infty}\overline{\rho}(\tau C_\alpha-\tau^nf)$$
$$\leq L\liminf_{n\to\infty}\overline{\rho}(C_\alpha-\tau^{n-1}f)=0.$$

Therefore, C_α is a fixed point of τ. Letting $x=\alpha^nx$ and $y=\alpha^ny$ in (2.8), we get

$$\rho\big(D_\alpha f(\alpha^nx,\alpha^ny)\big)\leq\varphi(\alpha^nx,\alpha^ny),$$

for all $x,y\in V$. Therefore

$$\rho\big(\frac{1}{\alpha^{3n}}D_\alpha f(\alpha^nx,\alpha^ny)\big)\leq\frac{1}{\alpha^{3n}}\varphi(\alpha^nx,\alpha^ny), \tag{2.21}$$

Employing the limit $n\to\infty$, we get

$$D_\alpha C_\alpha(x,y)=0,$$

for all $x,y\in V$. It follows from Lemma 2.2, that C_α is cubic. By using (2.20), we get (2.9).

To prove the uniqueness of C_α, let $C : V \to X_\rho$ be another cubic mapping satisfying (2.9). Then, C is a fixed point of τ.

$$\overline{\rho}\big(C_\alpha - C\big) = \overline{\rho}\big(\tau C_\alpha - \tau C\big) \leq L\overline{\rho}\big(C_\alpha - C\big),$$

which implies that $\overline{\rho}\big(C_\alpha - C\big) = 0$ or $C_\alpha = C$. This completes the proof.

Corollary 2.4 *Let X be a Banach space, $\varphi : V^2 \longrightarrow [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{\alpha^{3n}}\varphi(\alpha^n x, \alpha^n y) = 0,$$

and

$$\varphi(\alpha x, \alpha y) \leq L\alpha^3\varphi(x, y),$$

for all $x, y \in V$ with $L < 1$. Suppose that $f : V \longrightarrow X$ satisfies the following condition

$$\big\| D_\alpha f(x, y)\big\| \leq \varphi(x, y),$$

$x, y \in V$ and $f(x) = 0$. Then there exists a unique cubic mapping $C_\alpha : V \longrightarrow X$ such that

$$\big\| C_\alpha(x) - f(x)\big\| \leq \frac{1}{\alpha^3(1-L)}\varphi(x, 0),$$

for all $x \in V$.

Proof It is known that every normed space is modular space with the modular $\rho(x) = \|x\|$ and satisfies the Δ_α-condition with $\kappa = \alpha$. □

Theorem 2.5 *Let $\varphi : V^2 \longrightarrow [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} \kappa^{3n}\varphi\left(\frac{x}{\alpha^n}, \frac{y}{\alpha^n}\right) = 0, \tag{2.22}$$

and

$$\varphi\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) \leq \frac{L}{\alpha^3}\varphi(x, y), \tag{2.23}$$

for all $x, y \in V$ with $L < 1$. Suppose that $f : V \longrightarrow X_\rho$ satisfies the condition

$$\rho\big(D_\alpha f(x, y)\big) \leq \varphi(x, y), \tag{2.24}$$

for all $x, y \in V$ and $f(0) = 0$. Then there exists a unique mapping $C_\alpha : V \longrightarrow X_\rho$ such that

$$\rho\big(C_\alpha(x) - f(x)\big) \leq \frac{L}{\alpha^3(1-L)}\varphi(x, 0), \tag{2.25}$$

for all $x \in V$.

Proof We consider the set

$$M = \{g : V \to X_\rho\}$$

and define the function $\overline{\rho}$ on M as follows,

$$\overline{\rho}(g) =: \inf\{c > 0 : \ \rho(g(x)) \leq c\varphi(x, 0), \ \ \forall x \in V\}.$$

Similar to the proof of Theorem 2.3, we have:

1. $\overline{\rho}$ is a convex modular on M,
2. $M_{\overline{\rho}}$ is $\overline{\rho}$-complete.
3. $\overline{\rho}$ satisfies the Fatou property.

Now, we consider the function $\tau : M_{\overline{\rho}} \to M_{\overline{\rho}}$ defined by:

$$\tau g(x) = \alpha^3 g\left(\frac{x}{\alpha}\right),$$

for all $x \in V$ and $g \in M_{\overline{\rho}}$. Let $g, h \in M_{\overline{\rho}}$ and let $c \in [0, 1]$ be an arbitrary constant with $\overline{\rho}(g - h) < c$. From the definition of $\overline{\rho}$, we have $\rho(g(x) - h(x)) \le c\varphi(x, 0)$ for all $x \in V$. By the assumption and the last inequality, we get

$$\begin{aligned}\rho\left(\alpha^3 g\left(\frac{x}{\alpha}\right) - \alpha^3 h\left(\frac{x}{\alpha}\right)\right) &\le \kappa^3 \rho\left(g\left(\frac{x}{\alpha}\right) - g\left(\frac{x}{\alpha}\right)\right)\\ &\le \kappa^3 c\varphi\left(\frac{x}{\alpha}, 0\right)\\ &\le cL\varphi(x, 0),\end{aligned}$$

for all $x \in V$. Hence, $\overline{\rho}(\tau g - \tau h) \le L\overline{\rho}(g - h)$, for all $g, h \in \mathfrak{M}_{\overline{\rho}}$ that is, τ is a $\overline{\rho}$-contraction. Next, we show that τ has a bounded orbit at f. Letting $y = 0$ in (2.24), we get

$$\rho(\alpha^3 f(x) - f(\alpha x)) \le \varphi(x, 0), \tag{2.26}$$

for all $x \in V$. Replacing x with $\frac{x}{\alpha}$ in (2.26), we get

$$\rho\left(\alpha^3 f\left(\frac{x}{\alpha}\right) - f(x)\right) \le \varphi(\frac{x}{\alpha}, 0), \tag{2.27}$$

for all $x \in V$. Replacing x with $\frac{x}{\alpha}$ in (2.27), we get

$$\rho\left(\alpha^3 f\left(\frac{x}{\alpha^2}\right) - f\left(\frac{x}{\alpha}\right)\right) \le \varphi(\frac{x}{\alpha^2}, 0), \tag{2.28}$$

for all $x \in V$. By using (2.26), (2.27) and (2.28), we get

$$\begin{aligned}\rho(\alpha^6 f(\frac{x}{\alpha^2}) - f(x)) &\le \rho(\alpha^6 f(\frac{x}{\alpha^2}) - \alpha^3 f(\frac{x}{\alpha})) + \rho(\alpha^3 f(\frac{x}{\alpha}) - f(x))\\ &\le \kappa^3 \rho(\alpha^3 f(\frac{x}{\alpha^2}) - f(\frac{x}{\alpha})) + \rho(\alpha^3 f(\frac{x}{\alpha}) - f(x))\\ &\le \alpha^3 \varphi(\frac{x}{\alpha^2}, 0) + \varphi(\frac{x}{\alpha}, 0),\end{aligned} \tag{2.29}$$

for all $x \in V$. By induction, we can easily see that

$$\begin{aligned}\rho\big(\alpha^{3n} f(\frac{x}{\alpha^n}) - f(x)\big) &\le \frac{1}{\alpha^3}\sum_{i=1}^{n} \alpha^{3i}\varphi(\frac{x}{\alpha^i}, 0)\\ &\le \frac{1}{\alpha^3}\varphi(x, 0)\sum_{i=1}^{n} L^i\\ &\le \frac{L}{\alpha^3(1 - L)}\varphi(x, 0),\end{aligned} \tag{2.30}$$

for all $x \in V$. It follows from inequality (2.30) that

$$\rho\big(\alpha^{3n} f(\frac{x}{\alpha^n}) - \alpha^{3k} f(\frac{x}{\alpha^k})\big) \leq \frac{1}{2}\rho(2\alpha^{3n} f(\frac{x}{\alpha^n}) - 2f(x)) + \frac{1}{2}\rho(2\alpha^{3k} f(\frac{x}{\alpha^k}) - 2f(x)) \leq \frac{kL}{\alpha^3(1-L)}\varphi(x, 0), \tag{2.31}$$

for every $x \in V$ and $n, k \in \mathbb{N}$, By the definition of $\overline{\rho}$, we conclude that

$$\overline{\rho}(\tau^n f - \tau^k f) \leq \frac{kL}{\alpha^3(1-L)},$$

which implies the boundedness of an orbit of τ at f. It follows from Theorem 1.5 that, the sequence $\{\tau^n f\}$ $\overline{\rho}$-converges to $C_\alpha \in M_{\overline{\rho}}$. Now, by the $\overline{\rho}$-contractivity of τ, we have

$$\overline{\rho}(\tau^n f - \tau C_\alpha) \leq L\overline{\rho}(\tau^{n-1} f - C_\alpha).$$

Employing the limit $n \to \infty$ and applying the Fatou property of $\overline{\rho}$, we obtain that

$$\overline{\rho}(\tau C_\alpha - C_\alpha) \leq \liminf_{n\to\infty} \overline{\rho}(\tau C_\alpha - \tau^n f) \leq L \liminf_{n\to\infty} \overline{\rho}(C_\alpha - \tau^{n-1} f) = 0.$$

Therefore, C_α is a fixed point of τ. Letting $x = \dfrac{x}{\alpha^n}$ and $y = \dfrac{y}{\alpha^n}$ in (2.24), we get

$$\rho\big(D_\alpha f(\frac{x}{\alpha^n}, \frac{y}{\alpha^n})\big) \leq \varphi(\frac{x}{\alpha^n}, \frac{y}{\alpha^n}),$$

for all $x, y \in V$. Therefore

$$\rho\left(\alpha^{3n} D_\alpha f\left(\frac{x}{\alpha^n}, \frac{y}{\alpha^n}\right)\right) \leq \kappa^{3n}\varphi\left(\frac{x}{\alpha^n}, \frac{y}{\alpha^n}\right),$$

Passing to the limit $n \to \infty$, we get

$$D_\alpha C_\alpha(x, y) = 0$$

for all $x, y \in V$. It follows from Lemma 2.2 that C_α is cubic. By using (2.30), we get (2.25). □

Corollary 2.6 *Let X be a Banach space, $\varphi : V^2 \longrightarrow [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} \alpha^{3n}\varphi\left(\frac{x}{\alpha^n}, \frac{y}{\alpha^n}\right) = 0,$$

and

$$\varphi\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) \leq \frac{L}{\alpha^3}\varphi(x, y),$$

for all $x, y \in V$ with $L < 1$. Suppose that $f : V \longrightarrow X$ satisfies the condition

$$\left\|D_\alpha f(x, y)\right\| \leq \varphi(x, y),$$

for all $x, y \in V$ and $f(0) = 0$. Then there exists a unique cubic mapping $C_\alpha : V \longrightarrow X$ such that

$$\left\|C_\alpha(x) - f(x)\right\| \leq \frac{L}{\alpha^3(1-L)}\varphi(x, 0),$$

for all $x \in V$.

Proof It is known that every normed space is modular space with the modular $\rho(x) = \|x\|$ and satisfies the Δ_α-condition with $\kappa = \alpha$. □

Remark 2.7 In Corollaries 2.4 and 2.6, by replacing φ with:

$$\varphi(x, y) = \|x\|^p + \|y\|^p,$$
$$\varphi(x, y) = \|x\|^p \|y\|^q,$$
$$\varphi(x, y) = \|x\|^p + \|y\|^p + \|x\|^r \|y\|^s,$$

under suitable conditions, it is possible to obtain some corollaries.

Acknowledgements The first author was supported by University of Tabriz.

References

1. Aoki, T.: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Japan **2**, 64–66 (1950)
2. Bouikhalene, B., Eloqrachi, E.: Hyers-Ulam stability of spherical functions. Georgian Math. J. **23**(2), 181–189 (2016)
3. Czerwik, S.: Functional equations and inequalities in several variables. World Scientific, New Jersey, London, Singapore, Hong Kong (2002)
4. Eskandani, G.Z., Rassias, J.M.: Approximation of general α-cubic functional equations in 2-Banach spaces. Ukr. Math. J. **10**, 1430–1436 (2017)
5. Eskandani, G.Z., Rassias, J.M., Gavruta, P.: Generalized Hyers-Ulam stability for a general cubic functional equation in quasi–normed spaces, Asian-Eur. J. Math. 4(03), 413–425 (2011)
6. Găvruta, P.: On a problem of G. Isac and Th. M. Rassias concerning the stability of mappings. J. Math. Anal. Appl. **261**, 543–553 (2001)
7. Găvruta, P.: An answer to question of John M. Rassias concerning the stability of Cauchy equation, Advanced in Equation and Inequality, Edited by John M. Rassias, Hadronic Press Mathematics Series pp 67–71 (1999)
8. Găvruta, P.: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431–436 (1994)
9. Hyers, D.H.: On the stability of the linear functional equation. Proc. Nat. Acad. Sci. **27**, 222–224 (1941)
10. Hyers, D.H., Isac, G., Rassias, Th.M: Stability of functional equations in several variables. Birkhäuser, Basel (1998)
11. Jung, S.M.: Hyers-Ulam-Rassias stability of functional equations in mathematical analysis. Hadronic Press, Palm Harbor (2001)
12. Nakano, H.: Modulared semi-ordered linear spaces. Maruzen, Tokyo, Japan (1950)
13. Khamsi, M. A.: Quasicontraction Mapping in modular spaces without Δ_2-condition, Fixed Point Theory and Applications Volume, Artical ID 916187, 6 pages (2008)
14. Koshi, S., Shimogaki, T.: On F-norms of quasi-modular spaces. J. Fac. Sci. Hokkaido Univ. Ser. I **15**(3), 202–218 (1961)
15. Park, C.: Homomorphisms between Poisson JC^*-algebras. Bull. Braz. Math. Soc. **36**, 79–97 (2005)
16. Rassias, J.M.: On approximation of approximately linear mappings by linear mappings, J Funct Anal. 46(1), 126—130 (1982)
17. Rassias, J.M.: On approximation of approximately linear mappings by linear mappings. Bull. des Sci. Math. **108**(4), 445–446 (1984)
18. Rassias, J.M.: Solution of a problem of Ulam. J. Approx. Theory **57**(3), 268–273 (1989)
19. Rassias, J.M.: Solution of a stability problem of Ulam. Discussiones Mathematicae **12**, 95–103 (1992)
20. Rassias, Th.M.: On the stability of the linear mapping in Banach spaces. Proc. Amer. Math. Soc. **72**, 297–300 (1978)
21. Rassias, Th.M.: On a modified Hyers-Ulam sequence. J. Math. Anal. Appl. **158**, 106–113 (1991)
22. Sadeghi, G.: A fixed point approach to stability of functional equations in modular spaces. Bull. Malaysian Math. Sci. Soc. **37**, 333–344 (2014)
23. Ulam, S.M.: A collection of the mathematical problems, Interscience Publ. New York, 431–436 (1960)

24. Wongkum, K., Chaipunya, P., Kumam, P.: On the Generalized Ulam–Hyers–Rassias Stability of Quadratic Mappings in Modular Spaces without Δ_2-Conditions, Journal of Function Spaces, Volume, Article ID 461719, 7 pages (2014)
25. Yamamuro, S.: On conjugate spaces of Nakano spaces. Trans. Amer. Math. Soc. **90**, 291–311 (1959)