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On non-separable L^1-spaces of a vector measure

José Rodríguez[1]

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Abstract Let κ be an infinite cardinal. Let ν be a (countably additive Banach space-valued) vector measure defined on a σ-algebra Σ. We prove that if ν is homogeneous and $L^1(\nu)$ has density character κ, then there is a vector measure $\tilde{\nu} : \Sigma \to \ell^\infty_<(\kappa)$ such that $L^1(\nu) = L^1(\tilde{\nu})$ with equal norms. Here $\ell^\infty_<(\kappa)$ denotes the subspace of $\ell^\infty(\kappa)$ consisting of all $(x_\alpha)_{\alpha<\kappa} \in \ell^\infty(\kappa)$ such that $|\{\alpha < \kappa : |x_\alpha| > \varepsilon\}| < \kappa$ for every $\varepsilon > 0$. In this way, we extend to the non-separable setting a result of Curbera corresponding to the case $\kappa = \omega$. Some other results on non-separable L^1 spaces of vector measures are given.

Keywords Vector measure · Non-separable Banach space · Space of integrable functions · Maharam type · Space of bounded functions with countable support

Mathematics Subject Classification 46B26 · 46E30 · 46G10

1 Introduction

Every order continuous Banach lattice with a weak unit is lattice isometric to the L^1 space of a vector measure, [9, Proposition 2.4] (cf. [6, Theorem 8]). Such Banach lattices are weakly compactly generated [4, p. 193] (cf. [6, Theorem 2]) and admit an equivalent uniformly Gâteaux smooth norm, [19] (cf. [26, Theorem 2.2]). For an arbitrary Banach space X, the existence of such a norm is equivalent to being isomorphic to a subspace of a Hilbert generated

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✉ José Rodríguez
joserr@um.es

1 Dpto. de Ingeniería y Tecnología de Computadores, Facultad de Informática, Universidad de Murcia, 30100 Espinardo, Murcia, Spain

Banach space, and also to (B_{X^*}, w^*) being uniform Eberlein compact, [13, Theorem 2] (cf. [16, Theorem 6.30]).

Typical examples of Banach lattices arising as L^1 spaces of vector measures are all Banach spaces with unconditional basis, the classical spaces $L^p(\mu)$ (for $1 \leq p < \infty$ and a finite measure μ) and Orlicz spaces over a finite measure satisfying the Δ_2-condition. On the other hand, $C[0, 1]$ is a separable Banach lattice which is not isomorphic (as Banach space) to the L^1 space of any vector measure. In the non-separable setting, for an uncountable set Γ and $1 < p < \infty$, the space $\ell^p(\Gamma)$ is reflexive and embeds isomorphically into a Hilbert generated space, [12, Theorem 1]. As a Banach lattice, $\ell^p(\Gamma)$ is order continuous, but fails to have a weak unit and so it is not lattice isomorphic to the L^1 space of any vector measure. In fact, $\ell^p(\Gamma)$ is isomorphic to the L^1 space of a vector measure if and only if $p = 2$, see [26, Theorem 2.6]. Similarly, the Banach lattice $c_0(\Gamma)$ is order continuous and Hilbert generated, but it is not isomorphic to the L^1 space of any vector measure (see the Appendix).

Completely different vector measures can produce the same L^1 space, see [15] for a detailed discussion. The following result was proved in [7, Theorem 1] (cf. [21, Theorem 5]):

Theorem 1.1 (G.P. Curbera) *Let ν be a vector measure defined on a σ-algebra Σ and taking values in a Banach space. If ν is non-atomic and $L^1(\nu)$ is separable, then there is a vector measure $\tilde{\nu} : \Sigma \to c_0$ such that $L^1(\nu) = L^1(\tilde{\nu})$ with equal norms.*

The non-atomicity assumption in the result above cannot be dropped in general, [7, pp. 294–295]. At the conference "Integration, Vector Measures and Related Topics VI" (Będlewo, June 2014), Z. Lipecki asked whether a non-separable version of Theorem 1.1 can be obtained by using $c_0(\Gamma)$ as target space for large enough Γ. Here we address this question and provide some partial answers by using certain superspaces of $c_0(\Gamma)$. Our main results are:

Theorem 1.2 *Let κ be an infinite cardinal. Let ν be a vector measure defined on a σ-algebra Σ and taking values in a Banach space. If ν is homogeneous and $L^1(\nu)$ has density character κ, then there is a vector measure $\tilde{\nu} : \Sigma \to \ell^\infty_<(\kappa)$ such that $L^1(\nu) = L^1(\tilde{\nu})$ with equal norms.*

Theorem 1.3 *Let ν be a vector measure defined on a σ-algebra Σ and taking values in a Banach space. If ν is non-atomic and $L^1(\nu)$ has density character ω_1, then there is a vector measure $\tilde{\nu} : \Sigma \to \ell^\infty_c(\omega_1)$ such that $L^1(\nu) = L^1(\tilde{\nu})$ with equivalent norms.*

Given an infinite cardinal κ, we denote by $\ell^\infty_<(\kappa)$ the subspace of $\ell^\infty(\kappa)$ consisting of all $(x_\alpha)_{\alpha<\kappa} \in \ell^\infty(\kappa)$ such that $|\{\alpha < \kappa : |x_\alpha| > \varepsilon\}| < \kappa$ for every $\varepsilon > 0$. In general, $c_0(\kappa)$ is a subspace of $\ell^\infty_<(\kappa)$. The space $\ell^\infty_<(\kappa)$ was introduced by Pełczyński and Sudakov [24] and has been studied in [3] in connection with injectivity properties of Banach spaces. For $\kappa = \omega$ we have $\ell^\infty_<(\kappa) = c_0(\omega)$ and, therefore, Theorem 1.1 is a particular case of Theorem 1.2. If κ has uncountable cofinality, then $\ell^\infty_<(\kappa)$ coincides with the set of all $(x_\alpha)_{\alpha<\kappa} \in \ell^\infty(\kappa)$ such that $|\{\alpha < \kappa : x_\alpha \neq 0\}| < \kappa$. In particular, we have $\ell^\infty_<(\omega_1) = \ell^\infty_c(\omega_1)$, the Banach space of all bounded real-valued functions on ω_1 having countable support. For information on spaces of bounded functions on an uncountable set having countable support, see [3,17,18,23], [29, Section 16-1] and the references therein.

This paper is organized as follows. In Sect. 2 we introduce the basic terminology and present some preliminary results and examples of non-separable L^1 spaces of vector measures. In Sect. 3 we prove our main Theorems 1.2 and 1.3. To this end we use some ideas from the alternative proof of Theorem 1.1 given in [21], together with other ingredients like Maharam's theorem, which allows us to find a substitute for the Rademacher-type sequences used in both proofs of the separable case. We close the paper with an Appendix on linear injections into L^1 spaces of vector measures.

2 Preliminaries and examples

2.1 Terminology

Our standard references are [1,2,8]. All our Banach spaces are real. An *operator* is a linear continuous map between Banach spaces. By a *subspace* of a Banach space we mean a closed linear subspace. The closed unit ball of a Banach space Z is denoted by B_Z and the dual of Z is denoted by Z^*. By a *vector measure* we mean a countably additive Banach space-valued measure defined on a σ-algebra. The *density character* of a topological space T, denoted by $\mathrm{dens}(T)$, is the minimal cardinality of a dense subset of T.

Throughout this paper (Ω, Σ) is a measurable space and X is a Banach space. The set of all X-valued vector measures defined on Σ is denoted by $ca(\Sigma, X)$. The symbol $ca_+(\Sigma)$ stands for the subset of $ca(\Sigma) := ca(\Sigma, \mathbb{R})$ made up of all non-negative finite measures. The *Maharam type* of a non-atomic $\mu \in ca_+(\Sigma)$ is defined as $\mathrm{dens}(L^1(\mu))$ and coincides with the density character of its measure algebra equipped with the Fréchet–Nikodým metric.

Let $\nu \in ca(\Sigma, X)$. Given $A \in \Sigma$, we denote by ν_A the restriction of ν to the σ-algebra on A defined by $\Sigma_A := \{A \cap B : B \in \Sigma\}$. The composition of ν with any $x^* \in X^*$ is denoted by $x^*\nu$ and belongs to $ca(\Sigma)$. The *semivariation* of ν is the function $\|\nu\| : \Sigma \to \mathbb{R}$ defined by $\|\nu\|(A) = \sup_{x^* \in B_{X^*}} |x^*\nu|(A)$ for all $A \in \Sigma$ (as usual, $|x^*\nu|$ stands for the *variation* of $x^*\nu$). Given $\xi \in ca(\Sigma, Y)$ (where Y is a Banach space), we write $\nu \ll \xi$ to denote that ν is *absolutely continuous* with respect to ξ, meaning that $\lim_{\|\xi\|(A) \to 0} \|\nu(A)\| = 0$ or, equivalently, that $\nu(A) = 0$ whenever $\|\xi\|(A) = 0$. We say that $\lambda \in ca_+(\Sigma)$ is a *control measure* of ν if $\lambda \ll \nu$ and $\nu \ll \lambda$. A *Rybakov control measure* of ν is a control measure of the form $\lambda = |x^*\nu|$ for some $x^* \in B_{X^*}$; such control measures always exist, see e.g. [8, p. 268, Theorem 2]. We say that ν is *non-atomic* if some/every control measure of ν is non-atomic in the usual sense.

A Σ-measurable function $f : \Omega \to \mathbb{R}$ is said to be ν *-integrable* if the following two conditions are satisfied: (i) f is $|x^*\nu|$-integrable for all $x^* \in X^*$, and (ii) for each $A \in \Sigma$, there is a vector $\int_A f \, d\nu \in X$ such that $x^*(\int_A f \, d\nu) = \int_A f \, d(x^*\nu)$ for every $x^* \in X^*$. By identifying functions which coincide $\|\nu\|$-a.e. we obtain the Banach lattice $L^1(\nu)$ of all (equivalence classes of) ν-integrable functions, equipped with the $\|\nu\|$-a.e. order and the norm

$$\|f\|_{L^1(\nu)} := \sup_{x^* \in B_{X^*}} \int_\Omega |f| \, d|x^*\nu|, \quad f \in L^1(\nu).$$

Note that $\|f\|_{L^1(\nu)} = \|\nu_f\|(\Omega)$, where $\nu_f \in ca(\Sigma, X)$ is defined by $\nu_f(A) := \int_A f \, d\nu$ for all $A \in \Sigma$. The formula

$$|||f|||_\nu := \sup_{A \in \Sigma} \left\| \int_A f \, d\nu \right\|$$

defines a norm on $L^1(\nu)$ which is equivalent to $\|\cdot\|_{L^1(\nu)}$, since

$$|||f|||_\nu \le \|f\|_{L^1(\nu)} \le 2|||f|||_\nu \quad \text{for all } f \in L^1(\nu). \tag{2.1}$$

The basic properties of the space $L^1(\nu)$ can be found, for instance, in [22, Chapter 3]. As a Banach lattice, $L^1(\nu)$ is order continuous and has a weak unit. We write $\mathrm{sim}\,\Sigma$ to denote the set of all *simple functions* from Ω to $\mathbb{R}$, that is, linear combinations of characteristic functions 1_A where $A \in \Sigma$. Simple functions are ν-integrable and $\mathrm{sim}\,\Sigma$ is dense in $L^1(\nu)$ (after the $\|\nu\|$-a.e. identification). It is easy to check that $\mathrm{dens}(L^1(\nu))$ coincides with the Maharam type of any control measure of ν whenever ν is non-atomic. We say that ν is *homogeneous* if it is non-atomic and $\mathrm{dens}(L^1(\nu)) = \mathrm{dens}(L^1(\nu_A))$ for every $A \in \Sigma$ with $\|\nu\|(A) > 0$.

2.2 Examples

Obviously, the classical space $L^1(\mu)$ of a finite measure μ can be seen as the L^1 space of a vector measure. The following standard construction (see e.g. [22, Corollary 3.66]) shows that the same holds for $L^p(\mu)$ whenever $1 < p < \infty$.

Example 2.1 Let $\mu \in ca_+(\Sigma)$ and $1 \leq p < \infty$. Let $\nu \in ca(\Sigma, L^p(\mu))$ be defined by $\nu(A) := 1_A$ for all $A \in \Sigma$. Then $L^1(\nu) = L^p(\mu)$ with equal norms.

In Example 2.6 below we will show that, for any $1 < p < \infty$ and any infinite cardinal κ, the L^p space of the usual probability on the Cantor cube $\{-1, 1\}^\kappa$ can be realized as the L^1 space of a suitable $c_0(\kappa)$-valued vector measure.

To this end we need some lemmas which will also be applied in Sect. 3. The first one is based on some ideas from [14, Theorem 2.1].

Lemma 2.2 *Let $\nu \in ca(\Sigma, X)$ and let Δ be a w^*-dense subset of B_{X^*}. Then*

$$\|f\|_{L^1(\nu)} = \sup_{x^* \in \Delta} \int_\Omega |f|\, d|x^*\nu|$$

for every $f \in L^1(\nu)$.

Proof The statement is obvious for $f = 0$. Fix $f \in L^1(\nu)\setminus\{0\}$ and $\varepsilon > 0$. Let $g \in \mathrm{sim}\,\Sigma$ such that $\|f - g\|_{L^1(\nu)} \leq \varepsilon$ and $g \neq 0$. Write $g = \sum_{i=1}^p a_i 1_{A_i}$, where $a_i \in \mathbb{R}\setminus\{0\}$ and the A_i's are pairwise disjoint elements of Σ. Choose $x_1^* \in B_{X^*}$ such that

$$\|g\|_{L^1(\nu)} \leq \int_\Omega |g|\, d|x_1^*\nu| + \varepsilon. \tag{2.2}$$

Since Δ is w^*-dense in B_{X^*}, there is $x_0^* \in \Delta$ such that

$$|x_1^*\nu|(A_i) \leq |x_0^*\nu|(A_i) + \frac{\varepsilon}{|a_i|p} \quad \text{for every } \ i \in \{1, \ldots, p\}.$$

Then

$$\int_\Omega |g|\, d|x_1^*\nu| = \sum_{i=1}^p |a_i||x_1^*\nu|(A_i) \leq \sum_{i=1}^p |a_i||x_0^*\nu|(A_i) + \varepsilon = \int_\Omega |g|\, d|x_0^*\nu| + \varepsilon,$$

which combined with (2.2) yields

$$\|g\|_{L^1(\nu)} \leq \int_\Omega |g|\, d|x_0^*\nu| + 2\varepsilon.$$

Bearing in mind that $\|f - g\|_{L^1(\nu)} \leq \varepsilon$, we get

$$\|f\|_{L^1(\nu)} \leq \|g\|_{L^1(\nu)} + \varepsilon \leq \int_\Omega |g|\, d|x_0^*\nu| + 3\varepsilon \leq \int_\Omega |f|\, d|x_0^*\nu| + 4\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we have $\|f\|_{L^1(\nu)} = \sup_{x^* \in \Delta} \int_\Omega |f| \, d|x^*\nu|$. □

Lemma 2.3 *Let Γ be a non-empty set and Z a subspace of $\ell^\infty(\Gamma)$. For each $\gamma \in \Gamma$, denote by $e^*_\gamma \in B_{\ell^\infty(\Gamma)^*}$ the γ-th coordinate functional. Let $\nu \in ca(\Sigma, Z)$. Then*

$$\|f\|_{L^1(\nu)} = \sup_{\gamma \in \Gamma} \int_\Omega |f| \, d|e^*_\gamma \nu|$$

for every $f \in L^1(\nu)$.

Proof We denote by $e^*_\gamma|_Z$ the restriction of e^*_γ to Z. The set $\{e^*_\gamma|_Z : \gamma \in \Gamma\} \subseteq B_{Z^*}$ is 1-norming and so, by the Hahn-Banach separation theorem, its absolutely convex hull $\Delta := \mathrm{aco}(\{e^*_\gamma|_Z : \gamma \in \Gamma\})$ is w^*-dense in B_{Z^*}. Lemma 2.2 now applies to get

$$\|f\|_{L^1(\nu)} = \sup_{e^* \in \Delta} \int_\Omega |f| \, d|e^*\nu| = \sup_{\gamma \in \Gamma} \int_\Omega |f| \, d|e^*_\gamma \nu|$$

for every $f \in L^1(\nu)$. □

The following lemma can be found in [21, Lemma 6].

Lemma 2.4 *Let $\nu \in ca(\Sigma, X)$ and $\tilde{\nu} \in ca(\Sigma, Y)$, where Y is a Banach space. If $\|f\|_{L^1(\nu)} = \|f\|_{L^1(\tilde{\nu})}$ for every $f \in \mathrm{sim}\Sigma$, then $L^1(\nu) = L^1(\tilde{\nu})$ with equal norms.*

To deal with the next examples we need to introduce some terminology. Let κ be an infinite cardinal. For any set $I \subseteq \kappa$ we denote by $\rho_I : \{-1, 1\}^\kappa \to \{-1, 1\}^I$ the canonical projection. We say that a function $f : \{-1, 1\}^\kappa \to \mathbb{R}$ *depends on coordinates from I* if there is a function $f' : \{-1, 1\}^I \to \mathbb{R}$ such that $f = f' \circ \rho_I$. We say that f *depends on finitely many coordinates* if there is a finite set $I \subseteq \kappa$ such that f depends on coordinates from I. Dependence on finitely many coordinates is equivalent to being a linear combination of characteristic functions of clopen subsets of $\{-1, 1\}^\kappa$. We denote by $S(\kappa)$ the set of all real-valued functions on $\{-1, 1\}^\kappa$ depending on finitely many coordinates. We write $\pi_\alpha : \{-1, 1\}^\kappa \to \{-1, 1\}$ to denote the α-th coordinate projection for each $\alpha < \kappa$.

Example 2.5 Let κ be an infinite cardinal, λ the usual probability on $\{-1, 1\}^\kappa$ and Σ its domain. Then

$$\nu(A) := \left(\int_A \pi_\alpha \, d\lambda \right)_{\alpha < \kappa} \in c_0(\kappa) \quad \text{for every} \quad A \in \Sigma.$$

Moreover, $\nu \in ca(\Sigma, c_0(\kappa))$ and $L^1(\nu) = L^1(\lambda)$ with equal norms.

Proof The fact that ν takes values in $c_0(\kappa)$ follows from the density of $S(\kappa)$ in $L^1(\lambda)$, cf. the proof of [25, Lemma 2.1] for more details. Clearly, ν is finitely additive. From the inequality $\|\nu(A)\|_{c_0(\kappa)} \leq \lambda(A)$ for all $A \in \Sigma$ it follows that ν is countably additive. Lemma 2.3 ensures that $\|f\|_{L^1(\nu)} = \|f\|_{L^1(\lambda)}$ for every $f \in \mathrm{sim}\Sigma$, and then Lemma 2.4 applies to conclude that $L^1(\nu) = L^1(\lambda)$ with equal norms. □

The proof of the following example uses an argument which was kindly provided by G. Plebanek.

Example 2.6 Let κ be an infinite cardinal, λ the usual probability on $\{-1, 1\}^\kappa$ and Σ its domain. Let $1 < p < \infty$. Then there is $\nu \in ca(\Sigma, c_0(\kappa))$ such that $L^1(\nu) = L^p(\lambda)$ with equal norms.

Proof Write $K := \{-1,1\}^\kappa$. Let $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and write $\langle f, g\rangle := \int_K fg\,d\lambda$ for every $f \in L^p(\lambda)$ and $g \in L^q(\lambda)$. Since $S(\kappa)$ is norm dense in $L^q(\lambda)$ and $\mathrm{dens}(L^q(\lambda)) = \kappa$, there is a set $H \subseteq S(\kappa) \cap B_{L^q(\lambda)}$ of cardinality κ which is norm dense in $B_{L^q(\lambda)}$. Enumerate $H = \{h_\alpha : \alpha < \kappa\}$. Each h_α can be written as $h_\alpha = h'_\alpha \circ \rho_{I_\alpha}$, where $I_\alpha \subseteq \kappa$ is finite and $h'_\alpha : \{-1,1\}^{I_\alpha} \to \mathbb{R}$ is a function. Since κ is infinite and the I_α's are finite, we can construct (inductively) an injective map $\varphi : \kappa \to \kappa$ in such a way that $\varphi(\alpha) \notin I_\alpha$ for all $\alpha < \kappa$. Define $g_\alpha := h_\alpha \pi_{\varphi(\alpha)} \in B_{L^q(\lambda)}$ for every $\alpha < \kappa$.

CLAIM. *If* (α_n) *is a sequence in* κ *with* $\alpha_n \neq \alpha_m$ *whenever* $n \neq m$*, then* (g_{α_n}) *is weakly null in* $L^q(\lambda)$. Indeed, since $S(\kappa)$ is norm dense in $L^p(\lambda) = L^q(\lambda)^*$ and the sequence (g_{α_n}) is bounded, it suffices to check that $\langle f, g_{\alpha_n}\rangle \to 0$ as $n \to \infty$ for every $f \in S(\kappa)$. To this end, let us write $f = f' \circ \rho_I$ for some finite set $I \subseteq \kappa$ and some function $f' : \{-1,1\}^I \to \mathbb{R}$. Note that each fh_{α_n} depends on coordinates from $I \cup I_{\alpha_n}$. Choose $n_0 \in \mathbb{N}$ large enough such that for every $n \geq n_0$ we have $\varphi(\alpha_n) \notin I$. Then for every $n \geq n_0$ we have $\varphi(\alpha_n) \notin I \cup I_{\alpha_n}$ and so fh_{α_n} and $\pi_{\varphi(\alpha_n)}$ are stochastically independent, that is, $\int_K fh_{\alpha_n}\pi_{\varphi(\alpha_n)}\,d\lambda = 0$. Then

$$\langle f, g_{\alpha_n}\rangle = \int_K fg_{\alpha_n}\,d\lambda = 0 \quad \text{whenever} \quad n \geq n_0.$$

This proves the CLAIM.

From the previous claim it follows at once that for every $A \in \Sigma$ we have

$$\nu(A) := \Big(\int_A g_\alpha\,d\lambda\Big)_{\alpha<\kappa} \in c_0(\kappa).$$

Clearly, $\nu : \Sigma \to c_0(\kappa)$ is finitely additive and satisfies

$$\|\nu(A)\|_{c_0(\kappa)} = \sup_{\alpha<\kappa}\Big|\int_A g_\alpha\,d\lambda\Big| \leq \sup_{\alpha<\kappa}\|1_A\|_{L^p(\lambda)}\|g_\alpha\|_{L^q(\lambda)} \leq \lambda(A)^{\frac{1}{p}} \quad \text{for all} \quad A \in \Sigma,$$

hence $\nu \in ca(\Sigma, c_0(\kappa))$. By Lemma 2.3, the norm of any $f \in \mathrm{sim}\,\Sigma$ is

$$\begin{aligned}\|f\|_{L^1(\nu)} &= \sup_{\alpha<\kappa}\int_K |fg_\alpha|\,d\lambda \\ &= \sup_{\alpha<\kappa}\int_K |fh_\alpha|\,d\lambda = \sup_{\alpha<\kappa}\big\langle |f|, |h_\alpha|\big\rangle \overset{(*)}{=} \sup_{h\in B_{L^q(\lambda)}}\big\langle |f|, |h|\big\rangle = \|f\|_{L^p(\lambda)},\end{aligned}$$

where equality $(*)$ follows from the norm density of $\{h_\alpha : \alpha < \kappa\}$ in $B_{L^q(\lambda)}$. According to Example 2.1 and Lemma 2.4, we have $L^1(\nu) = L^p(\lambda)$ with equal norms. □

3 Proofs of Theorems 1.2 and 1.3

In order to prove Theorems 1.2 and 1.3 we need some lemmas.

Lemma 3.1 *Let* κ *be an infinite cardinal. If* $\mu \in ca_+(\Sigma)$ *is homogeneous of Maharam type* κ*, then there is a set* $\{\mu_\alpha : \alpha < \kappa\} \subseteq ca(\Sigma)$ *such that:*

(i) $|\mu_\alpha| = \mu$ *for all* $\alpha < \kappa$*;*
(ii) $(\mu_\alpha(E))_{\alpha<\kappa} \in c_0(\kappa)$ *for all* $E \in \Sigma$.

Proof We can suppose without loss of generality that $\mu(\Omega) = 1$. By Maharam's theorem (see e.g. [20, p. 122, Theorem 8]), the measure algebra of μ is isomorphic to the measure algebra of the usual probability on $\{-1,1\}^\kappa$. We can now find a set $\{g_\alpha : \alpha < \kappa\} \subseteq L^\infty(\mu)$

with $|g_\alpha| = 1$ for all $\alpha < \kappa$ such that, for every $E \in \Sigma$, we have $(\int_E g_\alpha \, d\mu)_{\alpha<\kappa} \in c_0(\kappa)$ (see Example 2.5). It is clear that the measures $\mu_\alpha \in ca(\Sigma)$ defined by $\mu_\alpha(E) := \int_E g_\alpha \, d\mu$ satisfy the required properties. □

Lemma 3.2 *Let κ be an infinite cardinal. Let $\lambda \in ca_+(\Sigma)$ be homogeneous of Maharam type κ. Let $\{\lambda_\alpha\}_{\alpha<\kappa}$ be a family in $ca_+(\Sigma)$ such that*

$$\lambda \ll \lambda_\alpha \quad \textit{for all} \quad \alpha < \kappa \quad \textit{and} \quad \lim_{\lambda(A)\to 0} \sup_{\alpha<\kappa} \lambda_\alpha(A) = 0.$$

Then there is a family $\{\mu_\alpha\}_{\alpha<\kappa}$ in $ca(\Sigma)$ such that:

(i) $|\mu_\alpha| = \lambda_\alpha$ *for all* $\alpha < \kappa$*;*
(ii) $(\mu_\alpha(E))_{\alpha<\kappa} \in \ell^\infty_<(\kappa)$ *for all* $E \in \Sigma$.

Proof Each λ_α is homogeneous of Maharam type κ, so for each $\alpha < \kappa$ we can apply Lemma 3.1 to obtain a set $\{\mu_{\alpha,\beta} : \beta < \kappa\} \subseteq ca(\Sigma)$ such that:

- $|\mu_{\alpha,\beta}| = \lambda_\alpha$ for all $\beta < \kappa$;
- $(\mu_{\alpha,\beta}(E))_{\beta<\kappa} \in c_0(\kappa)$ for all $E \in \Sigma$.

Fix a family $\{A_\gamma\}_{\gamma<\kappa}$ in Σ such that $\inf_{\gamma<\kappa} \lambda(E \triangle A_\gamma) = 0$ for all $E \in \Sigma$. We now distinguish two cases:

Case 1: $\kappa = \omega$. By allowing infinitely many repetitions, we can assume further that for every $m < \omega$ and every $E \in \Sigma$ we have $\inf_{n\geq m} \lambda(E \triangle A_n) = 0$. For each $n < \omega$, the set

$$B(n) := \bigcup_{m\leq n} \left\{ k < \omega : |\mu_{n,k}(A_m)| > \frac{1}{n+1} \right\}$$

is finite and we choose $\beta(n) \in \omega \setminus B(n)$. Define $\mu_n := \mu_{n,\beta(n)} \in ca(\Sigma)$ for every $n < \omega$, so that $\{\mu_n\}_{n<\omega}$ satisfies (i). We next check that (ii) holds. To this end, fix $E \in \Sigma$ and $\varepsilon > 0$. Take $\delta > 0$ such that $\sup_{n<\omega} \lambda_n(A) \leq \varepsilon$ whenever $\lambda(A) \leq \delta$. Choose $m < \omega$ such that $\frac{1}{m+1} \leq \varepsilon$ and $\lambda(E \triangle A_m) \leq \delta$. For each $n < \omega$ we have

$$|\mu_n(E) - \mu_n(A_m)| \leq |\mu_n|(E \triangle A_m) = \lambda_n(E \triangle A_m) \leq \varepsilon. \tag{3.1}$$

Bearing in mind that $|\mu_n(A_m)| = |\mu_{n,\beta(n)}(A_m)| \leq \frac{1}{n+1} \leq \varepsilon$ whenever $n \geq m$, from (3.1) we conclude that $|\mu_n(E)| \leq 2\varepsilon$ for every $n \geq m$. As $\varepsilon > 0$ is arbitrary, this proves that $(\mu_n(E))_{n<\omega} \in c_0(\omega)$. The proof of Case 1 is finished.

Case 2: κ is uncountable. For each $\alpha < \kappa$, the set

$$B(\alpha) := \bigcup_{\gamma\leq\alpha} \{\beta < \kappa : \mu_{\alpha,\beta}(A_\gamma) \neq 0\}$$

has cardinality $|B(\alpha)| < \kappa$, because κ is uncountable and $\{\beta < \kappa : \mu_{\alpha,\beta}(A_\gamma) \neq 0\}$ is countable for every $\gamma < \kappa$. Then for every $\alpha < \kappa$ we can choose $\beta(\alpha) \in \kappa \setminus B(\alpha)$ and we define $\mu_\alpha := \mu_{\alpha,\beta(\alpha)} \in ca(\Sigma)$. An argument similar to that of Case 1 shows that the family $\{\mu_\alpha\}_{\alpha<\kappa}$ satisfies the required properties. □

Lemma 3.3 *Let κ be an infinite cardinal. Let $\nu \in ca(\Sigma, X)$ with* $\mathrm{dens}(L^1(\nu)) = \kappa$*. Then there is $C \subseteq B_{X^*}$ with $|C| \leq \kappa$ such that:*

(i) $\nu \ll |x^*\nu|$ *for all* $x^* \in C$*;*
(ii) $\|f\|_{L^1(\nu)} = \sup_{x^*\in C} \int_\Omega |f| \, d|x^*\nu|$ *for all* $f \in L^1(\nu)$.

Proof Fix a norm dense set $\mathcal{F} \subseteq L^1(\nu)$ with $|\mathcal{F}| = \kappa$. By the Rybakov–Walsh theorem (see e.g. [8, pp. 268–269]), the set $\Delta := \{x^* \in B_{X^*} : \nu \ll |x^*\nu|\}$ is norm dense (hence w^*-dense) in B_{X^*}. Then for every $f \in L^1(\nu)$ there is a countable set $\Delta_f \subseteq \Delta$ such that

$$\|f\|_{L^1(\nu)} = \sup_{x^* \in \Delta_f} \int_\Omega |f|\, d|x^*\nu|$$

(apply Lemma 2.2). It is easy to check that $C := \bigcup_{f\in\mathcal{F}} \Delta_f$ fulfills the required properties. □

We arrive at the proofs of our main results:

Proof of Theorem 1.2 The Banach space in which ν takes values is denoted by X. By Lemma 3.3 there is a collection $\{x^*_\alpha\}_{\alpha<\kappa}$ in B_{X^*} such that $\nu \ll |x^*_\alpha\nu|$ for all $\alpha < \kappa$ and

$$\|f\|_{L^1(\nu)} = \sup_{\alpha<\kappa} \int_\Omega |f|\, d|x^*_\alpha\nu| \quad \text{for all } f \in L^1(\nu). \tag{3.2}$$

Lemma 3.2 can now be applied to $\lambda_\alpha := |x^*_\alpha\nu|$ and $\lambda := |x^*_0\nu|$ to find a family $\{\mu_\alpha\}_{\alpha<\kappa}$ in $ca(\Sigma)$ such that $|\mu_\alpha| = |x^*_\alpha\nu|$ for all $\alpha < \kappa$ and

$$\tilde{\nu}(E) := (\mu_\alpha(E))_{\alpha<\kappa} \in \ell^\infty_<(\kappa) \quad \text{for all } E \in \Sigma.$$

The function $\tilde{\nu} : \Sigma \to \ell^\infty_<(\kappa)$ is finitely additive. Moreover, since

$$\|\tilde{\nu}(E)\|_{\ell^\infty_<(\kappa)} = \sup_{\alpha<\kappa} |\mu_\alpha(E)| \le \sup_{\alpha<\kappa} |x^*_\alpha\nu|(E) \le \|\nu\|(E) \quad \text{for all } E \in \Sigma,$$

we have $\lim_{\lambda(E)\to 0} \|\tilde{\nu}(E)\|_{\ell^\infty_<(\kappa)} = 0$. It follows that $\tilde{\nu} \in ca(\Sigma, \ell^\infty_<(\kappa))$.

In order to prove that $L^1(\nu) = L^1(\tilde{\nu})$ with equal norms, it suffices to check that $\|f\|_{L^1(\nu)} = \|f\|_{L^1(\tilde{\nu})}$ for every $f \in \mathrm{sim}\,\Sigma$ (Lemma 2.4). Write $e^*_\alpha \in B_{\ell^\infty_<(\kappa)^*}$ to denote the α-th coordinate projection for every $\alpha < \kappa$. Lemma 2.3 applies to compute the norm of any $f \in \mathrm{sim}\,\Sigma$ as

$$\|f\|_{L^1(\tilde{\nu})} = \sup_{\alpha<\kappa} \int_\Omega |f|\, d|e^*_\alpha\tilde{\nu}| = \sup_{\alpha<\kappa} \int_\Omega |f|\, d|x^*_\alpha\nu| \overset{(3.2)}{=} \|f\|_{L^1(\nu)}.$$

The proof is complete. □

Proof of Theorem 1.3 Let μ be a Rybakov control measure of ν. Then μ is non-atomic and has Maharam type ω_1. Therefore, there exist disjoint $A, B \in \Sigma$ with $\Omega = A \cup B$ such that $L^1(\mu_A)$ is separable and μ_B is homogeneous and has Maharam type ω_1 (see e.g. [20, p. 122, Theorem 7]). So, $L^1(\nu_A)$ is separable, ν_B is homogeneous and $\mathrm{dens}(L^1(\nu_B)) = \omega_1$. By Theorems 1.1 and 1.2 applied to ν_A and ν_B, respectively, there exist $\xi \in ca(\Sigma_A, c_0)$ and $\psi \in ca(\Sigma_B, \ell^\infty_c(\omega_1))$ such that

$$L^1(\nu_A) = L^1(\xi) \quad \text{and} \quad L^1(\nu_B) = L^1(\psi)$$

with equal norms. Write $Z := c_0 \oplus_1 \ell^\infty_c(\omega_1)$ and define $\varphi \in ca(\Sigma, Z)$ by

$$\varphi(E) := (\xi(E \cap A), \psi(E \cap B)) \quad \text{for all } E \in \Sigma.$$

Fix $f \in \mathrm{sim}\,\Sigma$ and denote by $f|_A$ (resp. $f|_B$) its restriction to A (resp. B). Then

$$\int_E f\, d\varphi = \left(\int_{E\cap A} f|_A\, d\xi, \int_{E\cap B} f|_B\, d\psi \right) \quad \text{for all } E \in \Sigma$$

and so

$$\sup_{E\in\Sigma}\left\|\int_E f\,d\varphi\right\|_Z=\sup_{E\in\Sigma}\left\|\int_{E\cap A} f|_A\,d\xi\right\|_{c_0}+\sup_{E\in\Sigma}\left\|\int_{E\cap B} f|_B\,d\psi\right\|_{\ell_c^\infty(\omega_1)}. \tag{3.3}$$

On one hand, we have

$$\begin{aligned}\|f\|_{L^1(\varphi)} &\overset{(2.1)}{\leq} 2\sup_{E\in\Sigma}\left\|\int_E f\,d\varphi\right\|_Z\\ &\overset{(3.3)}{=} 2\sup_{E\in\Sigma}\left\|\int_{E\cap A} f|_A\,d\xi\right\|_{c_0}+2\sup_{E\in\Sigma}\left\|\int_{E\cap B} f|_B\,d\psi\right\|_{\ell_c^\infty(\omega_1)}\\ &\overset{(2.1)}{\leq} 2\|f|_A\|_{L^1(\xi)}+2\|f|_B\|_{L^1(\psi)}\\ &= 2\|f|_A\|_{L^1(\nu_A)}+2\|f|_B\|_{L^1(\nu_B)}\\ &\leq 4\|f\|_{L^1(\nu)}.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}\|f\|_{L^1(\nu)} &= \|f1_A+f1_B\|_{L^1(\nu)}\\ &\leq \|f|_A\|_{L^1(\nu_A)}+\|f|_B\|_{L^1(\nu_B)}\\ &= \|f|_A\|_{L^1(\xi)}+\|f|_B\|_{L^1(\psi)}\\ &\overset{(2.1)}{\leq} 2\sup_{E\in\Sigma}\left\|\int_{E\cap A} f|_A\,d\xi\right\|_{c_0}+2\sup_{E\in\Sigma}\left\|\int_{E\cap B} f|_B\,d\psi\right\|_{\ell_c^\infty(\omega_1)}\\ &\overset{(3.3)}{=} 2\sup_{E\in\Sigma}\left\|\int_E f\,d\varphi\right\|_Z\\ &\overset{(2.1)}{\leq} 2\|f\|_{L^1(\varphi)}.\end{aligned}$$

It follows that

$$\frac{1}{4}\|f\|_{L^1(\varphi)}\leq\|f\|_{L^1(\nu)}\leq 2\|f\|_{L^1(\varphi)}\quad\text{for every } f\in\operatorname{sim}\Sigma.$$

The proof of Lemma 2.4 given in [21, Lemma 6] can now be adapted straightforwardly to prove that $L^1(\nu)=L^1(\varphi)$ with equivalent norms.

Since c_0 embeds isomorphically into $\ell_c^\infty(\omega_1)$ and $\ell_c^\infty(\omega_1)$ is isomorphic to its square, the space $Z=c_0\oplus_1\ell_c^\infty(\omega_1)$ embeds isomorphically into $\ell_c^\infty(\omega_1)$. Take any isomorphic embedding $j:Z\to\ell_c^\infty(\omega_1)$ and define $\tilde\nu:=j\circ\varphi\in ca(\Sigma,\ell_c^\infty(\omega_1))$. It is easy to check that $L^1(\tilde\nu)=L^1(\varphi)$ with equivalent norms. Then $L^1(\nu)=L^1(\tilde\nu)$ with equivalent norms and the proof is complete. □

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Appendix: Linear injections into L^1 of a vector measure

As we mentioned in the introduction, given an uncountable set Γ, the space $\ell^p(\Gamma)$ is not isomorphic to the L^1 space of a vector measure for any $p\neq 2$, see [26, Theorem 2.6].

However, we should note that any order continuous Banach lattice, like $\ell^p(\Gamma)$, is lattice isometric to the L^1 space of a "vector measure" defined on a δ *-ring* (a structure which is weaker than σ-algebra), see [5, pp. 22–23].

On the other hand, for an uncountable set Γ, the space $\ell^p(\Gamma)$ *embeds isomorphically* into the L^1 space of a finite measure if and only if $1 < p \leq 2$, see [10, Theorem 2.1]. For the range $2 < p < \infty$ the situation is different:

Proposition 3.4 *Let Γ be a non-empty set and let Z be either $c_0(\Gamma)$ or $\ell^p(\Gamma)$ for some $2 < p < \infty$. If there is an injective operator from Z into $L^1(\nu)$ for some $\nu \in ca(\Sigma, X)$, then Γ is countable.*

Proof Let $S : Z \to L^1(\nu)$ be an injective operator. Let μ be a Rybakov control measure of ν and $i : L^1(\nu) \to L^1(\mu)$ the inclusion operator, which is injective. Then $T := i \circ S : Z \to L^1(\mu)$ is injective as well. Since $Z^* = \ell^q(\Gamma)$ for some $1 \leq q < 2$, the adjoint operator $T^* : L^\infty(\mu) \to Z^*$ is compact, by a result of Rosenthal (see [27, p. 211, Remark 2]). By Schauder's theorem, T is compact and so $T(Z)$ is separable. Therefore, there is a countable set $\Delta \subseteq L^\infty(\mu)$ separating the points of $T(Z)$. Since T is injective, the countable set $T^*(\Delta) \subseteq Z^*$ separates the points of Z, hence (Z^*, w^*) is separable. This clearly implies that Γ is countable. □

In particular, for any uncountable set Γ the space $c_0(\Gamma)$ does not embed isomorphically into the L^1 space of a vector measure. This assertion can be extended to all infinite-dimensional $C(K)$ spaces except c_0 itself, see Corollary 3.6 below.

A Banach space Z is said to be *weakly countably determined* (WCD) if there is a sequence (K_n) of w^*-compact subsets of Z^{**} such that, for every $z \in Z$ and $z^{**} \in Z^{**} \setminus Z$, there is $n \in \mathbb{N}$ such that $z \in K_n$ and $z^{**} \notin K_n$. The class of WCD Banach spaces includes all weakly compactly generated spaces and their subspaces. For complete information on WCD spaces, we refer the reader to [11, Chapter 7].

A Banach space Z is said to have the *Dunford–Pettis property* if every weakly compact operator T from Z to a Banach space is Dunford–Pettis (i.e. $T(C)$ is norm compact whenever $C \subseteq Z$ is weakly compact).

Proposition 3.5 *Let Z be a WCD Banach space with the Dunford–Pettis property such that Z^* contains no subspace isomorphic to c_0. If there is an injective operator from Z into $L^1(\nu)$ for some $\nu \in ca(\Sigma, X)$, then Z is separable.*

Proof The proof is similar to that of Proposition 3.4. Fix an injective operator $S : Z \to L^1(\nu)$. Let μ be a Rybakov control measure of ν, let $i : L^1(\nu) \to L^1(\mu)$ be the inclusion operator and define $T := i \circ S : Z \to L^1(\mu)$. Observe that the adjoint $T^* : L^\infty(\mu) \to Z^*$ is weakly compact, because $L^\infty(\mu)$ is a $C(K)$ space and Z^* contains no subspace isomorphic to c_0 (see e.g. [1, Theorem 5.5.3]). By Gantmacher's theorem, T is weakly compact and so the Dunford–Pettis property of Z ensures that T is a Dunford–Pettis operator. Since every Dunford–Pettis operator from a WCD Banach space has separable range (see [28, Theorem 7.1]), $T(Z)$ is separable. The rest of the proof follows the argument of Proposition 3.4, bearing in mind that a WCD Banach space is separable if (and only if) it has w^*-separable dual (see [28, Theorem 6.1] or [30, Corollary 2]). □

For any compact Hausdorff topological space K, the Banach space $C(K)$ has the Dunford–Pettis property (see e.g. [1, Theorem 5.4.5]) and its dual $C(K)^*$ contains no subspace isomorphic to c_0 (combine [1, Theorem 5.5.3] and [2, Theorem 4.68]). These facts and Proposition 3.5 yield the following:

Corollary 3.6 *Let K be an infinite compact Hausdorff topological space. If $C(K)$ is isomorphic to a subspace of $L^1(\nu)$ for some $\nu \in ca(\Sigma, X)$, then $C(K)$ is isomorphic to c_0.*

Proof Such $C(K)$ space is WCD, because every subspace of a weakly compactly generated Banach space (like $L^1(\nu)$) is WCD. Proposition 3.5 applies to deduce that $C(K)$ is separable, i.e. K is metrizable. On the other hand, since every subspace of an order continuous Banach lattice (like $L^1(\nu)$) has the so-called Pełczyński's property (u) (see e.g. [2, Theorems 4.54 and 4.56]), so does $C(K)$. It follows that $C(K)$ is isomorphic to c_0 (see e.g. [1, Theorem 4.5.2]). □

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