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Spaces splittable over the class of Eberlein and descriptive compact spaces

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Abstract We study splittability over some classes of compact spaces which are useful in functional analysis and general topology. Among other things we show that a scattered pseudocompact space splittable over the class of Eberlein compact spaces is Eberlein compact. We also prove that a compact space splittable over the class of Eberlein compact spaces is hereditarily σ-metacompact, and that if X is a compact space splittable over the class of Corson compact spaces, then $d(X) = w(X)$. We also obtain several results on Rosenthal and on descriptive compact spaces. For instance: (1) a compact space is Rosenthal if and only if it is splittable over the class of Rosenthal compacta, (2) $\mathfrak{c}$-metrizable countably compact spaces which split over the class of descriptive compacta are descriptive compact spaces, (3) if X is a compact space splittable over the class of descriptive compact spaces, then $hd(X) = w(X)$, (4) scattered compact spaces splittable over the class of descriptive compact spaces are σ-discrete descriptive compacta, and (5) it is consistent with ZFC that a compact space which splits over the class of descriptive compacta is a descriptive compact space.

Keywords Splittability · Eberlein compact · Rosenthal compact · Descriptive compact · Corson compact · Gul'ko compact · Scattered space · (maximal) Pseudocompact spaces · Countably compact spaces · Preiss–Simon property

Mathematics Subject Classification 54C35 · 54D30 · 54G12 · 54C10 · 54D20

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1 Introduction

All spaces under consideration are Tychonoff. The study of subspaces of function spaces embraces a number of interesting aspects of General Topology and Functional Analysis. In this framework, Arhangelskii [3] defined Eberlein-Grothendieck spaces as those spaces which are homeomorphic to a subspace of the space $C_p(K)$ where K is some compact space. Eberlein-Grothendieck spaces include Eberlein compacta (compact topological spaces homeomorphic to a subset of a Banach space with the weak topology): indeed, a compact space is an Eberlein compact space if and only if it is an Eberlein-Grothendieck compact space. An equivalent useful description: a compact space K is Eberlein compact if and only if there exists in $C_p(K)$ a dense σ-compact subspace. The advantage of this criterion is that it describes the class of Eberlein compacta by means of a topological property of $C_p(K)$.

This last characterization points out that arguments involving *density* in $C_p(K)$ (or in $\mathbb{R}^K$) play a key role in Functional Analysis. In this set-up Tkachuk defined in [20] a space X as splittable if for each $f \in \mathbb{R}^X$, there exists a countable set $A \subset C_p(X)$ such that f belongs to the closure of A in $\mathbb{R}^X$. Arhangel'skii and Shakhmatov proved in [6] that a Tychonoff space X is splittable if and only if for any $A \subset X$, there exists a continuous function from $f: X \to \mathbb{R}^\omega$ such that $f^{-1}(f(A)) = A$. By using this helpful characterization, they also showed that a pseudocompact splittable space is metrizable. Taking as a point of starting these results, Jardon [12] introduced the class of weakly splittable spaces: a Tychonoff space X is weakly splittable if, for each function $f \in \mathbb{R}^X$, there exists a σ-compact subspace $F \subset C_p(X)$ such that f is in the closure of F in $\mathbb{R}^X$. Weakly splittable compact spaces are called weakly Eberlein compact. It was proved in [12] that if X is a weakly Eberlein compact space then $C_p(X)$ is Lindelöf and that a weakly Eberlein compact space is Eberlein compact whenever $|X| \leq \mathfrak{c}$.

Arhangel'skii-Shakmatov characterization can be generalized in the following way. Given a class $\mathcal{P}$ of topological spaces, a space X is splittable over $\mathcal{P}$ if for any $A \subset X$ there exist a space $Y \in \mathcal{P}$ and a continuous function $f: X \to Y$ such that $f^{-1}(f(A)) = A$. In [12] it was proved that a Tychonoff space is weakly splittable if and only if it is splittable over the class of Eberlein–Grothendieck spaces. Moreover, if $\mathcal{P}$ is the class of Eberlein compacta, Jardón and Tkachuk showed in [13] that if X is a scattered or a σ-metrizable compact space splittable over the class $\mathcal{P}$, then X is an Eberlein compact space, and that under the axiom of constructibility, X is an Eberlein compact space.

Some other results which are helpful in this paper are the outcomes on descriptive compact spaces obtained by Oncina and Raja. Raja proved in [17] that a compact space is descriptive if and only if it is fragmentable and hereditarily weakly θ-refinable (weakly submetacompact). Oncina and Raja [15] proved that the class of descriptive compact spaces is stable by closed subspaces, countable products and continuous images. They also proved that any countably compact subspace of a descriptive compact space is compact and that a hereditarily separable closed subset of a descriptive compact space is metrizable.

In this work we study splittability over, among others, the class of Eberlein compacta, Rosenthal compacta and descriptive compacta. Among other things, we show in Sect. 3 that a scattered pseudocompact space splittable over the class of Eberlein compact spaces is Eberlein compact and that a countably compact space splittable over the class of Eberlein compact spaces is compact. These results provide, respectively, partial answers to Questions 4.16 and 4.17 of [13]. We also prove that a compact space splittable over the class of Eberlein compact spaces is hereditarily σ-metacompact which generalizes a result of Yakovlev [21] on Eberlein compact spaces, and that if X is a compact space splittable over the class of Corson compact spaces, then $d(X) = w(X)$. This outcome answers in the positive Question 4.2 of [13]. In the fourth section we deal whit splittability over other classes of compact spaces. Some of the results obtained are: (1) a compact space is Rosenthal if and only if it is splittable over the class of Rosenthal compacta, (2) if a compact space X of cardinality $\leq \mathfrak{c}$ is splittable over the class of fragmentable (respectively, Radon-Nikodym) compact spaces, then it is a fragmentable (respectively, a Radon-Nikodym) compact space, (3) $\mathfrak{c}$-metrizable countably compact spaces which split over the class of descriptive compacta are descriptive compact spaces, (4) if X is a compact space splittable over the class of descriptive compact spaces, then $hd(X) = w(X)$, (5) scattered compact spaces splittable over the class of descriptive compact spaces are σ-discrete descriptive compact spaces, and (6) under $ACP^\sharp$ or $V = L$, compact spaces splittable over the class of descriptive compacta are descriptive compact spaces.

2 Notation and terminology

Given a cardinal $\kappa > \omega$ we denote by $L(\kappa)$ the one-point Lindelöfication of a discrete space of cardinality κ. Continuous images of a closed subset of $(L(\kappa))^\omega$ are called *primarily Lindelöf* spaces. A family $\mathcal{F}$ is said to be *a network modulo a cover* $\mathcal{C}$ if for any $C \in \mathcal{C}$ and every open subset U of X with $C \subset U$ there exists $F \in \mathcal{F}$ such that $C \subset F \subset U$. A space X is *Lindeöf* Σ if there exists a countable family $\mathcal{F}$ of subsets of X which is a network modulo a compact cover $\mathcal{C}$ of the space X.

Some interesting subclasses of Eberlein compacta we will also deal with are Gul'ko compacta and Corson compacta. Recall that a space X is called a *Gul'ko* (respectively, a *Corson*) compact space if $C_p(X)$ is a Lindelöf Σ–space (respectively, a primarily Lindelöf space).

We also discuss splittability over other classes of compact spaces. Recall that a compact space is said to be *Rosenthal* if it can be represented as a space of functions of the first Baire class on some Polish space, equipped with the pointwise convergence topology. They emerge from the study, initiated by Rosenthal, of Banach spaces in which the classical space ℓ^1 embeds isomorphically.

Let $\varphi\colon X \times X \to \mathbb{R}$ be a non negative function such that $\varphi(x, y) = 0$ if and only if $x = y$. A topological space (X, τ) is called fragmentable by φ if for any non-empty $A \subset X$ and $\epsilon > 0$ there exists a τ-open set U such that $U \cap A \neq \emptyset$ and φ-diam$(U \cap A) < \epsilon$. As usual, the φ-diameter of a set $B \subset X$ is defined by φ-diam$(B) = \sup\{\varphi(x, y)\colon x, y \in B\}$. A function $\rho\colon X \times X \to [0, \infty)$ is lower semicontinuous if $\{(x, y) : \rho(x, y) \leq r\}$ is closed for any $r \in \mathbb{R}$. A space is called fragmentable if it is fragmentable by some metric on X. A compact space X is said to be *Radon-Nikodym* if it is a fragmentable space by a lower semicontinuous metric.

A family $\mathcal{H}$ is called isolated if it is discrete in its union endowed with the relative topology; that means that for every $i \in I$ and each $x \in H_i$ there exists an open neighborhood U of x such that $H_j \cap U = \emptyset$ for every $j \in I$ with $j \neq i$. A compact space X is called *descriptive* if its topology has a σ-isolated network.

Given two spaces X and Y, $C_p(X, Y)$ stands for the space of all continuous functions from X to Y endowed with the pointwise convergence topology. When $Y = \mathbb{R}$, the reals endowed with the usual topology, we write $C_p(X)$ instead of $C_p(X, \mathbb{R})$. If Y is a subspace of $C_p(X)$, e^Y denotes the canonical evaluation function $e^Y : X \to C_p(Y)$ defined as $e^Y(x) = e_x : Y \to \mathbb{R}$ with $e_x(f) = f(x)$ for any $f \in Y$. A function $f\colon X \to Y$ is called a *condensation* if it is a continuous bijection; in this case we say that X condenses onto Y.

As usual, ω stands for the set of all natural numbers, ω_1 is the first uncountable ordinal number and $\mathfrak{c}$ for the continuum. Given a space X, the well-known cardinal functions *density*, *weight* and *cellularity* are denoted by $d(X)$, $w(X)$ and $c(X)$, respectively. For the cardinal function $d(X)$, we denote by $hd(X)$ the cardinal function whose value on a space X is equal to $\sup d(A)$, where the sup is taken over all subspaces A of the space X; the function hd is called *hereditary density*. The *tightness of a point* x in a topological space X is the smallest cardinal number $\mathfrak{m} \geq \aleph_0$ with the property that if $x \in \overline{C}$, then there exists $C_0 \subset C$ such that $|C_0| \leq \mathfrak{m}$ and $x \in \overline{C_0}$. If we denote this cardinal number by $t(x, X)$, the tightness of X, $t(X)$, is the supremum of all numbers $t(x, X)$ for $x \in X$. A space X is Fréchet–Urysohn, if for any $A \subset X$ and $x \in \overline{A}$, there exists a sequence in A converging to x. Recall that X is a *Preiss–Simon* space if for every closed subset A of X, each $x \in A$ is a limit of a sequence of non-empty open subsets of A. Notice that every Preiss–Simon space is Fréchet-Urysohn.

The rest of our notation and terminology is standard and follows [3,9,18].

3 Pseudocompact spaces splittable over the class of Eberlein–Grothendieck spaces

A space X is called point-splittable over the class of spaces $\mathcal{P}$ if for any $x \in X$ there exist $Y \in \mathcal{P}$ and a continuous function $f: X \to Y$ such that $f^{-1}(f(x)) = x$. It is evident that splittability over the class $\mathcal{P}$ implies point-splittability over the class $\mathcal{P}$. Our first result states the following

Proposition 1 *Let X be space. The following hold:*

(i) *If X is a pseudocompact space point-splittable over the class of Eberlein–Grothendieck spaces, then X is Preiss–Simon.*
(ii) *If X is a countable space point-splittable over the class of Gul'ko compacta, then X is Fréchet–Uryshon.*

Proof (i) Let B be a closed subset of X and pick any $x \in B$. By the point-splittability assumption on X, there exist an Eberlein–Grothendieck space Y and an onto continuous function $f: X \to Y$ such that $f^{-1}(f(x)) = x$. Observe that Y is a pseudocompact Eberlein–Grothendieck space and, consequently, it is an Eberlein compact space. Thus, Y is Preiss-Simon and there exists a sequence $\{U_n : n \in \omega\}$ of open sets in $\overline{f(B)}$ converging to $f(x)$. Now consider the sequence $\left\{f^{-1}(U_n) \cap B : n \in \omega\right\}$ of open sets in B. We will prove that this sequence converges to x. For this, let W be an open set in B such that $x \in W$ and take an open set E in X such that $E \cap B = W$. Then there exists an open set H for which $x \in H \subset \overline{H} \subset E$. It is clear that the set $X \backslash \overline{H}$ is open, $\overline{X \backslash \overline{H}}$ is pseudocompact and $x \in X \backslash \overline{X \backslash \overline{H}} \subset E$. Being the subspace $Z = f(\overline{X \backslash \overline{H}})$ a pseudocompact Eberlein–Grothendieck space, it is compact. Since $f(x) \notin Z$, the set $V = Y \backslash Z$ is an open neighborhood of $f(x)$. Thus, there exists $m \in \omega$ such that $U_n \subset V \cap B$ for any $n \geq m$. Let us observe now that $f^{-1}(V) \subset E$ because $f^{-1}(V) = (X \backslash f^{-1}(Z))$, $\overline{X \backslash \overline{H}} \subset f^{-1}(f(\overline{X \backslash \overline{H}})) = f^{-1}(Z)$, and $X \backslash \overline{X \backslash \overline{H}} \subset E$. Hence $W_n = f^{-1}(U_n) \cap B \subset E \cap B = W$ for any $n \geq m$. We have just showed that X is a Preiss–Simon space.

(ii) Since any Gul'ko compact space is Fréchet–Urysonh, the results follows from [4, Theorem 16].

□

As we commented in the Introduction, a space is weakly splittable if and only if it is splittable over the class of Eberlein–Grothendieck spaces, and a compact space is weakly Eberlein compact if and only if it is splittable over the class of Eberlein compact spaces (see [12]). We can use these facts to show

Proposition 2 *Let X be a weakly splittable pseudocompact space. If $A \subset X$ is pseudocompact, then it is closed.*

Proof By the above comment, there exist an Eberlein–Grothendieck space Y and a continuous function $f: X \to Y$ such that $f^{-1}(f(A)) = A$. Since $f(A) \subset Y$ is a pseudocompact Eberlein-Grothendieck space, it is compact. Thus $f(A)$ is closed in Y and so is $A = f^{-1}(f(A))$ in X. □

Proposition 2 provides a connection between weakly splittable pseudocompact spaces and maximal pseudocompact spaces. Let us remember that a Tychonoff pseudocompact space (X, τ) is called *maximal pseudocompact* if any stronger Tychonoff topology on X fails to be pseudocompact. In [1, Theorem 3.12] the authors show that every pseudocompact subset of X is maximal pseudocompact if and only if every pseudocompact subset of X is closed. In particular, Eberlein compact spaces are hereditarily maximal pseudocompact ([16, Corollary 6]). Thus, we have

Corollary 1 *If X is a weakly splittable pseudocompact space, then X is hereditarily maximal pseudocompact.*

Recall that a compact space is called *dyadic* if X is a continuous image of $\mathbb{D}^\kappa$ for some cardinal κ where $\mathbb{D}$ is the discrete space $\{0, 1\}$. Taking into account [1, Theorem 3.8, Corollary 3.2], we get

Corollary 2 *Any weakly splittable pseudocompact (para)topological group is a metrizable Eberlein campact space.*

Corollary 3 *Any weakly splittable dyadic space is a metrizable Eberlein campact.*

The following proposition is useful in the sequel.

Proposition 3 *Let $\mathcal{P}$ be a class of spaces which is countably productive and suppose that X is a space splittable over $\mathcal{P}$. If a family $\{A_\alpha \subset X : \alpha \in I\}$ of non-empty subsets of X with $|I| = \kappa \leq 2^\omega$ satisfies:*

(i) $X = \bigcup\{A_\alpha : \alpha \in I\}$, *and*
(ii) $A_\alpha \cap A_\beta = \emptyset$ *for any distinct index* $\alpha, \beta \in I$,

then there exist $Y \in \mathcal{P}$ and a continuous function $f: X \to Y$ such that $f^{-1}(f(A_\alpha)) = A_\alpha$ for any $\alpha \in I$.

Proof There exists a sequence $\{I_n \subset I : n \in \omega\}$ such that for any distinct index $\alpha, \beta \in I$ there are $n, m \in \omega$ with $\alpha \in I_n$, $\beta \in I_m$ and $I_n \cap I_m = \emptyset$. For any $n \in \omega$, consider the set $A_n = \bigcup\{A_\alpha : \alpha \in I_n\}$ and a space $Y_n \in \mathcal{P}$. Choose a continuous map $f_n : X \to Y_n$ such that $f_n^{-1}(f_n(A_n)) = A_n$ $(n \in \omega)$ and let $f = \prod_{n\in\omega} f_n : X \to Y = \prod_{n\in\omega} Y_n \in \mathcal{P}$. It is apparent that $f^{-1}(f(A_\alpha)) = A_\alpha$ for any $\alpha \in I$. □

We now apply the previous proposition to obtain

Proposition 4 *Let X be a space. The following hold:*

(i) *If X is a pseudocompact space of cardinality $\leq \mathfrak{c}$ splittable over the class of Eberlein compact spaces, then X is Eberlein compact. Moreover, if X is separable, then it is metrizable.*
(ii) *If X is a countably compact space of cardinality $\leq \mathfrak{c}$ splittable over the class of Gul'ko compact spaces, then X is Gul'ko compact. Moreover, if X is separable, then it is metrizable.*

Proof (i) Since any countable product of Eberlein compact spaces is Eberlein compact, Proposition 3 tells us that there exists an Eberlein compact space Y and an injective continuous function $f : X \to Y$. The function $f : X \to f(X) \subset Y$ defines a condensation onto the pseudocompact subspace $f(X) \subset Y$. Being the space $f(X)$ Eberlein compact, it is a Preiss-Simon compact space. Condensations from a pseudocompact space to a Preiss-Simon compact space are homeomorphisms so that X and $f(X)$ are homeomorphic. Therefore X is an Eberlein compact space. Moreover, if X is separable, then it is a separable Eberlein compact space. Thus, X is metrizable.

(ii) Any countable product of Gul'ko compact spaces is a Gul'ko compact space. Thus, if we argue as in i), there is a Gul'ko compact space Y and an injective continuous function $f : X \to Y$. The function $f : X \to f(X) \subset Y$ defines a condensation from X to the countably compact subspace $f(X)$ of Y. Observe that $f(X)$ is a Gul'ko compact space because it is a countably compact subspace of a Gul'ko compact space. We will prove that the function $f : X \to Z$ is closed. For this, take any closed subset $A \subset X$. It is clear that $f(A)$ is a countably compact subspace of a Gul'ko compact space and, consequently, it is compact. This shows that f is closed. Therefore X and $f(X)$ are homeomorphic which proves that X is a Gul'ko compact space. Since separable Gul'ko compact spaces are metrizable, X is a separable metrizable space provided that X is separable. □

Yakovlev proved in [21] that an Eberlein compact space is hereditarily σ-metacompact. We extend this property to spaces which split over the class of Eberlein compacta.

Proposition 5 *If X is a compact space splittable over the class of Eberlein compact spaces, then X is hereditarily σ-metacompact.*

Proof Take any $A \subset X$. There exist an Eberlein compact space Y and an onto continuous function $f : X \to Y$ such that $f^{-1}(f(A)) = A$. Since Y is hereditarily σ-metacompact, $f(A)$ is σ-metacompact. Notice now that the function $g = f|_A : A \to f(A)$ is perfect. Since σ-metacompactness is inverse invariant by perfect functions (see [9, Problem 5.3H]), A is σ-metacompact. Thus, X is hereditarily σ-metacompact. □

As we say above, pseudocompact Eberlein-Grothendieck spaces are Eberlein compacta. We study whether a pseudocompact space splittable over the class of Eberlein compact spaces is compact (see questions 4.16 and 4.17 of [13]). The next proposition provides a partial answer to Question 4.17 of [13].

Proposition 6 *If X is a countably compact space splittable over the class of Gul'ko compact spaces, then X is compact.*

Proof Suppose that X is not Lindelöf. Then there exists an open cover $\mathcal{B}$ of X such that $\bigcup \mathcal{D} \neq X$ for any countable subfamily $\mathcal{D} \subset \mathcal{B}$. It is clear that $\kappa = |\mathcal{B}|$ is a non countable cardinal. It follows from Proposition 1 that the cardinality of $\overline{A}$ is $\leq \mathfrak{c}$ for any countable subset $A \subset X$. Thus, Proposition 4 implies that $\overline{A}$ is a (metrizable) compact space. We will define by transfinite induction a free sequence of cardinality $\omega_1 \leq \kappa$. For the first step of the induction, choose any non-empty set $U_0 \in \mathcal{B}$ and take any point $x_0 \in U_0$. If $0 < \alpha < \omega_1$, our induction hypothesis is that two sequences $\{x_\lambda\}_{\lambda<\alpha}$ and $\{U_\lambda\}_{\eta<\alpha}$ have been defined with, for all $\lambda < \alpha$, $x_\lambda \in U_\lambda$ and U_λ a non-empty open subset of $\mathcal{B}$. Now consider, for any $\beta \leq \alpha$, a finite subcover $\mathcal{V}_\beta \subset \mathcal{B}$ of the compact subspace $\overline{A_\beta}$ with $A_\beta = \{x_\lambda : \lambda < \beta\}$.

Since X is not Lindelof, $\bigcup\{U_\lambda : \lambda < \alpha\} \cup \bigcup\{\mathcal{V}_\beta : \beta \leq \alpha\} \neq X$ for all $\alpha < \omega_1$. Thus, we can now choose $U_\alpha \in \mathcal{B}$ and $x_\alpha \in U_\alpha \setminus \left[\bigcup\{U_\lambda : \lambda < \alpha\} \cup \bigcup\{\mathcal{V}_\beta : \beta \leq \alpha\}\right]$. This completes the transfinite induction.

Notice that the definition of the sequence $\{x_\lambda : \lambda < \alpha\}$ implies that $\overline{\{x_\lambda : \lambda < \alpha\}} \cap \overline{\{x_\lambda : \lambda \geq \alpha\}} = \emptyset$ for any $\alpha < \omega_1$. Thus, $A_{\omega_1} = \{x_\lambda : \lambda < \omega_1\}$ is a free sequence.

By Proposition 1 $\overline{A_{\omega_1}}$ is a countably compact Fréchet–Urysohn space so that $|\overline{A_{\omega_1}}| \leq \mathfrak{c}$. From Proposition 4, $\overline{A_{\omega_1}}$ is a Gul'ko compact space. Since the tightness of a Gul'ko compact space is countable, we have that the tightness of $\overline{A_{\omega_1}}$ is countable as well. This implies that $\overline{A_{\omega_1}}$ does not have free sequences of cardinality ω_1, a contradiction. Thus, X is compact. □

From [1, Theorem 3.8], a linearly ordered space is hereditarily maximal pseudocompact if and only if it is countably compact and first countable. Thus, Propositions 2 and 6 imply

Corollary 4 *If a pseudocompact linearly ordered space splits over the class of Eberlein compacta, then it is compact.*

Example 1 Reznichenko constructs a Gul'ko compact space Z for which there exists a point $z_0 \in Z$ such that $X = Z \setminus \{z_0\}$ is pseudocompact (see Problem 222 of [19]). The space X is a pseudocompact non-countably compact space and there exists a condensation $\varphi: X \to Y$ onto a Gul'ko compact space Y (see Problem 224 of [19]). Notice that φ splits X over the Gul'ko compact space Y. Therefore X is a pseudocompact non-countably compact space splittable over the class of Gul'ko compact spaces. Thus, we can not replace *countably compact* by *pseudocompact* in Proposition 6. □

The following proposition answers partially Question 4.16 of [13]. It also generalizes [13, Theorem 3.3] and the well-known result of Alster [2] stating that every scattered Corson compact is Eberlein compact. Recall that a space X is *scattered* if every non-empty subspace $Y \subset X$ has an isolated point.

Proposition 7 *If X is a scattered pseudocompact space which splits over the class of Eberlein compact spaces, then X is Eberlein compact.*

Proof Let $X_0 \subset X$ be the subset of all isolated points of X. It is easy to see that X_0 is open and dense. First we will prove that X is countably compact. Suppose the contrary and let $A = \{a_n : n \in \omega\} \subset X$ be an infinite countable closed subset of X. Proposition 1 tells us that for any $n \in \omega$ there exists a countable set $B_n \subset X_0$ such that $a_n \in \overline{B_n}$. Consider now the open set $B = \bigcup\{B_n : n \in \omega\}$. Then $\overline{B}$ is a pseudocompact spaces which splits over the class of Eberlein-Grothendieck spaces. Notice that Proposition 1 implies that its cardinality is $\leq \mathfrak{c}$. Now Proposition 4 implies that $\overline{B}$ is an Eberlein compact space. It is clear that $A \subset \overline{B}$ so that A is compact. This contradiction shows that X is a countably compact space. Thus, it follows from Proposition 6 that X is compact. Now Proposition 5 tells us that X is a hereditarily σ-metacompact space. Therefore X is an Eberlein compact because any hereditarily σ-metacompact scattered compact space is Eberlein compact (see [21, Theorem 7]). □

A space is said to be *submaximal* if every dense subset is open. One of the reasons to consider submaximal spaces is provided by the theory of maximal spaces (a space X is *maximal* if it is dense-in-itself and no larger topology on the set X is dense-in-itself). Submaximal spaces were characterized by Bourbaki as spaces that do not admit a larger topology with the same semi–regularisation. By above proposition and [5, Corollary 5.13, Corollary 7.9], we have

Corollary 5 *Suppose that X is a pseudocompact submaximal space. If X splits over the class of Eberlein compact spaces, then X is Eberlein compact and the set of all non isolated points of X is finite.*

If X is the set of real numbers endowed with the discrete topology, then the identity map id splits X over $\mathbb{R}$ with its natural topology. Thus, id sends any closed discrete space of cardinality $\mathfrak{c}$ to a non discrete subspace of $\mathbb{R}$. This fact and the techniques used in the proof of Propositions 6 and 7 point out the usefulness and interest of discrete subspaces of pseudocompact spaces which split over the class of Eberlein compact spaces.

Proposition 8 *If X is a pseudocompact space splittable over the class of Eberlein compact spaces and $A \subset X$ is a discrete subspace of cardinality $\kappa \leq \mathfrak{c}$, then there exist an Eberlein compact space Y and a continuous onto function $f : X \to Y$ such that $f^{-1}(f(x)) = x$ for any $x \in A$ and $f(A)$ is a discrete set of cardinality κ.*

Proof Let $A = \{x_\alpha : \alpha \in I\}$ be a discrete subspace of cardinality κ such that $x_\alpha \neq x_\beta$ for distinct $\alpha, \beta \in I$. By Proposition 3, there exist an Eberlein compact space Y and a continuous function $f : X \to Y$ such that $f^{-1}(f(x_\alpha)) = x_\alpha$ for any $\alpha \in I$. Being $f(X)$ a pseudocompact subspace of an Eberlein compact space, it is compact. Assume, without loss of generality, that the function f is surjective. Take, for any x_α, an open set U_α such that $\{x_\alpha\} = U_\alpha \cap A$. From the regularity of X there exists an open set V_α with $x_\alpha \in V_\alpha \subset \overline{V_\alpha} \subset U_\alpha$. Since $X \backslash \overline{V_\alpha}$ is open and $V_\alpha \cap (X \backslash \overline{V_\alpha}) = \emptyset$, the set $\overline{X \backslash \overline{V_\alpha}}$ is pseudocompact and $x_\alpha \notin \overline{X \backslash \overline{V_\alpha}}$. Hence $Z_\alpha = f(\overline{X \backslash \overline{V_\alpha}})$ is compact because it is a pseudocompact subspace of an Eberlein compact space. Now, since $f^{-1}(f(x_\beta)) = x_\beta$ for any $\beta \in I$, we have $y_\alpha = f(x_\alpha) \notin Z_\alpha$ and $y_\beta = f(x_\beta) \in Z_\alpha$ for each $\beta \in I \backslash \{\alpha\}$. Note that the set $W = Y \backslash Z_\alpha$ is open in Y and $W \cap \{y_\gamma : \gamma \in I\} = \{y_\alpha\}$. Therefore $\{y_\gamma : \gamma \in I\}$ is a discrete subset of cardinality κ. □

Any discrete space splits over the two point discrete space. It is worth noting that compact spaces have a different behavior. The following result is implicit in the proof of Theorem 1 of [10].

Proposition 9 *If X is a compact space splittable over the class of compact spaces of cardinality $\leq \kappa$, then $|X| \leq \kappa$.*

The following result answers Question 4.2 of [13] in the positive. Surprisingly, the proof is not a complicated task.

Proposition 10 *If X is a compact space splittable over the class of Corson compact spaces, then $d(X) = w(X)$.*

Proof If $|X| \leq \mathfrak{c}$, then from Proposition 3.12 of [13] it follows that X is a Corson compact space. By [19, Problem 121], we have $d(X) = w(X)$.

Suppose that $|X| = \kappa > \mathfrak{c}$. Since X is Fréchet-Urysohn (see Proposition 3.12 of [13]), $d(X) > \mathfrak{c}$ and the cardinality of X is $\leq d(X)^\omega = d(X)$. On the other hand, it is a well-known fact that, for any space X, we have $d(X) \leq w(X)$. Since for any compact space the inequality $w(X) \leq |X|$ holds, we obtain $d(X) = w(X)$. □

The following result generalizes [12, Proposition 2.7] and [13, Theorem 3.13].

Proposition 11 *Let $\mathcal{P}$ be a class of compact spaces which is κ-productive, stable under closed subspaces and such that $t(Y) \leq \kappa$ for any $Y \in \mathcal{P}$. If $c(Y) = d(Y) = w(Y)$ for any $Y \in \mathcal{P}$, then a compact space X which splits over $\mathcal{P}$ satisfies the equalities $c(X) = d(X) = w(X)$.*

Proof If $|X| \leq 2^\kappa$, then there is a family $\mathcal{B} = \{B_\lambda : \lambda < \kappa\}$ of subsets of X such that for every distinct $x, y \in X$ there exists $B_{\lambda_1}, B_{\lambda_2} \in \mathcal{B}$ such that $B_{\lambda_1} \cap B_{\lambda_2} = \emptyset$, $x \in B_{\lambda_1}$ and $y \in B_{\lambda_2}$. By hypothesis, for any $B_\lambda \in \mathcal{B}$ there exist $Y_\lambda \in \mathcal{P}$ and a continuous function $f_\lambda : X \to Y_\lambda$ such that $f_\lambda^{-1}(f_\lambda(B_\lambda)) = B_\lambda$. Since the continuous function $f : X \to \prod_{\lambda<\kappa} Y_\lambda$ defined by $f(x) = (f_\lambda(x))_{\lambda<\kappa}$ is injective, X is homeomorphic to a closed subspace of a product of κ spaces in $\mathcal{P}$ wihch implies that $X \in \mathcal{P}$. Therefore $c(X) = d(X) = w(X)$.

Suppose now that $|X| > 2^\kappa$. For any $A \subset X$ there exist $Y \in \mathcal{P}$ and a continuous onto function $f : X \to Y$ such that $f^{-1}(f(A)) = A$. It is evident that $w(Y) = c(Y) \leq c(X)$. We will prove that $2^\kappa < c(X)$. Suppose contrary we claim that $c(X) \leq 2^\kappa$. Then $w(Y) \leq 2^\kappa$ and $|Y| \leq (2^\kappa)^\kappa = 2^\kappa$. Thus, X splits over the class of compact spaces of cardinality $\leq 2^\kappa$. Proposition 9 tells us that $|X| \leq 2^\kappa$ which is a contradiction; thus, $2^\kappa < c(X)$. Since the tightness of Y is $\leq \kappa$, the cardinality of Y is $\leq d(Y)^\kappa = c(Y)^\kappa \leq c(X)^\kappa = c(X)$. Hence X is splittable over the class of compact spaces of cardinality $\leq c(X)$ which implies that $|X| \leq c(X)$. Being the tightness of $X \leq \kappa$, we have $|X| \leq w(X)^\kappa = w(X)$ because $|X| > 2^\kappa$. It follows from compactness of X that $w(X) \leq |X|$. Therefore $w(X) = |X|$ and, consequently, $c(X) = d(X) = w(X)$. □

4 Splittability over the classes of Rosenthal and descriptive compacta

In this section, our first result concerns splittability over Rosenthal compact spaces. For further information on the class of Rosenthal compacta, the interested reader can consult Section C.17 in [11]. The properties of Rosenthal compact spaces suggest that it is reasonable that its behavior should in some rough sense similar to the behaviour of metrizable compact spaces. In this spirit we have:

Proposition 12 *If X is a compact space splittable over the class of Rosenthal compact spaces, then X is Rosenthal compact.*

Proof The cardinality of any Rosenthal compact space is $\leq \mathfrak{c}$. Hence X is a compact space splittable over the class of compact spaces of cardinality $\leq \mathfrak{c}$. Proposition 9 implies that the cardinality of X is $\leq \mathfrak{c}$. Since the class of Rosenthal compact spaces is stable by countable products and closed subsets, Proposition 3 tells us that X is homeomorphic to a closed subspace of a countable product of Rosenthal compact spaces. Therefore X is a Rosenthal compact space. □

We now turn our attention to descriptive compact spaces. It is well known that any Gul'ko compact space is descriptive compact. Recall that a space X is called weakly θ-refinable (or weakly submetacompact) if every open cover of X has a σ-isolated refinement.

Raja proved in Proposition 2.3 of [17] the following characterization of descriptive compact spaces.

Theorem 1 *A compact space is descriptive if and only if it is fragmentable and hereditarily weakly θ-refinable.*

Since every countably compact weakly θ-refinable space is compact (see Theorem 9.2 in [8]), the previous results implies that every countably compact subspace of a descriptive compact is compact. For countably compact spaces of cardinality not greater than the continuum, we have

Proposition 13 *If X is a countably compact space of cardinality $\leq \mathfrak{c}$ which splits over the class of descriptive compact spaces, then it is descriptive compact.*

Proof Since the class of descriptive compacta is closed under countable products and closed subspaces, Proposition 3 tells us that there exist a descriptive compact space Y and an injective continuous function $f: X \to Y$. Then the space $f(X)$ is descriptive compact because it is a countably compact subspace of a descriptive compact space. Choose a closed subset E of X. Since X is countably compact, E and $h(E) \subset f(X)$ are countably compact and, *a posteriori*, they are compact. Thus, $h(E)$ is closed in $f(X)$. Hence $f: X \to f(X)$ is closed which implies that it is homeomorphism. Therefore X is a descriptive compact space. □

In Proposition 13 *descriptive* can not be replaced by *Radon–Nikodym* as the following example shows.

Example 2 The identity map $i: \omega_1 \to \omega_1 + 1$ splits the countably compact space ω_1 over the compact space $\omega_1 + 1$. Being a scattered space, the compact space $\omega_1 + 1$ is Radon–Nikodym compact. However, ω_1 is not compact.

The previous example is in marked contrast to the behaviour of compact spaces.

Proposition 14 *If X is a compact space of cardinality $\leq \mathfrak{c}$ splittable over the class of fragmentable (respectively, Radon–Nikodym) compact spaces, then it is fragmentable (Radon–Nikodym) compact.*

Proof Analogous to the proof of Proposition 13. □

The following definition is helpful in the proof of Proposition 15.

Definition 1 (see [13]) Let τ_1 and τ_2 topologies on a set X. The set X is said to have $\mathcal{P}(\tau_1, \tau_2)$ if there exists a sequence $\{A_n : n \in \omega\}$ of subsets of X such that for every $x \in X$ and every $V \in \tau_1$ with $x \in V$, there is $n \in \omega$ and $U \in \tau_2$ such that $x \in A_n \cap U \subset V$. If τ_1 is the topology defined by a metric ρ on X we denote $\mathcal{P}(\tau_1, \tau_2)$ by $\mathcal{P}(\rho, \tau_2)$.

It is a well-known fact that there exists a separable non metrizable scattered descriptive compact space. If X is a hereditarily separable descriptive compact space, then its weight is countable (see Proposition 4.2 in [15]). Introducing the natural modifications, the arguments used in [15] can be generalized to obtain a similar result independent of the weight of X.

Proposition 15 *If X is a descriptive compact space, then $hd(X) = w(X)$.*

Proof It is clear that $hd(X) \leq w(X)$. Suppose that $\kappa = hd(X)$. From Theorem 2.5 of [15] it follows that there exists a finer fragmentable metric ρ such that X has $\mathcal{P}(\rho, \tau)$ with a sequence of τ-closed sets $\{A_n: n \in \omega\}$, where τ is the topology of X. By the definition of κ, we have τ-density of A_n is $\leq \kappa$; hence for every $n \in \omega$ we can choose a τ-dense set B_n of cardinality $\leq \kappa$ with $B_n \subset A_n$. Since X has $\mathcal{P}(\rho, \tau)$ with the sequence of τ-closed sets $\{A_n: n \in \omega\}$, the set $B = \bigcup\{B_n: n \in \omega\}$ is ρ-dense in X.

It follows from fragmentability of X that for any $n \in \omega$ there exists a family $\{U^n_\alpha : \alpha < \gamma_n\}$ of non-empty F_σ-open (in (X, τ)) sets such that the ρ-diameter of $D^n_\alpha = U^n_\alpha \setminus \bigcup\{U^n_\beta : \beta < \alpha\}$ is $< \frac{1}{n}$ and $X = \bigcup\{D^n_\alpha : \alpha < \gamma_n\}$. It is worth mentioning that every U^n_α is open in (X, ρ). The Lindelöf number of (X, ρ) is less or equal to the cardinality of B so that it is $\leq \kappa$. Thus, we can assume without loss of generality that $|\gamma_n| \leq \kappa$. Next observe that the sets $\{D^n_\alpha : \alpha < \gamma_n, n \in \omega\}$ separates the points of X. Since X is normal, for any D^n_α ($n \in \omega$, $\alpha < \gamma_n$) there exists a continuous function f^n_α from X to $[0, 1]$ such that $f^n_\alpha(X \setminus U^n_\alpha) = 0$ and $f^n_\alpha(U^n_\alpha) \subset (0, 1]$. It is easy to see that $Y = \{f^n_\alpha : \alpha < \gamma_n, n \in \omega\} \subset C_p(X)$ separates the points of X and $|Y| \leq \kappa$. Then the evaluation map $e^Y : X \to C_p(Y)$ is injective (see problem 166 in [18]). Thus, compactness of X implies that X and $e^Y(X)$ are homeomorphic. Since $w(C_p(Y))$ is the cardinality of Y, we have $w(X) = w(e^Y(X)) \leq w(C_p(Y)) = |Y| \leq \kappa = hd(X)$. Therefore $w(X) = hd(X)$ □

Descriptive can be replaced by *splittable over the class of descriptive compacta*.

Proposition 16 *If X is a compact space splittable over the class of descriptive compact spaces, then $hd(X) = w(X)$.*

Proof Observe first that the space X has countable tightness. Now, if $hd(X) \leq \mathfrak{c}$, then $|X| \leq d(X)^\omega \leq hd(X)^\omega \leq \mathfrak{c}^\omega \leq \mathfrak{c}$. It follows from Proposition 13 that X is a descriptive compact space. Therefore Proposition 15 tells us that $hd(X) = w(X)$.

Suppose now that $hd(X) > \mathfrak{c}$. It is easy to see that $\mathfrak{c} < d(X) \leq hd(X) \leq w(X) \leq |X|$. On the other hand, the inequalities $|X| \leq d(X)^\omega \leq hd(X)^\omega = hd(X)$ hold. Therefore $hd(X) = w(X)$. □

The following proposition is surely folklore.

Proposition 17 *If $f : X \to Y$ is a perfect function and Y is weakly θ-refinable, then X is weakly θ-refinable.*

Proof Let $\mathcal{U}$ be an open cover of X. For any $y \in Y$, consider a finite subcover $\mathcal{U}_y \subset \mathcal{U}$ of the compact subspace $f^{-1}(y)$. Notice that $X \setminus \bigcup \mathcal{U}_y$ is closed in X; hence $f(X \setminus \bigcup \mathcal{U}_y)$ is closed in Y and $y \notin f(X \setminus \bigcup \mathcal{U}_y)$. It is clear that $\mathcal{W} = \{Y \setminus f(X \setminus \bigcup \mathcal{U}_y) : y \in Y\}$ is an open cover of the space Y. Then there exists a σ-isolated refinement $\mathcal{V} = \{\mathcal{V}_n : n \in \omega\}$ of $\mathcal{W}$ where $\mathcal{V}_n$ is discrete for all $n \in \omega$. It is easy to see that $\mathcal{E}_n = \{f^{-1}(W) \cap U : W \in \mathcal{V}_n, U \in \mathcal{U}_y \text{ for some } y \in Y\}$ is a σ-isolated refinement of $\mathcal{U}$. Therefore X is weakly θ-refinable. □

Proposition 18 *If a compact space X is splittable over the class of descriptive compact spaces, then it is hereditarily weakly θ-refinable.*

Proof It follows from Proposition 17 and an argument similar to the one used in the proof of Proposition 5. □

In [13, Theorem 3.3] it was showed that scattered compact spaces which split over the class of Corson compact spaces are Eberlein compact. For descriptive compacta, the following result holds:

Proposition 19 *If X is a scattered compact space splittable over the class of descriptive compact spaces, then X is a σ-discrete descriptive compact space.*

Proof From the previous proposition, X is a hereditarily weakly θ-refinable space. Since scattered compact spaces are fragmentable, X is a hereditarily weakly θ-refinable fragmentable space. By Theorem 1, X is a descriptive compact space. It follows from Theorem 3.4 of [14] that X is σ-discrete. □

If we remove *scattered* in Proposition 19, some particular properties are to be expected.

Proposition 20 *If X is a compact space splittable over the class of descriptive compact spaces, then the following hold:*

(i) *If $A \subset X$ is countably compact, then it is compact.*
(ii) *If $W \subset X$ is closed, then W is splittable over the class of descriptive compact spaces.*
(iii) *The tightness of X is countable.*

Proof (i) There exist a compact descriptive space Y and a continuous function $f : X \to Y$ such that $f^{-1}(f(A)) = A$. The subspace $f(A)$ is countably compact and hence it is compact. Since the function

$$g = f|_{f^{-1}(f(A))} : f^{-1}(f(A)) \to f(A)$$

is perfect, $f^{-1}(f(A))$ is compact.

(ii) For any $A \subset W$ there exist a descriptive compact space Y and a continuous onto function $f : X \to Y$ such that $f^{-1}(f(A)) = A$. Define $g = f|_W : W \to f(W)$ as $g(x) = f(x)$ for each $x \in W$. Since any closed subspace of a descriptive compact space is descriptive compact, $f(W)$ is descriptive compact as well. It is easy to see that g is continuous and $g^{-1}(g(A)) = A$. Therefore W is splittable over the class of descriptive compact spaces.

(iii) It suffices to note that X is a compact space splittable over the class of compact spaces of countable tightness (see [15, Corollary 4.3]) and apply [4, Theorem 15]. □

The following result generalizes Corollary 3.2 of [13].

Proposition 21 *Suppose that $\mathcal{P}$ is a κ productive class. If X is splittable over the class $\mathcal{P}$ and there exists a continuous function from X into a space Y in $\mathcal{P}$ such that the cardinality of $f^{-1}(y)$ is $\leq 2^\kappa$ for any $y \in Y$, then X condenses into a space of $\mathcal{P}$.*

Proof For any $y \in Y$ there exists a family $\mathcal{F}_y = \{A_\alpha^y : \alpha < \kappa\}$ of subsets X which separates the points of $f^{-1}(y)$. For any $\alpha < \kappa$ define the set $A_\alpha = \bigcup\{A_\alpha^y : y \in Y\}$. The family $\{f^{-1}(y) : y \in Y\} \cup \{A_\alpha : \alpha < \kappa\}$ separates the points of X; indeed, given two distinct points $x, z \in X$, we have

(a) If $f(x) \neq f(z)$, then $f^{-1}(f(x))$ separates the points x and z because $z \notin f^{-1}(f(x))$.
(b) If $f(x) = f(z)$, then $x, z \in f^{-1}(f(x))$. It follows from $x, z \in f^{-1}(f(x))$ that there exists $\alpha < \kappa$ such that $A_\alpha^y \cap \{x, z\} = \{x\}$; hence A_α separates the points x and z.

Now, for any $\alpha < \kappa$ there exist $Y_\alpha \in \mathcal{P}$ and a continuous function $f_\alpha : X \to Y_\alpha$ such that $f_\alpha^{-1}(f_\alpha(A_\alpha)) = A_\alpha$. Consider the continuous function $h = f \times \prod_{\alpha<\kappa}\{f_\alpha\} : X \to Z = Y \times \prod_{\alpha<\kappa} Y_\alpha$. It is evident that $Z \in \mathcal{P}$. If x, z are distinct points of X, then $h(x) \neq h(z)$ because the family $\{f^{-1}(y) : y \in Y\} \cup \{A_\alpha : \alpha < \kappa\}$ separates the points of X. Therefore X condenses into Z. □

Given a cardinal number κ, a space X is called κ-metrizable if it is the union of at most κ metrizable subspaces. For descriptive compact spaces, the following proposition strengthens [13, Theorem 3.8].

Proposition 22 *Let X be a countably compact space splittable over the class of descriptive compact spaces. If there exists a continuous function g from X into a descriptive compact space Y such that $f^{-1}(y)$ is $\mathfrak{c}$-metrizable for any $y \in Y$, then X is a descriptive compact space.*

Proof Take $y \in Y$ and suppose that $f^{-1}(y) = \bigcup\{A_\alpha^y : \alpha < \mathfrak{c}\}$, where A_α^y is metrizable for any $\alpha < \mathfrak{c}$. Assume, without loss of generality, that $A_{\alpha_1}^y \cap A_{\alpha_2}^y = \emptyset$ for distinct $\alpha_1, \alpha_2 < \mathfrak{c}$. For any $\alpha < \mathfrak{c}$ define the set $B_\alpha = \bigcup\{A_\alpha^y : y \in Y\}$. It is clear that $B_{\alpha_1} \cap B_{\alpha_2} = \emptyset$ for distinct $\alpha_1, \alpha_2 < \mathfrak{c}$. It follows from Proposition 3 that there exist a descriptive compact space Z and a continuous function $g : X \to Z$ such that $g^{-1}(g(B_\alpha)) = B_\alpha$ for any $\alpha < \mathfrak{c}$. The function $f : X \to Y \times Z$ defined as $f(x) = (g(x), h(x))$ is continuous and $f(X) \subset Y \times Z$ is descriptive compact. For any $y \in f(X)$, the subspace $f^{-1}(y)$ is compact because it is a metrizable countably compact space. Thus, $|f^{-1}(y)| \leq \mathfrak{c}$. The previous proposition tells us that there exits a condensation $\varphi : X \to Q$, where Q is a descriptive compact space. Take now a closed subset E of X. Since X is countably compact, E is countably compact as well. It follows From Proposition 20 that $\varphi(E) \subset Q$ is compact so that $h(E)$ is closed in Q. Hence φ is a closed injective continuous function which implies that it is a homeomorphism. Therefore X is a descriptive compact space. □

Corollary 6 *Let X be a $\mathfrak{c}$–metrizable countably compact space. If X is splittable over the class of descriptive compact spaces, then X is descriptive compact.*

Proof Consider the (unique) function f from X into the descriptive compact space $\{0\}$ and apply Proposition 22. □

It was showed in [13, Theorem 3.11] that it is consistent with ZFC that splittability of a compact space over the class of Eberlein (Corson, Gul'ko) compact spaces is an Eberlein (Corson, Gul'ko) compact space. We extend this result to descriptive compacta. Recall that the axiom of constructibility asserts that $V = L$ where V and L denote the von Neumann universe and the constructible universe, respectively. We will also consider the statement $ACP^\sharp$ introduced in [7]: given a cardinal μ, let $s_0(\mu) = \mu$; if α is an ordinal and we have $s_\alpha(\mu)$, then $s_{\alpha+1}(\mu) = (s_\alpha(\mu))^+$. If α is a limit ordinal and we have $s_\beta(\mu)$ for any $\beta < \alpha$, then $s_\alpha(\mu) = \sup\left\{s_\beta(\mu) : \beta < \alpha\right\}$. Let us consider the following statement:

(ACP) for every cardinal $\mu \geq \mathfrak{c}$, there is a cardinal $\alpha < \mathfrak{c}$ such that $\mu^\omega \leq s_\alpha(\mu)$.

The assumption $ACP\&(\omega_1 < \mathfrak{c})$ is denoted by $ACP^\sharp$. It is well known that both $ACP^\sharp$ and $V = L$ are consistent with ZFC.

Proposition 23 [7, Corollaries 8.9 and 9.7] *Assume that $ACP^\sharp$ or $V = L$ holds. For any Hausdorff space X there exists disjoint sets $X_1, X_2 \subset X$ such that $X = X_1 \cup X_2$ and any compact $K \subset X_i$ is scattered for each $i \in \{1, 2\}$.*

Proposition 24 *Assume that $ACP^\sharp$ or $V = L$ holds. If X is a compact space splittable over the class of descriptive compact spaces, then X is descriptive compact.*

Proof Proposition 23 implies that there exist disjoint sets $X_1, X_2 \subset X$ such that $X = X_1 \cup X_2$ and each compact space $K \subset X_i$ is scattered for any $i \in \{1, 2\}$. Then there exist a descriptive compact space Y and an onto continuous function $f : X \to Y$ such that $f^{-1}(f(X_i)) = X_i$ for any $i \in \{1, 2\}$. Notice that, for each $y \in Y$, the space $f^{-1}(y) \subset X$ is compact and either $f^{-1}(y) \subset X_1$ or $f^{-1}(y) \subset X_2$. Hence $f^{-1}(y)$ is a scattered compact space splittable over the class of descriptive compact spaces. It follows from Proposition 19 that $f^{-1}(y)$ is σ-metrizable. Thus, Proposition 22 implies that X is a descriptive compact space. □

References

1. Alas, O.T., Tkachuk, V.V., Wilson, R.G.: Maximal pseudocompact spaces and the Preiss–Simon property. Cent. Eur. J. Math. **12**(3), 500–509 (2014)

2. Alster, K.: Some remarks on Eberlein compacts. Fund. Math. **104**, 43–46 (1979)
3. Arkhangel'skii, A.V.: Topological function spaces. In: Mathematics and its Applications (Soviet Series), vol. 78. Kluwer Academic Publishers, Dordrecht (1992)
4. Arhangel'skii, A.V.: The general concept of cleavability of a topological space. Topol. Appl. **44**, 27–36 (1992)
5. Arhangel'skii, A.V., Collins, P.J.: On submaximal spaces. Topol. Appl. **64**(3), 219–241 (1995)
6. Arhangelskii, A. V., Shakhmatov, D. B.: Pointwise approximation of arbitrary functions by countable families of continuous functions. (Russian) Trudy Sem. Petrovsk. 13:206–227 (1988); translation in J. Soviet Math. 50(2):1497–1512 (1990)
7. Bregman, J.U., Šapirovskij, B., Šostak, A.: On partition of topological spaces. Čas. Pěst. Mat. 109(1):27–53 (1984)
8. Burke, D.: Covering properties. In: Vaughan, K. (ed.) Handbook of set theoretic topology, pp. 347–423. Elsevier Science Publishers, Netherlands (1984)
9. Engelking, R.: General topology. Heldermann Verlag (1989)
10. Gerlits, J., Hajnal, A., Szentmiklóssy, Z.: On the cardinality of certain Hausdorff space. Discrete Math. **108**, 31–35 (1992)
11. Hart, K.P., Nagata, J., Vaughan, J.E.: Encyclopedia of General Topology. Elseiver Science Publisher (2004)
12. Jardon, D.: Weakly Eberlein compact spaces. Top. Proc. **346**, 695–707 (2002)
13. Jardon, D., Tkachuk, V.V.: Splittability over some classes of Corson compact spaces. Topol. Appl. **184**, 41–49 (2015)
14. Nyikoz, P.: Covering properties on σ-scattered spaces. Top. Proc. **2**, 509–542 (1977)
15. Oncina, L., Raja, M.: Descriptive compact spaces and renorming. Studia Math. **165**, 39–52 (2004)
16. Preiss, D., Simon, P.: A weakly pseudocompact subspace of Banach space is weakly compact. Comment. Math. Univ. Carolinae **15**(4), 603–609 (1974)
17. Raja, M.: Weak* locally uniformly rotund norms and descriptive compact spaces. J. Funct. Anal. **197**, 1–13 (2003)
18. Tkachuk, V.V.: A C_p-theory problem book: topological and function spaces. Springer, Heidelberg (2010)
19. Tkachuk, V.V.: A C_p-theory problem book: compactness in function spaces. Springer, Heildeberg (2015)
20. Tkachuk, V.V.: A note on splittable spaces. Comment. Math. Univ. Carolinae **33**(3), 551–555 (1992)
21. Yakovlev, N.N.: On bicompacta in Σ-products and related spaces. Comment. Math. Univ. Carolinae **21**, 263–282 (1980)