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ORIGINAL PAPER

Lyapunov type inequalities for a fractional thermostat model

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Abstract In this paper, we present some Lyapunov-type inequalities for a nonlinear fractional heat equation with nonlocal boundary conditions depending on a positive parameter. As an application, we obtain a lower bound for the eigenvalues of corresponding equations.

Keywords Lyapunov's inequality · Caputo fractional derivative · Green's function · Eigenvalue

Mathematics Subject Classification 34A08 · 34A40 · 26D10

1 Introduction

Consider the boundary value problem with Dirichlet conditions

$$\begin{cases} x''(t) + q(t)x(t) = 0, \quad a < t < b, \\ x(a) = x(b) = 0, \end{cases} \tag{1}$$

where $q : [a, b] \to \mathbb{R}$ is a continuous function. Lyapunov in [1] proved that if Problem (1) has a nontrivial solution then

$$\int_a^b |q(s)|\, ds > \frac{4}{b-a}.$$

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In [2], Hartman and Wintner proved that if Problem (1) has a nontrivial solution then

$$\int_a^b (b-s)(s-a)q^+(s)\,ds > b-a,$$

where $q^+(s) = \max\{q(s), 0\}$.

Inequalities of this type have appeared in the literature for other classes of boundary value problems and we refer the reader to [3–7] and the references therein for more details.

Recently, some Lyapunov-type inequalities have been obtained by some authors for different fractional boundary value problems (see [8–12], for example).

In this paper, we are concerned with the problem of finding some Lyapunov-type inequalities for the following fractional boundary value problem

$$\begin{cases} -{}^CD_a^\alpha u(t) = y(t), \quad a < t < b, \\ u'(a) = 0, \ \beta\, {}^CD_a^{\alpha-1}u(b) + u(\eta) = 0, \end{cases} \tag{2}$$

where ${}^CD_a^\alpha$ denotes the Caputo fractional derivative of order α, $1 < \alpha \le 2$, $\beta > 0$ and $a \le \eta \le b$.

As an application of our results, we obtain a lower bound for the eigenvalues of the cor-respondig problem.

The above mentioned fractional boundary value problem can be considered as the fractional version of the nonlocal boundary value problem

$$\begin{cases} -u''(t) = y(t), \quad 0 < t < 1, \\ u'(0) = 0, \quad \beta u'(1) + u(\eta) = 0, \end{cases}$$

with $0 \le \eta \le 1$ which has been studied in the special case with $\eta = 0$ in [13] and this problem models a thermostat insulated at $t = 0$ with a controller dissipating heat at $t = 1$ depending on the temperature detected by a sensor at $t = \eta$.

2 Background

In this section, we present the basic results about fractional calculus theory which be used later. For more details, we refer the reader to [14, 15].

Definition 1 Let $f : [a, b] \longrightarrow \mathbb{R}$ be a given function. For $\alpha > 0$, the Riemann-Liouville fractional integral of order α of f is defined by

$$(I_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,ds,$$

where $\Gamma(\alpha)$ denotes the classical gamma function.

Definition 2 Let $f : [a, b] \longrightarrow \mathbb{R}$ be a given function. For $\alpha > 0$, the Caputo derivative of fractional order $\alpha > 0$ of f is given by

$$({}^CD_a^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 1 *Suppose that* $f \in C(a,b) \cap L^1(a,b)$ *with a fractional derivative of order* $\alpha > 0$ *belonging to* $C(a,b) \cap L^1(a,b)$. *Then*

$$I_a^\alpha({}^C D_a^\alpha f)(t) = f(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1},$$

for $t \in [a,b]$, *where* $c_i \in \mathbb{R}$ $(i = 0, 1, \ldots, n-1)$ *and* $n = [\alpha] + 1$.

Lemma 2 *Suposse* $f \in L^1(a,b)$ *and* $\alpha > 0$, $\beta > 0$. *Then*

1. ${}^C D_a^\alpha I_a^\alpha f(t) = f(t)$
2. $I_a^\alpha (I_a^\beta) f(t) = (I_a^{\alpha+\beta} f)(t)$

3 Main results

Our starting point in this section is the following lemma which gives us an expression for the Green's function of the boundary value problem (2). The case for $a = 0$ and $b = 1$ appears in [16, Lemma 2.4].

Lemma 3 *Suppose* $y \in C[a,b]$. *A function* $u \in C[a,b]$ *is a solution of Problem* (2) *if and only if it satisfies the integral equation*

$$u(t) = \int_a^b G(t,s) y(s)\, ds,$$

where $G(t,s)$ *is the Green's function given by*

$$G(t,s) = \beta + H_\eta(s) - H_t(s),$$

where for $r \in [a,b]$, $H_r : [a,b] \to \mathbb{R}$ *is the function defined as*

$$H_r(s) = \begin{cases} \frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{for} \quad a \le s \le r \le b, \\ 0, & \text{for} \quad a \le r < s \le b. \end{cases}$$

Proof Using Lemma 2, we have

$$u(t) = -I_a^\alpha y(t) + c_0 + c_1(t-a) = -\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\, ds + c_0 + c_1(t-a),$$

for some constants $c_0, c_1 \in \mathbb{R}$.

This gives us

$$u'(t) = -\frac{1}{\Gamma(\alpha)} \int_a^t (\alpha-1)(t-s)^{\alpha-2} y(s)\, ds + c_1.$$

From the boundary condition $u'(a) = 0$, we get $c_1 = 0$.

This gives us

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\, ds + c_0.$$

By using the fact that ${}^C D_a^{\alpha-1} c_0 = 0$, and Lemma 2, we have

$${}^C D_a^{\alpha-1} u(t) = -{}^C D_a^{\alpha-1} I_a^\alpha y(t) = -{}^C D_a^{\alpha-1} I_a^{\alpha-1} I_a y(t) = -I_a y(t) = -\int_a^t y(s)\, ds.$$

This gives us

$$\beta\, {}^C D_a^{\alpha-1} u(b) = -\beta \int_a^b y(s)\, ds.$$

Taking into account the boundary condition

$$\beta\, {}^C D_a^{\alpha-1} u(b) + u(\eta) = 0,$$

we have

$$0 = -\beta \int_a^b y(s)\, ds - \frac{1}{\Gamma(\alpha)} \int_a^\eta (\eta - s)^{\alpha-1} y(s)\, ds + c_0$$

and, from this, it follows

$$c_0 = \beta \int_a^b y(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_a^\eta (\eta - s)^{\alpha-1} y(s)\, ds.$$

Consequently,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} y(s)\, ds + \beta \int_a^b y(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_a^\eta (\eta - s)^{\alpha-1} y(s)\, ds.$$

Therefore,

$$u(t) = \beta \int_a^b y(s)\, ds + \int_a^b H_\eta(s) y(s)\, ds - \int_a^b H_t(s) y(s)\, ds$$

or, equivalently,

$$u(t) = \int_a^b \big(\beta + H_\eta(s) - H_t(s)\big)\, y(s)\, ds.$$

This completes the proof. □

Remark 1 Notice that the Green's function can be expressed as

$$G(t,s) = \begin{cases} \beta + H_\eta(s) - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{for } a \le s \le t \le b, \\ \beta + H_\eta(s), & \text{for } a \le t \le s \le b. \end{cases}$$

In the following proposition, we present some properties about Green's function

Proposition 1 *The Green's function satisfies:*

(i) $\max\{G(t,s) : t, s \in [a,b]\} = \beta + \dfrac{(\eta - a)^{\alpha-1}}{\Gamma(\alpha)}$.

(ii) $\min\{G(t,s) : t, s \in [a,b]\} = \beta - \dfrac{(b - \eta)^{\alpha-1}}{\Gamma(\alpha)}$.

Proof (i) Notice that for $s \in [a,b]$ fixed, we have

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} 0, & \text{for } a \le t \le s, \\ -\dfrac{(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}, & \text{for } a \le t \le s \le b. \end{cases}$$

From this, it follows that $G(t,s)$ is a decreasing function in t, and this gives us

$$\max\{G(t,s) : t,s \in [a,b]\} = G(a,s) = \begin{cases} \beta + \dfrac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{for } a \le s \le \eta \le b, \\ \beta, & \text{for } a \le \eta \le s \le b. \end{cases}$$

On the other hand, if we put $\varphi(s) = \beta + \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}$ for $s \in [a,\eta]$, since $\varphi'(s) = -\frac{(\alpha-1)(\eta-s)^{\alpha-2}}{\Gamma(\alpha)} < 0$, φ is a decreasing function and we infer that $\max\{\varphi(s) : s \in [a,\eta]\} = \varphi(a) = \beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}$.
Therefore,

$$\max\{G(t,s) : t,s \in [a,b]\} = \max\left\{\beta, \beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}\right\} = \beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}$$

and this proves (i).

(ii) Since $G(t,s)$ is a decreasing function in t, we have

$$\min\{G(t,s) : t,s \in [a,b]\} = G(b,s) = \begin{cases} \beta + \dfrac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{for } a \le s \le \eta \le b, \\ \beta - \dfrac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{for } a \le \eta \le s \le b. \end{cases}$$

Put $\psi(s) = \beta - \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}$ for $s \in [\eta,b]$. Since $\psi'(s) = \frac{(\alpha-1)(b-s)^{\alpha-2}}{\Gamma(\alpha)} \ge 0$, ψ is a nondecreasing, and, consequently, $\min\{\psi(s) : s \in [\eta,b]\} = \psi(\eta) = \beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}$.
On the other hand, put $\alpha(s) = \beta + \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}$ for $s \in [a,\eta]$, since $\alpha'(s) = -\frac{(\alpha-1)(\eta-s)^{\alpha-2}}{\Gamma(\alpha)} + \frac{(\alpha-1)(b-s)^{\alpha-2}}{\Gamma(\alpha)} = \frac{(\alpha-1)}{\Gamma(\alpha)}\left[(b-s)^{\alpha-2} - (\eta-s)^{\alpha-2}\right] \le 0$, (because $1 < \alpha \le 2$), α is decreasing on $[a,\eta]$ and, therefore, $\min\{\alpha(s) : s \in [a,\eta]\} = \alpha(\eta) = \beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}$.
These facts say us that

$$\min\{G(t,s) : t,s \in [a,b]\} = \beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}$$

and this completes the proof.

□

Remark 2 Notice that if $\beta\Gamma(\alpha) \ge (b-\eta)^{\alpha-1}$ then $G(t,s) \ge 0$.
In the case $\beta\Gamma(\alpha) < (b-\eta)^{\alpha-1}$ then, since

$$\beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)} \le G(t,s) \le \beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)},$$

we have that

$$|G(t,s)| \le \max\left\{\beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}, \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)} - \beta\right\}, \quad \text{for } t,s \in [0,1].$$

Our main result is the following Lyapunov-type inequality.

Theorem 1 *Suppose that the fractional boundary value problem*

$$\begin{cases} -{}^C D_a^{\alpha} u(t) = q(t)u(t), \quad a < t < b, \\ u'(a) = 0, \ \beta\, {}^C D_a^{\alpha-1} u(b) + u(\eta) = 0, \end{cases}$$

with $1 < \alpha \le 2$*,* $\beta > 0$*,* $a \le \eta \le b$ *and* $\beta \ge \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}$*, where* $q : [a, b] \to \mathbb{R}$ *is a continuous function, has a nontrivial continuous solution then*

$$\int_a^b |q(s)|\, ds \ge \frac{\Gamma(\alpha)}{\beta\Gamma(\alpha) + (\eta - a)^{\alpha-1}}.$$

Proof Consider the Banach space $C[a, b] = \{x : [a, b] \to \mathbb{R} : x \text{ continuous}\}$ with the standard norm $\|x\|_\infty = \max\{|x(t)| : a \le t \le b\}$, for $x \in C[a, b]$.

By Lemma 3,

$$u(t) = \int_a^b G(t, s)q(s)u(s)\, ds, \quad \text{for } a \le t \le b,$$

where $G(t, s)$ is the Green's function appearing in Lemma 3.

Using Remark 2, since $\beta \ge \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}$, $G(t, s) \ge 0$ and, moreover $\max\{G(t, s) : t, s \in [a, b]\} = \beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)}$, we infer, for any $t \in [a, b]$,

$$|u(t)| \le \int_a^b G(t, s)|q(s)||u(s)|\, ds \le \|u\|_\infty \int_a^b \frac{\beta\Gamma(\alpha) + (\eta - a)^{\alpha-1}}{\Gamma(\alpha)} |q(s)|\, ds,$$

and, this gives us

$$\|u\|_\infty \le \|u\|_\infty \frac{\beta\Gamma(\alpha) + (\eta - a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b |q(s)|\, ds.$$

Since the solution u is nontrivial, we get

$$1 \le \frac{\beta\Gamma(\alpha) + (\eta - a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b |q(s)|\, ds$$

and this gives us the desired result. □

Theorem 1 gives us the following corollary.

Corollary 1 *Suppose that the boundary value problem*

$$\begin{cases} -u''(t) = q(t)u(t), \ a < t < b, \\ u'(a) = 0, \ \beta u'(b) + u(\eta) = 0, \end{cases}$$

where $\beta > 0$*,* $a \le \eta \le b$ *and* $\beta \ge (b - \eta)$ *and* $q : [a, b] \to \mathbb{R}$ *is a continuous function, has a nontrivial continuous solution then*

$$\int_a^b |q(s)|\, ds \ge \frac{1}{\beta + (\eta - a)}.$$

Proof Apply Theorem 1 for $\alpha = 2$. □

4 Application

In this section, we present some applications of the results obtained in Sect. 3 to eigenvalue problem.

$\lambda \in \mathbb{R}$ is said to be an eigenvalue of the fractional boundary value problem

$$\begin{cases} -{}^{C}D_a^{\alpha} u(t) = \lambda u(t), \quad a < t < b, \\ u'(a) = 0, \ \beta\, {}^{C}D_a^{\alpha-1} u(b) + u(\eta) = 0, \end{cases} \tag{3}$$

where $1 < \alpha \leq 2$, $\beta > 0$ and $a \leq \eta \leq b$ if Problem (3) has at least a nontrivial continuous solution x_λ. In this case, we say that x_λ is an eigenvector associated to the eigenvalue λ.

Corollary 2 *Under assumption $\beta \geq \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}$ and suppose that λ is an eigenvalue of Problem* (3) *then*

$$|\lambda| \geq \frac{\Gamma(\alpha)}{(\beta\Gamma(\alpha) + (\eta - a)^{\alpha-1})(b-a)}.$$

Proof As λ is an eigenvalue of Problem (3), this means that Problem (3) has a nontrivial continuous solution x_λ and, by using Theorem 1, we have

$$\int_a^b |\lambda|\, ds \geq \frac{\Gamma(\alpha)}{\beta\Gamma(\alpha) + (\eta - a)^{\alpha-1}}.$$

Therefore,

$$|\lambda| \geq \frac{\Gamma(\alpha)}{(\beta\Gamma(\alpha) + (\eta - a)^{\alpha-1})(b-a)},$$

which yields the desired result. □

Corollary 3 *Suppose that λ is an eigenvalue of the ordinary boundary value problem*

$$\begin{cases} -u''(t) = \lambda u(t), \quad a < t < b, \\ u'(a) = 0, \ \beta u'(b) + u(\eta) = 0, \end{cases} \tag{4}$$

where $\beta > 0$, $a \leq \eta \leq b$ and $\beta \geq (b-a)$, then

$$|\lambda| \geq \frac{1}{(\beta + (\eta - a))(b-a)}.$$

Proof Since λ is an eigenvalue of Problem (4), this says that Problem (4) admits a nontrivial continuous solution x_λ. Now, by using Corollary 1, we get

$$|\lambda|(b-a) \geq \frac{1}{\beta + (\eta - a)}.$$

This gives us the desired result. □

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