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Characterizations of complete spacelike submanifolds in the (n + p)-dimensional anti-de Sitter space of index q

Henrique F. de Lima[1] · **Fábio R. dos Santos**[1] · **Marco Antonio L. Velásquez**[1]

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Abstract Our purpose in this paper is to study the geometry of n-dimensional complete spacelike submanifolds immersed in the $(n+p)$-dimensional anti-de Sitter space $\mathbb{H}_q^{n+p}$ of index q, with $1 \leq q \leq p$. Under suitable constraints on the Ricci curvature and the second fundamental form, we show that a complete maximal spacelike submanifold of $\mathbb{H}_q^{n+p}$ must be totally geodesic. Furthermore, we establish sufficient conditions to guarantee that a complete spacelike submanifold with nonzero parallel mean curvature vector in $\mathbb{H}_p^{n+p}$ must be pseudo-umbilical, which means that its mean curvature vector is an umbilical direction.

Keywords Anti-de Sitter space · Complete spacelike submanifolds · Totally geodesic submanifolds · Parallel mean curvature vector · Pseudo-umbilical submanifolds

Mathematics Subject Classification Primary 53C42; Secondary 53A10 · 53C20 · 53C50

1 Introduction

Apart from their physical importance (see, for example, [25,34]), the interest in the study of spacelike submanifolds immersed in a Lorentzian space is motivated by their nice Bernstein-type properties. For instance, it was proved by Calabi [10] (for $n \leq 4$) and by Cheng and Yau [14] (for all n) that the only complete maximal spacelike hypersurfaces of the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. In [29], Nishikawa proved that a complete maximal spacelike hypersurface (that is, with mean curvature identically zero) in

✉ Henrique F. de Lima
henrique@mat.ufcg.edu.br

Fábio R. dos Santos
fabio@mat.ufcg.edu.br

Marco Antonio L. Velásquez
marco.velasquez@pq.cnpq.br

[1] Departamento de Matemática, Universidade Federal de Campina Grande, Campina Grande, Paraíba 58429-970, Brazil

the de Sitter space $\mathbb{S}_1^{n+1}$ must be totally geodesic. In [18], Goddard conjectured that the complete spacelike hypersurfaces of $\mathbb{S}_1^{n+1}$ with constant mean curvature H must be totally umbilical. Ramanathan [32] proved Goddard's conjecture in $\mathbb{S}_1^3$ for $0 \leq H \leq 1$. Moreover, for $H > 1$, he showed that the conjecture is false, as can be seen from an example due to Dajczer and Nomizu in [16]. Independently, Akutagawa [2] proved that Goddard's conjecture is true when either $n = 2$ and $H^2 \leq 1$ or $n \geq 3$ and $H^2 < \frac{4(n-1)}{n^2}$. He also constructed complete spacelike rotation surfaces in $\mathbb{S}_1^3$ having constant mean curvature $H > 1$ and which are not totally umbilical. Next, Montiel [26] showed that Goddard's conjecture is true provided that M^n is compact. Furthermore, he exhibited examples of complete spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature H satisfying $H^2 \geq \frac{4(n-1)}{n^2}$ and being non totally umbilical, the so-called hyperbolic cylinders.

In higher codimension, Cheng [12] extended Akutagawa's result for complete spacelike submanifolds with parallel mean curvature vector (that is, the mean curvature vector field is parallel as a section of the normal bundle) in the de Sitter space $\mathbb{S}_p^{n+p}$ of index p. Afterwards, Aiyama [1] studied compact spacelike submanifolds in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector and proved that if the normal connection of M^n is flat, then M^n is totally umbilical. Furthermore, she proved that a compact spacelike submanifold in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector and nonnegative sectional curvature must be totally umbilical. Meanwhile, Alías and Romero [4] developed some integral formulas for compact spacelike submanifolds in $\mathbb{S}_p^{n+p}$ which have a very clear geometric meaning and, as application, they obtained a Bernstein type result for complete maximal submanifolds in $\mathbb{S}_q^{n+p}$, extending a previous result due to Ishihara [19]. Moreover, they extended Ramanathan's result [32] showing that the only compact spacelike surfaces in $\mathbb{S}_p^{2+p}$ with parallel mean curvature vector are the totally umbilical ones and, in particular, they also reproved Cheng's result [14] establishing that every complete spacelike surface in $\mathbb{S}_p^{2+p}$ with parallel mean curvature vector such that $H^2 < 1$ is totally umbilical. Next, Li [22] showed that Montiel's result [26] still holds for higher codimensional spacelike submanifolds in $\mathbb{S}_p^{n+p}$. More recently, Araújo and Barbosa [6], assuming appropriated controls on the second fundamental form and on the scalar curvature, extended the techniques developed in [23,33,35] and proved that a compact spacelike submanifold in $\mathbb{S}_p^{n+p}$ with nonzero mean curvature and parallel mean curvature vector must be isometric to a sphere.

When the ambient spacetime is the anti-de Sitter space $\mathbb{H}_1^{n+1}$, Choi et al. [15] used the generalized maximum principle of Omori [30] and Yau [36] in order to obtain a Myers type theorem [28] concerning complete maximal spacelike hypersurfaces. More precisely, they showed that if the height function with respect to a timelike vector of such a hypersurface obeys a certain boundedness, then it must be totally geodesic. Extending a technique due to Yau [37], the first author jointly with Camargo [7] obtained another rigidity results to complete maximal spacelike hypersurfaces in $\mathbb{H}_1^{n+1}$, imposing suitable conditions on both the norm of the second fundamental form and a certain height function naturally attached to the hypersurface. Afterwards, working with a suitable warped product model for an open subset of $\mathbb{H}_1^{n+1}$, the same authors jointly with Caminha and Parente [8] extended the main result of [7] showing that if M^n is a complete spacelike hypersurface with constant mean curvature and bounded scalar curvature in $\mathbb{H}_1^{n+1}$, such that the gradient of its height function with respect to a timelike vector has integrable norm, then M^n must be totally umbilical. More recently, the first author jointly Aquino [5] obtained another characterizations theorems concerning complete constant mean curvature spacelike hypersurfaces of $\mathbb{H}_1^{n+1}$, under suitable constraints on the behavior of the Gauss mapping. In higher codimension, Ishihara [19] proved that a n-dimensional complete maximal spacelike submanifold immersed in the anti-de Sitter space

$\mathbb{H}_p^{n+p}$ of index p must have the squared norm of the second fundamental form bounded from above by np. Moreover, the only ones that attain this estimate are the hyperbolic cylinders. Later on, Cheng [13] obtained a refinement of Ishihara's result [19] for the case of complete maximal spacelike surfaces immersed in $\mathbb{H}_p^{2+p}$.

Motivated by the works above described, our purpose in this paper is to study the geometry of complete spacelike submanifolds immersed in the anti-de Sitter space $\mathbb{H}_q^{n+p}$ of index q. In this setting, we extend the technique due to Alías and Romero in [4] and, under appropriated constraints on the Ricci curvature and second fundamental form, we show that a complete maximal spacelike submanifold M^n of $\mathbb{H}_q^{n+p}$ must be totally geodesic (see Theorem 1 and Corollaries 1 and 2). Furthermore, we establish sufficient conditions to guarantee that a complete spacelike submanifold with nonzero parallel mean curvature vector $\mathbf{H}$ in $\mathbb{H}_p^{n+p}$ must be pseudo-umbilical, which means that $\mathbf{H}$ is an umbilical direction (see Theorem 2 and Corollary 3). Our approach is based on a generalized form of a maximum principle at the infinity of Yau [37] (see Lemma 1 and Remark 1).

2 Preliminaries

Let $\mathbb{R}_{q+1}^{n+p+1}$ be the $(n+p+1)$-dimensional semi-Euclidean space endowed with metric tensor $\langle , \rangle$ of index q, with $1 \leq q \leq p$, given by

$$\langle v, w\rangle = \sum_{i=1}^{n+p-q} v_i w_i - \sum_{j=n+p-q+1}^{n+p+1} v_j w_j,$$

and let $\mathbb{H}_q^{n+p}$ be the $(n+p)$-dimensional unitary anti-de Sitter space of index q, that is,

$$\mathbb{H}_q^{n+p} = \{x \in \mathbb{R}_{q+1}^{n+p+1}\,;\ \langle x, x\rangle = -1\},$$

which has constant sectional curvature equal to -1.

Along this work, we will consider $x\colon M^n \to \mathbb{H}_q^{n+p} \subset \mathbb{R}_{q+1}^{n+p+1}$ a spacelike submanifold isometrically immersed in $\mathbb{H}_q^{n+p}$. We recall that a submanifold immersed is said to be *spacelike* if its induced metric is positive definite. In this setting, we will denote by ∇°, $\overline{\nabla}$ and ∇ the Levi-Civita connections of $\mathbb{R}_{q+1}^{n+p+1}$, $\mathbb{H}_q^{n+p}$ and M^n, respectively, and $\nabla^\perp$ will stand for the normal connection of M^n in $\mathbb{H}_q^{n+p}$.

We will denote by α the second fundamental form of M^n in $\mathbb{H}_q^{n+p}$ and by A_ξ the shape operator associated to a fixed vector field ξ normal to M^n in $\mathbb{H}_q^{n+p}$. We note that, for each $\xi \in \mathfrak{X}^\perp(M)$, A_ξ is a symmetric endomorphism of the tangent space T_xM at $x \in M^n$. Moreover, A_ξ and α are related by

$$\langle A_\xi X, Y\rangle = \langle \alpha(X, Y), \xi\rangle, \tag{2.1}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$.

We also recall that the Gauss and Weingarten formulas of M^n in $\mathbb{H}_q^{n+p}$ are given by

$$\nabla^\circ_X Y = \overline{\nabla}_X Y + \langle X, Y\rangle x = \nabla_X Y + \alpha(X, Y) + \langle X, Y\rangle x, \tag{2.2}$$

and

$$\nabla^\circ_X \xi = \overline{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{2.3}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$ and normal vector field $\xi \in \mathfrak{X}^\perp(M)$.

As in [31], the curvature tensor R of the spacelike submanifold M^n is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where [,] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(M)$.

A well known fact is that the curvature tensor R of M^n can be described in terms of its second fundamental form α and the curvature tensor $\overline{R}$ of the ambient spacetime $\mathbb{H}_q^{n+p}$ by the so-called Gauss equation, which is given by

$$\begin{aligned}\langle R(X,Y)Z, W\rangle &= \langle Y, Z\rangle\langle X, W\rangle - \langle X, Z\rangle\langle Y, W\rangle \\ &\quad + \langle \alpha(X,W), \alpha(Y,Z)\rangle - \langle \alpha(X,Z), \alpha(Y,W)\rangle, \end{aligned} \tag{2.4}$$

for all tangent vector fields $X, Y, Z, W \in \mathfrak{X}(M)$. Moreover, Codazzi equation asserts that

$$(\nabla_X A_\xi)Y = (\nabla_Y A_\xi)X, \tag{2.5}$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}^\perp(M)$.

We will define the mean curvature vector of M^n in $\mathbb{H}_q^{n+p}$ by

$$\mathbf{H} = \frac{1}{n}\mathrm{tr}(\alpha).$$

We recall that M^n is called *maximal* when $\mathbf{H} \equiv 0$ and we say that M^n has *parallel mean curvature vector* when $\nabla_X^\perp \mathbf{H} \equiv 0$, for every $X \in \mathfrak{X}(M)$. In this last case, when $q = p$ and $\mathbf{H} \neq 0$, we have that $\langle \mathbf{H}, \mathbf{H}\rangle$ is a negative constant along M^n. Moreover, M^n is called *totally geodesic* when its second fundamental form α vanishes identically and it is called *totally umbilical* when

$$\alpha(X,Y) = \langle X, Y\rangle \mathbf{H}, \tag{2.6}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$.

We close this section describing the main analytical tool which is used along the proofs of our results in the next sections. In [37] Yau, generalizing a previous result due to Gaffney [17], established the following version of Stokes' Theorem on an n-dimensional, complete noncompact Riemannian manifold M^n: *if* $\omega \in \Omega^{n-1}(M)$ *is an integrable* $(n-1)$-*differential form on* M^n, *then there exists a sequence* B_i *of domains on* M^n *such that* $B_i \subset B_{i+1}$, $M^n = \bigcup_{i\geq 1} B_i$ and

$$\lim_{i\to+\infty}\int_{B_i} d\omega = 0.$$

Suppose that M^n is oriented by the volume element dM. If $\omega = \iota_X dM$ is the contraction of dM in the direction of a smooth vector field X on M^n, then Caminha obtained a suitable consequence of Yau's result, which can be regarded as an extension of Hopf's maximum principle for complete Riemannian manifolds (cf. Proposition 2.1 of [9]). In what follows, $\mathcal{L}^1(M)$ and div denote the space of Lebesgue integrable functions and the divergence on M^n, respectively.

Lemma 1 *Let X be a smooth vector field on the n-dimensional complete noncompact oriented Riemannian manifold* M^n, *such that* $\mathrm{div}X$ *does not change sign on* M^n. *If* $|X| \in \mathcal{L}^1(M)$, *then* $\mathrm{div}X = 0$.

Remark 1 Lemma 1 can also be seen as a consequence of the version of Stokes' Theorem given by Karp in [20]. In fact, using Theorem in [20], condition $|X| \in \mathcal{L}^1(M)$ can be weakened to the following technical condition:

$$\liminf_{r\to+\infty} \frac{1}{r}\int_{B(2r)\setminus B(r)} |X| dM = 0,$$

where $B(r)$ denotes the geodesic ball of radius r center at some fixed origin $o \in M^n$. See also Corollary 1 and Remark in [20] for some another geometric conditions guaranteing this fact.

Remark 2 Reasoning in a similar way of that in the beginning of Section 4 of [5] (see also Section 4 in [3]), it is not difficult to verify that there exist no n-dimensional compact (without boundary) spacelike submanifolds immersed in $\mathbb{H}_p^{n+p}$. Motivated by this fact, along this paper we will deal with complete spacelike submanifolds.

3 Complete maximal submanifolds immersed in $\mathbb{H}_q^{n+p}$

Let $a \in \mathbb{R}_{q+1}^{n+p+1}$ be a fixed arbitrary vector and put

$$a = a^\top + a^N - \langle a, x\rangle x, \tag{3.1}$$

where $a^\top \in \mathfrak{X}(M)$ and $a^N \in \mathfrak{X}^\perp(M)$ denote, respectively, the tangential and normal components of a with respect to $M^n \hookrightarrow \mathbb{H}_q^{n+p}$. By taking covariant derivative in (3.1) and using (2.2) and (2.3), we get for all tangent vector field $X \in \mathfrak{X}(M)$ that

$$\nabla_X a^\top = A_{a^N} X + \langle a, x\rangle X \tag{3.2}$$

and

$$\nabla_X^\perp a^N = -\alpha(a^\top, X). \tag{3.3}$$

Hence, from (2.1) and (3.2) we obtain

$$\mathrm{div}(a^\top) = \mathrm{tr}(A_{a^N}) + n\langle a, x\rangle = n\langle a, \mathbf{H}\rangle + n\langle a, x\rangle. \tag{3.4}$$

Moreover, we also have that

$$\begin{aligned}\mathrm{tr}(\nabla_{a^\top} A_\xi) = & \sum_i \langle \nabla_{a^\top} A_\xi e_i, e_i\rangle - \sum_i \langle \nabla_{a^\top} e_i, A_\xi e_i\rangle \\ & + n\langle \nabla_{a^\top}^\perp \mathbf{H}, \xi\rangle - \sum_i a^\top \langle A_\xi e_i, e_i\rangle.\end{aligned}$$

So, considering a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $A_\xi e_i = \lambda_i^\xi e_i$, with a straightforward computation we can verify that

$$\mathrm{tr}(\nabla_{a^\top} A_\xi) = n\langle \nabla_{a^\top}^\perp \mathbf{H}, \xi\rangle. \tag{3.5}$$

From Codazzi Eq. (2.5) jointly with (3.2) and (3.5) we obtain, for all $\xi \in \mathfrak{X}^\perp(M)$,

$$\begin{aligned}\mathrm{div}(A_\xi a^\top) = & \, n\langle \nabla_{a^\top}^\perp \mathbf{H}, \xi\rangle + \mathrm{tr}(A_{a^N} \circ A_\xi) + \langle a, x\rangle \mathrm{tr}(A_\xi) \\ & + \sum_i \langle \alpha(a^\top, e_i), \nabla_{e_i}^\perp \xi\rangle.\end{aligned} \tag{3.6}$$

On the other hand, taking the trace in Gauss Eq. (2.4), we have

$$\mathrm{Ric}(X, Y) = -(n-1)\langle X, Y\rangle + n\langle \alpha(X, Y), \mathbf{H}\rangle - \sum_i \langle \alpha(X, e_i), \alpha(Y, e_i)\rangle, \tag{3.7}$$

where Ric denotes the Ricci curvature of M^n. Considering $X = Y = a^\top$ in (3.7), we obtain

$$\mathrm{Ric}(a^\top, a^\top) = -(n-1)\left|a^\top\right|^2 + n\langle\alpha(a^\top, a^\top), \mathbf{H}\rangle - \sum_i \langle\alpha(a^\top, e_i), \alpha(a^\top, e_i)\rangle. \tag{3.8}$$

Furthermore, from (3.3) and (3.6) we get

$$\mathrm{div}(A_{a^N} a^\top) = n\langle\nabla^\perp_{a^\top}\mathbf{H}, a^N\rangle + \mathrm{tr}(A^2_{a^N}) + \langle a, x\rangle \mathrm{tr}(A_{a^N}) - \sum_i \langle\alpha(a^\top, e_i), \alpha(a^\top, e_i)\rangle. \tag{3.9}$$

Hence, from (3.8) and (3.9) we conclude that

$$\mathrm{div}(A_{a^N} a^\top) = n\langle\nabla^\perp_{a^\top}\mathbf{H}, a^N\rangle + \mathrm{tr}(A^2_{a^N}) + \langle a, x\rangle \mathrm{tr}(A_{a^N}) + \mathrm{Ric}(a^\top, a^\top) + (n-1)\left|a^\top\right|^2 - n\langle\alpha(a^\top, a^\top), \mathbf{H}\rangle. \tag{3.10}$$

Based on the previous computations, we obtain the following Bernstein type result concerning maximal submanifolds immersed in $\mathbb{H}^{n+p}_q$

Theorem 1 *Let M^n be a complete maximal spacelike submanifold immersed in $\mathbb{H}^{n+p}_q$, with $1 \le q \le p$. Suppose that $Ric \ge -(n-1)$ on M^n. If there exist p vectors $a_1, \dots, a_p \in \mathbb{R}^{n+p+1}_{q+1}$ such that $a^N_1, \dots, a^N_p$ are linearly independent, with $A_{a^N_i}$ bounded on M^n and $|a_i^\top| \in \mathcal{L}^1(M)$ for each $1 \le i \le p$, then M^n is totally geodesic.*

Proof Let us consider $a = a_i$ for some $i \in \{1, \dots, p\}$. Provided that $\mathbf{H} = 0$, from (2.1) we see that Eq. (3.10) can be rewritten as follows

$$\mathrm{div}(A_{a^N} a^\top) = \mathrm{tr}(A^2_{a^N}) + \mathrm{Ric}(a^\top, a^\top) + (n-1)\left|a^\top\right|^2. \tag{3.11}$$

Thus, since $\mathrm{Ric}(a^\top, a^\top) \ge -(n-1)\left|a^\top\right|^2$, from (3.11) we obtain that

$$\mathrm{div}(A_{a^N} a^\top) \ge 0. \tag{3.12}$$

Moreover, whereas A_{a^N} is bounded on M^n, $|A_{a^N}| \le C_1$, for some constant $C_1 > 0$. Thus, as we are assuming that $|a^\top| \in \mathcal{L}^1(M)$, we have

$$|A_{a^N} a^\top| \le |A_{a^N}||a^\top| \le C_1 |a^\top| \in \mathcal{L}^1(M). \tag{3.13}$$

Hence, taking into account (3.12) and (3.13), we can apply Lemma 1 to guarantee that $\mathrm{div}(A_{a^N} a^\top) = 0$. Consequently, returning to Eq. (3.11), we conclude that $A_{a^N} \equiv 0$. Therefore, since $\alpha(X, Y) = \sum_{i=1}^p \langle A_{a_i^N} X, Y\rangle a_i^N$, we have that M^n must be totally geodesic. □

Remark 3 Despite our assumption on the a_i^N in Theorem 1 to be a technical hypothesis, it is motivated by the fact that it occurs in a natural way in the context of spacelike hypersurfaces (see Corollary 2). In this sense, it is a mild hypothesis.

In the case that $p = q$, being M^n a maximal submanifold of $\mathbb{H}^{n+p}_p$, a classical result due to Ishihara [19] assures us that $|A|^2 \le np$ (see also Cheng [13] for the case $n = 2$). Moreover, each maximal submanifold in $\mathbb{H}^{n+p}_p$ meets the condition Ric $\ge -(n-1)$. Thus, as a consequence of Theorem 1 we obtain

Corollary 1 *Let M^n be a complete maximal spacelike submanifold immersed in $\mathbb{H}^{n+p}_p$. If there exist p vectors $a_1, \ldots, a_p \in \mathbb{R}^{n+p+1}_{p+1}$ such that $a_1^N, \ldots, a_p^N$ are linearly independent, with $|a_i^\top| \in \mathcal{L}^1(M)$ for each $1 \leq i \leq p$, then M^n is totally geodesic.*

Taking into account that the warped product model $-(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} \mathbb{H}^n$, which is considered in [7] in order to prove their results, models just only an open subset of $\mathbb{H}^{n+1}_1$ (cf. Example 4.3 of [27]), from Corollary 1 we obtain the following improvement of Theorem 1.2 of [7]

Corollary 2 *Let M^n be a complete maximal spacelike hypersurface immersed in $\mathbb{H}^{n+1}_1$. If there exists a vector $a \in \mathbb{R}^{n+2}_2$ such that a^N does not vanish on M^n and $|a^\top| \in \mathcal{L}^1(M)$, then M^n is a totally geodesic hyperbolic space.*

4 Submanifolds with parallel mean curvature vector in $\mathbb{H}^{n+p}_p$

In this section, we study the rigidity of a complete spacelike submanifold M^n of $\mathbb{H}^{n+p}_p$ with nonzero parallel mean curvature vector $\mathbf{H}$. For this, fixed a nonzero vector $a \in \mathbb{R}^{n+p+1}_{p+1}$, we observe that Eq. (3.4) gives us

$$\begin{aligned} \operatorname{div}\Big(\langle a, \mathbf{H}\rangle a^\top\Big) = &\frac{1}{n}\operatorname{tr}(A_{a^N})^2 + \langle a, x\rangle \operatorname{tr}(A_{a^N}) \\ &-\langle \alpha(a^\top, a^\top), \mathbf{H}\rangle + \langle a, \nabla^\perp_{a^\top}\mathbf{H}\rangle. \end{aligned} \tag{4.1}$$

Thus, from (3.10) jointly with (4.1) we obtain

$$\begin{aligned} \operatorname{div}\Big[(A_{a^N} - \langle a, \mathbf{H}\rangle I)a^\top\Big] = &(n-1)\langle \nabla^\perp_{a^\top}\mathbf{H}, a^N\rangle + \operatorname{tr}(A^2_{a^N}) \\ &-\frac{1}{n}\operatorname{tr}(A_{a^N})^2 + T(a^\top, a^\top), \end{aligned} \tag{4.2}$$

where I denotes the identity operator in the algebra of smooth vector fields on M^n and, following the terminology established in [4], T stands for a covariant tensor on M^n which is given by

$$T(X, X) = \operatorname{Ric}(X, X) + (n-1)\big|X\big|^2 - (n-1)\langle \alpha(X, X), \mathbf{H}\rangle. \tag{4.3}$$

According to [4,11], a spacelike submanifold M^n of $\mathbb{H}^{n+p}_p$ with nonzero mean curvature vector $\mathbf{H}$ is said *pseudo-umbilical* if $\mathbf{H}$ is an umbilical direction. From (2.6) we see that a totally umbilical spacelike submanifold is always pseudo-umbilical. Conversely, we get

Proposition 1 *Let M^n be a complete pseudo-umbilical spacelike submanifold with nonzero parallel mean curvature vector $\mathbf{H}$ in $\mathbb{H}^{n+p}_p$. If there exist p vectors $a_1, \ldots, a_p \in \mathbb{R}^{n+p+1}_{p+1}$ such that $a_1^N, \ldots, a_p^N$ are linearly independent, with $\langle a_i, \mathbf{H}\rangle$ and $A_{a_i^N}$ bounded on M^n and $|a_i^\top| \in \mathcal{L}^1(M)$ for each $1 \leq i \leq p$, then M^n is totally umbilical.*

Proof Let us consider $a = a_i$ for some $i \in \{1, \ldots, p\}$. We have that

$$\big|(A_{a^N} - \langle a, \mathbf{H}\rangle I)a^\top\big| \leq \big(|A_{a^N}| + |\langle a, \mathbf{H}\rangle|\big)\,|a^\top| \leq C_2|a^\top| \in \mathcal{L}^1(M). \tag{4.4}$$

Since we are assuming that M^n is pseudo-umbilical of $\mathbb{H}^{n+p}_p$, Lemma 4.1 of [4] assures that

$$\operatorname{Ric}(X, X) \geq -(n-1)|X|^2 + (n-1)\langle \alpha(X, X), \mathbf{H}\rangle, \tag{4.5}$$

for all $X \in \mathfrak{X}(M)$. Thus, from (4.3) and (4.5) we get that $T(a^\top, a^\top) \geq 0$. Moreover, we observe that the function $u = \mathrm{tr}(A^2_{a^N}) - \frac{1}{n}\mathrm{tr}(A_{a^N})^2$ is always nonnegative with $u = 0$ if, and only if, a^N is a umbilical direction. From (4.2), we obtain

$$\mathrm{div}\left[(A_{a^N} - \langle a, \mathbf{H}\rangle I)a^\top\right] \geq 0. \tag{4.6}$$

Thus, from (4.4) and (4.6), Lemma 1 assure us

$$\mathrm{tr}(A^2_{a^N}) - \frac{1}{n}\mathrm{tr}(A_{a^N})^2 + T(a^\top, a^\top) = 0.$$

Then, $\mathrm{tr}(A^2_{a^N}) - \frac{1}{n}\mathrm{tr}(A_{a^N})^2 = 0$ and, hence, a^N is a umbilical direction of M^n. Therefore, since we are supposing the existence of such vectors $a_1, \ldots, a_p \in \mathbb{R}^{n+p+1}_{p+1}$ whose normal projections $a_1^N, \ldots, a_p^N$ with respect to M^n are linearly independent, we conclude that (2.6) holds, that is, M^n must be totally umbilical. □

Proceeding, we establish sufficient conditions to guarantee that a spacelike submanifold immersed in $\mathbb{H}^{n+p}_p$ with nonzero parallel mean curvature vector must be pseudo-umbilical.

Theorem 2 *Let M^n be a complete spacelike submanifold immersed in $\mathbb{H}^{n+p}_p$ with nonzero parallel mean curvature vector $\mathbf{H}$ and bounded normalized scalar curvature R. If there exists a nonzero vector $a \in \mathbb{R}^{n+p+1}_{p+1}$ such that a^N is timelike, i.e., $\langle a^N, a^N\rangle < 0$, collinear to $\mathbf{H}$ and $|a^\top| \in \mathcal{L}^1(M)$, then M^n is pseudo-umbilical.*

Proof Initially, taking a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n, from (3.7) we get that the squared norm of second form fundamental α of M^n satisfies

$$|\alpha|^2 = \sum_{i,j} |\alpha(e_i, e_j)|^2 = n^2\langle \mathbf{H}, \mathbf{H}\rangle + n(n-1)(R+1). \tag{4.7}$$

Now, let us consider a nonzero vector $a \in \mathbb{R}^{n+p+1}_{p+1}$ such that a^N is timelike, collinear to $\mathbf{H}$ and with $|a^\top| \in \mathcal{L}^1(M)$. Since M^n has bounded normalized scalar curvature and nonzero parallel mean curvature vector $\mathbf{H}$, from (4.7) we conclude that $|\alpha|^2$ is bounded on M^n. So, taking $\xi = \mathbf{H}$ in (3.6) we get

$$\mathrm{div}(A_{\mathbf{H}}a^\top) = \mathrm{tr}(A_{a^N} \circ A_{\mathbf{H}}) + \langle a, x\rangle \mathrm{tr}(A_{\mathbf{H}}), \tag{4.8}$$

where $A_{\mathbf{H}}$ denotes the Weingarten operator associated to $\mathbf{H}$.

On the other hand, from (3.4) we have

$$\langle a, x\rangle = \frac{1}{n}\mathrm{div}(a^\top) - \langle a, \mathbf{H}\rangle. \tag{4.9}$$

Consequently, from (4.8) and (4.9)

$$\mathrm{div}(A_{\mathbf{H}}a^\top) = \mathrm{tr}(A_{a^N} \circ A_{\mathbf{H}}) + \mathrm{tr}(A_{\mathbf{H}})\frac{1}{n}\mathrm{div}(a^\top) - \frac{1}{n}\mathrm{tr}(A_{a^N})\mathrm{tr}(A_{\mathbf{H}}). \tag{4.10}$$

Since

$$\mathrm{div}\left(\mathrm{tr}(A_{\mathbf{H}})a^\top\right) = \mathrm{tr}(A_{\mathbf{H}})\mathrm{div}(a^\top), \tag{4.11}$$

from (4.10) and (4.11) we obtain

$$\mathrm{div}\, V = \mathrm{tr}(A_{a^N} \circ A_{\mathbf{H}}) - \frac{1}{n}\mathrm{tr}(A_{a^N})\mathrm{tr}(A_{\mathbf{H}}), \tag{4.12}$$

where V is a tangent vector field on M^n given by

$$V = \left(A_{\mathbf{H}} - \frac{1}{n}\mathrm{tr}(A_{\mathbf{H}})I \right) a^{\top}.$$

We note that, since we are supposing a^N timelike and collinear to $\mathbf{H}$, there exists on M^n a smooth function λ having strict sign such that $a^N = \lambda\mathbf{H}$. Thus, from (2.3) and (4.12) we get

$$\mathrm{div}\, V = \lambda \left(\mathrm{tr}(A_{\mathbf{H}}^2) - \frac{1}{n}\mathrm{tr}(A_{\mathbf{H}})^2 \right). \tag{4.13}$$

Consequently, from (4.13) we conclude that div V does not change sign on M^n. Moreover, we also have that

$$|V| \le (|A_{\mathbf{H}}| + |\langle \mathbf{H}, \mathbf{H} \rangle|)\, |a^{\top}| \in \mathcal{L}^1(M).$$

Hence, we can apply once more Lemma 1 to assure that div $V = 0$ on M^n.

Therefore, returning to (4.13) we obtain that

$$\lambda \left(\mathrm{tr}(A_{\mathbf{H}}^2) - \frac{1}{n}\mathrm{tr}(A_{\mathbf{H}})^2 \right) = 0,$$

which implies that $\mathbf{H}$ is an umbilical direction. □

We observe that, in the case $p = 1$, the notion of pseudo-umbilical coincides with that of totally umbilical. Moreover, we note that the hypothesis that a^N is timelike amounts to the support function $f_a = \langle a, \nu \rangle$ having strict sign on the spacelike hypersurface $M^n \hookrightarrow \mathbb{H}_1^{n+1}$, where ν stands for the Gauss mapping of M^n. Consequently, taking into account the classification of the totally umbilical hypersurfaces of $\mathbb{H}_1^{n+1}$ (see, for instance, Example 1 of [24]) and that Theorem 1 of [21] assures us that a complete constant mean curvature spacelike hypersurface of $\mathbb{H}_1^{n+1}$ must have bounded second fundamental form (or, equivalently, bounded normalized scalar curvature), from Theorem 2 we obtain the following

Corollary 3 *Let M^n be a complete spacelike hypersurface immersed in $\mathbb{H}_1^{n+1}$ with nonzero constant mean curvature. If there exists a nonzero vector $a \in \mathbb{R}_2^{n+2}$ such that the support function f_a has strict sign on M^n and $|a^{\top}| \in \mathcal{L}^1(M)$, then M^n is a totally umbilical hyperbolic space.*

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