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ORIGINAL PAPER

The weak Banach–Saks property for function spaces

Guillermo P. Curbera[1] · **Werner J. Ricker**[2]

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Abstract We establish the weak Banach–Saks property for function spaces arising as the optimal domain of an operator.

Keywords Weak Banach–Saks property · Subsequence splitting property · Banach function space · Optimal domain · Vector measure · Ultraproduct

Mathematics Subject Classification Primary 46E30 · 46B20 · 46G10; Secondary 46B08

1 Introduction

Astashkin and Maligranda proved that the Banach function space (B.f.s.)

$$Ces_p[0,1] := \left\{ f : x \mapsto \frac{1}{x}\int_0^x |f(y)|\,dy \in L^p([0,1]) \right\}, \quad 1 \le p < \infty,$$

has the *weak Banach–Saks property*, namely, every weakly null sequence in $Ces_p[0,1]$ admits a subsequence whose arithmetic means converge to zero in the norm of $Ces_p[0,1]$, [2, §7]. The space $L^p([0,1])$, for $1 \le p < \infty$, itself has the weak Banach–Saks property. This is due to Banach and Saks for $1 < p < \infty$ [3], and to Szlenk for $p = 1$ [25]. An important step in the proof of the above result in [2] is to first establish that $Ces_p[0,1]$ satisfies the

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✉ Guillermo P. Curbera
curbera@us.es

Werner J. Ricker
werner.ricker@ku.de

1 Facultad de Matemáticas and IMUS, Universidad de Sevilla, Aptdo. 1160, 41080 Sevilla, Spain

2 Math.-Geogr. Fakultät, Katholische Universität Eichstätt-Ingolstadt, 85072 Eichstätt, Germany

subsequence splitting property. This property goes back to a celebrated paper of Kadec and Pełczyński [16], where they observed that in $L^p([0,1])$, $1 \leq p < \infty$, every norm bounded sequence $\{f_n\}$ has a subsequence $\{f'_n\}$ that can be split in the form $f'_n = g_n + h_n$, where the functions $\{h_n\}$ have pairwise disjoint support and the sequence $\{g_n\}$ has uniformly absolutely continuous (a.c.) norm in $L^p([0,1])$, that is, $\sup_n \|g_n \chi_A\|_p \to 0$ when $\lambda(A) \to 0$, where λ is Lebesgue measure on [0, 1]. Characterizations of the subsequence splitting property (in terms of ultraproducts) are due to Weis [26]; they play a crucial role in Sect. 3.

The above results raise the question of whether the subsequence splitting property and the weak Banach–Saks property are also satisfied in analogous B.f.s.', such as, for example,

$$\{f \in L^1(G) : \nu * |f| \in L^p(G)\}, \quad 1 < p < \infty, \tag{1}$$

where ν is a positive, finite Borel measure on a compact abelian group G; or for

$$\left\{ f : [0,1] \to \mathbb{R} : I_\alpha(f)(x) := \int_0^1 \frac{|f(y)|}{|x-y|^{1-\alpha}}\, dy \in X \right\}, \tag{2}$$

where $0 < \alpha < 1$ and X is a rearrangement invariant (r.i.) space on [0, 1]; or for

$$\left\{ f : [0,1] \to \mathbb{R} : T(f)(x) := \int_x^1 y^{(1/n)-1} |f(y)|\, dy \in X \right\}, \tag{3}$$

where $n \geq 2$ and X is a r.i. space on [0, 1].

The common feature for these types of B.f.s.' is that each one is the *optimal extension domain* of an appropriate linear operator. Indeed, in the case of $Ces_p[0,1]$ this is so for the Cesàro operator

$$f \mapsto \mathcal{C}(f) : x \mapsto \frac{1}{x} \int_0^x f(y)\, dy;$$

see [2]; in the case of the B.f.s. in (1) for the operator of convolution with the measure ν, that is, for

$$f \mapsto T_\nu(f) = f * \nu : x \mapsto \int_G f(y-x)\, d\nu(y);$$

see [21]; in the case of the B.f.s. in (2) for the Riemann–Liouville fractional integral of order α, that is, for

$$f \mapsto I_\alpha(f) : x \mapsto \int_0^1 \frac{f(y)}{|x-y|^{1-\alpha}}\, dy;$$

see [6]; and in the case of the B.f.s. in (3) for the kernel operator associated to the n-dimensional Sobolev inequality, that is, for

$$f \mapsto T_n(f) : x \mapsto \int_x^1 y^{(1/n)-1} f(y)\, dy;$$

see [7,8,13].

Concerning the optimal domain of an operator, consider a finite measure space (Ω, Σ, μ), a Banach space X and an X-valued linear map T defined on an ideal $\mathcal{D} \subseteq L^0(\mu)$ which contains $L^\infty(\mu)$. Here $L^0(\mu)$ is the space of classes of all a.e. $\mathbb{R}$-valued, measurable functions defined on Ω. Then the optimal domain for T, taking its values in X, is the linear space defined by

$$[T, X] := \{f \in L^0(\mu) : T(|f|) \in X\},$$

which becomes a B.f.s. when endowed with the norm

$$\|f\|_{[T,X]} := \|T(|f|)\|_X, \quad f \in [T, X].$$

Note that $\mathcal{D} \subseteq [T, X]$. In this notation, we have $Ces_p = [\mathcal{C}, L^p([0, 1])]$, the B.f.s. in (1) is $[T_\nu, L^p(G)]$, the B.f.s. in (2) is $[I_\alpha, X]$, and the B.f.s. in (3) is $[T_n, X]$.

The aim of this paper is to extend the above mentioned results of Astashkin and Maligranda to the setting of operators other than the Cesàro operator and B.f.s.' other than $L^p([0, 1])$. For this we need to determine conditions on the Banach space X and on the operator T which guarantee that the space $[T, X]$ has the subsequence splitting property and the weak Banach–Saks property. This is achieved in Theorems 4, 5 and 6. The combination of these theorems leads to the following result.

Recall that a linear operator between function spaces is said to be *positive* if it maps positive functions to positive functions. If both function spaces are B.f.s.', then the operator is automatically continuous.

Theorem 1 *Let* (Ω, Σ, μ) *be a separable, finite measure space, X be a B.f.s. which possesses the weak Banach–Saks property and such that both X and X^* have the subsequence splitting property. Let* $T : \mathcal{D} \to X$ *be a positive, linear operator with* $L^\infty(\mu) \subseteq \mathcal{D} \subseteq L^0(\mu)$. *Then the B.f.s.* $[T, X]$ *has both the subsequence splitting property and the weak Banach–Saks property.*

Given a measurable function $K : (x, y) \in [0, 1] \times [0, 1] \mapsto K(x, y) \in [0, \infty]$, recall that the associated kernel operator T_K is defined by

$$T_K f(x) := \int_0^1 K(x, y) f(y)\, dy, \quad x \in [0, 1], \tag{4}$$

for any function $f \in L^0$ for which it is meaningful to do so, i.e., such that $T_K f \in L^0$.

As a consequence of Theorem 1 we have the following result.

Corollary 2 *Let X be a r.i. space on* $[0, 1]$ *which possesses the weak Banach–Saks property and such that both X and X^* have the subsequence splitting property. Let* $K : (x, y) \in [0, 1] \times [0, 1] \mapsto K(x, y) \in [0, \infty]$ *be a measurable kernel such that* $T_K(\chi_{[0,1]}) \in X$, *where T_K is as in* (4). *Then the B.f.s.* $[T_K, X]$ *has both the subsequence splitting property and the weak Banach–Saks property.*

In particular, the result holds whenever X is reflexive and possesses the weak Banach–Saks property.

From Corollary 2 it follows, for example, that the B.f.s $[I_\alpha, X]$ in (2) corresponding to the kernel $K(x, y) := |x - y|^{\alpha-1}$, the B.f.s. $[T_n, X]$ in (3) generated by the Sobolev kernel $K(x, y) = y^{(1/n)-1}\chi_{[x,1]}(y)$, and the Cesàro space $[\mathcal{C}, X]$ corresponding to the kernel $K(x, y) := (1/x)\chi_{[0,x]}(y)$, all have the subsequence splitting property and the weak Banach–Saks property, whenever X is a reflexive r.i. space with the weak Banach–Saks property. We refer to Sect. 5 for the details and further examples, also including convolution operators by measures.

A comment regarding the techniques is in order. There is a (somewhat unexpected, although classical) tool available for treating optimal domains in a unified way: there always exists an underlying vector measure associated with the operator together with its corresponding L^1-space consisting of all the scalar functions which are integrable with respect to that vector measure (in the sense of Bartle, Dunford and Schwartz). Accordingly, Theorems 4 and 5 are formulated for the subsequence splitting property and the weak Banach–Saks property for L^1-spaces of a general vector measure, respectively. For instance, in the case of the Cesàro operator, the associated $L^p([0,1])$-valued vector measure is given by

$$m_{L^p} : A \mapsto m_{L^p}(A) := \mathcal{C}(\chi_A), \quad A \subseteq [0,1] \text{ measurable.}$$

For this vector measure it turns out that $Ces_p[0,1] = L^1(m_{L^p})$.

2 Preliminaries

A *Banach function space* (B.f.s.) X over a measure space (Ω, Σ, μ) is a Banach space of classes of measurable functions on Ω satisfying the ideal property, that is, $g \in X$ and $\|g\|_X \leq \|f\|_X$ whenever $f \in X$ and $|g| \leq |f|$ μ-a.e. We denote by X^+ the cone in X consisting in all $f \in X$ satisfying $f \geq 0$ μ-a.e. The B.f.s. X has *absolutely continuous* (a.c.) norm if $\lim_{\mu(A)\to 0} \|f\chi_A\|_X = 0$ for $f \in X$; here χ_A denotes the characteristic function of a set $A \in \Sigma$. An equivalent condition is that order bounded, increasing sequences in X are norm convergent. A subset $K \subseteq X$ is said to have *uniformly a.c.* norm if $\lim_{\mu(A)\to 0} \sup_{f\in K} \|f\chi_A\|_X = 0$. Sets with uniform a.c. norm are also called almost order bounded sets or L-weakly compact sets. In B.f.s.' with a.c. norm, all relatively compact sets have uniform a.c. norm, and all sets with uniform a.c. norm are relatively weakly compact; see [20, §3.6].

A *rearrangement invariant* (r.i.) space X on [0, 1] is a B.f.s. on [0, 1] such that if $g^* \leq f^*$ and $f \in X$, then $g \in X$ and $\|g\|_X \leq \|f\|_X$. Here f^* is the decreasing rearrangement of f, that is, the right continuous inverse of its distribution function: $\mu_f(\tau) := \mu(\{t \in [0,1] : |f(t)| > \tau\})$. If a r.i. space X has a.c. norm, then the dual space X^* is again r.i. A r.i. space X always satisfies $L^\infty \subseteq X \subseteq L^1$.

We recall briefly the theory of integration of real functions with respect to a vector measure, due to Bartle et al. [4]. Let (Ω, Σ) be a measurable space, X be a Banach space with dual space X^* and closed unit ball B_{X^*}, and $m \colon \Sigma \to X$ be a σ-additive vector measure. The *semivariation* of m is defined by

$$A \mapsto \|m\|(A) := \sup\{|x^*m|(A) : x^* \in B_{X^*}\}, \quad A \in \Sigma,$$

where $|x^*m|$ is the variation measure of the scalar measure $x^*m : A \mapsto \langle x^*, m(A)\rangle$ for $A \in \Sigma$. A *Rybakov control measure* for m is a measure of the form $\eta = |x_0^*m|$ for a suitable element $x_0^* \in X^*$ such that $\eta(A) = 0$ if and only if $\|m\|(A) = 0$; see [12, Theorem IX.2.2].

A measurable function $f \colon \Omega \to \mathbb{R}$ is called *m-scalarly integrable* if $f \in L^1(|x^*m|)$, for every $x^* \in X^*$. The function f is *m-integrable* if, in addition, for each $A \in \Sigma$ there exists a vector in X (denoted by $\int_A f\,dm$) such that $\langle \int_A f\,dm, x^*\rangle = \int_A f\,dx^*m$, for every $x^* \in X^*$. The m-integrable functions form a linear space in which

$$\|f\|_{L^1(m)} := \sup\left\{\int_\Omega |f|\,d|x^*m| : x^* \in B_{X^*}\right\}$$

is a seminorm. Identifying functions which differ $\|m\|$-a.e., we obtain a Banach space (of classes) of m-integrable functions, denoted by $L^1(m)$. It is a B.f.s. over (Ω, Σ, η) relative to any Rybakov control measure η for m. Simple functions are dense in $L^1(m)$, the m-essentially bounded functions are contained in $L^1(m)$, and $L^1(m)$ has a.c. norm. This last property implies that $L^1(m)^*$ can be identified with its associate space, that is, with the space of all measurable functions g satisfying $fg \in L^1(\eta)$ for all $f \in L^1(m)$; the identification is given by $f \in L^1(m) \mapsto \int_\Omega fg\,d\eta \in \mathbb{R}$. In particular, $L^\infty \subseteq L^1(m)^*$. An equivalent norm for $L^1(m)$ is given by $|||f||| := \sup\{\| \int_A f\,dm\|_X : A \in \Sigma\}$, which satisfies $|||f||| \leq ||f||_{L^1(m)} \leq 2|||f|||$ for $f \in L^1(m)$. The integration operator $I_m : L^1(m) \to X$ is defined by $f \mapsto \int_\Omega f\,dm$. It is continuous, linear and has operator norm one. It should be noted that the spaces $L^1(m)$ can be quite different to the classical L^1-spaces of scalar measures. Indeed, every Banach lattice with a.c. norm and having a weak unit (e.g., $L^2([0, 1])$) is the L^1-space of some vector measure [5, Theorem 8].

For further details concerning B.f.s.' and r.i. spaces we refer to [19]. For further facts related to the spaces $L^1(m)$ see [23].

The following result is implicit in the construction of the Bartle, Dunford, Schwartz integral (cf. the proof of Theorem 2.6(a) of [4]), although it is not explicitly stated; see also [23, Theorem 3.5]. We include a proof for the sake of completeness.

Lemma 3 *Let $\{f_n\} \subseteq L^1(m)$ be a sequence satisfying*

(a) *$f_n(x) \to f(x)$ for $||m||$-a.e. $x \in \Omega$, and*
(b) *$\{\int_A f_n\,dm\}$ is convergent in X, for each $A \in \Sigma$.*

Then, $f \in L^1(m)$ and $\{f_n\}$ converges to f in the norm of $L^1(m)$.

Proof For each $f_n \in L^1(m)$, $n \in \mathbb{N}$, the set function $A \mapsto \int_A f_n\,dm \in X$, $A \in \Sigma$, is a σ-additive measure (due to the Orlicz–Pettis Theorem), which is absolutely continuous with respect to a control measure for m. This fact, together with (b) implies, via the Vitali–Hahn–Saks Theorem [12, Theorem I.5.6], that the convergence in (b) is uniform with respect to the sets $A \in \Sigma$. Accordingly, $f \in L^1(m)$ [4, Theorem 2.7]. The convergence $f_n \to f$ in norm in $L^1(m)$ follows directly by considering the equivalent norm $||| \cdot |||$ in $L^1(m)$. □

3 The subsequence splitting property for $L^1(m)$

In [26, 2.1 Definition] Weis gives a general definition of the subsequence splitting property. Let X be a B.f.s. with a.c. norm defined over a measure space (S, σ, μ). Then X has the *subsequence splitting property* if for every norm bounded sequence $\{f_n\} \subseteq X$ there is a subsequence $\{f_n'\}$ of $\{f_n\}$ and sequences $\{g_n\}$ and $\{h_n\}$ in X such that:

(a) For $n \in \mathbb{N}$ we have $f_n' = g_n + h_n$, with g_n and h_n having disjoint support.
(b) The sequence $\{g_n\}$ has uniformly a.c. norm in X.
(c) The functions $\{h_n\}$ have pairwise disjoint support.

Weis gives several characterizations of the subsequence splitting property, [26, 2.5 Theorem]. We select only those which are required in the sequel. Namely,

(i) X has the subsequence splitting property,
(ii) $\tilde{X}$ has a.c. norm,
(iii) $\tilde{X}$ does not contain a copy of c_0,

where the space $\tilde{X}$ is constructed as follows; see [26]. Let $\mathcal{U}$ be a free ultrafilter in $\mathbb{N}$. Consider the ultraproduct of X via $\mathcal{U}$ given by the quotient

$$X_{\mathcal{U}} := \ell_\infty(X)\big/N_{\mathcal{U}}, \quad \text{where } N_{\mathcal{U}} = \left\{\{f_n\} \in \ell_\infty(X) : \lim_{\mathcal{U}} \|f_n\|_X = 0\right\}$$

and $\ell_\infty(X)$ is the space of all bounded sequences in X. Denote by $[f_n] \in X_{\mathcal{U}}$ the equivalence class of the element $\{f_n\} \in \ell_\infty(X)$. The space $X_{\mathcal{U}}$ becomes a Banach lattice for the following norm and order:

$$\|[f_n]\|_{\mathcal{U}} := \lim_{\mathcal{U}} \|f_n\|_X, \quad \inf\{[f_n],[g_n]\} := [\inf\{f_n, g_n\}].$$

For details on ultraproducts of Banach spaces see [15]. Let χ_S be the characteristic function of the underlying set S. Then, $[\chi_S]$ is the equivalence class of the constant sequence $\{\chi_S\}$. Let $\tilde{X}$ denote the band in $X_{\mathcal{U}}$ generated by $[\chi_S]$, that is, $\tilde{X} = [\chi_S]^{\perp\perp}$. Recall that a band M in a Banach lattice Z is a closed subspace which is an order ideal (i.e., $f \in M$ and $g \in Z$ with $|g| \le |f|$ imply $g \in M$) and is closed under the formation of suprema [19, p. 3].

Theorem 4 *Let X be a B.f.s. and $m\colon \Sigma \to X$ be a σ-additive vector measure. The following conditions are assumed to hold.*

(a) *X and X^* have the subsequence splitting property.*
(b) *The range $m(\Sigma)$ of m has uniformly a.c. norm in X.*

Then, the B.f.s. $L^1(m)$ has the subsequence splitting property.

Proof Recall, since X has the subsequence splitting property, that it has a.c. norm. In order to prove the result we construct an $\tilde{X}$-valued σ-additive measure $\tilde{m}$ with the property that $(L^1(m))\tilde{}$ is order isomorphically contained in the B.f.s. $L^1(\tilde{m})$. A general result asserts that every L^1-space of a vector measure has a.c. norm [5, Theorem 1], and hence, $(L^1(m))\tilde{}$ has a.c. norm. Then, by the characterization (ii) recorded above, it follows that $L^1(m)$ has the subsequence splitting property.

Let η be a Rybakov control measure for m. Then, with continuous inclusions, we have

$$L^\infty(\Omega,\Sigma,\eta) \subseteq L^1(m) \subseteq L^1(\Omega,\Sigma,\eta). \tag{5}$$

Fix a free ultrafilter $\mathcal{U}$ in $\mathbb{N}$. Then the ultraproduct of $L^1(\Omega,\Sigma,\eta)$ via $\mathcal{U}$ can be identified as

$$L^1(\Omega,\Sigma,\eta)_{\mathcal{U}} = L^1(\tilde{\Omega},\tilde{\Sigma},\tilde{\eta}) \oplus \Delta',$$

where $(\tilde{\Omega},\tilde{\Sigma},\tilde{\eta})$ is a measure space and the elements of Δ' are disjoint from $[\chi_\Omega]$; see [10, §4], [11], [14, §3]. Thus, it follows that

$$(L^1(\Omega,\Sigma,\eta))\tilde{} = L^1(\tilde{\Omega},\tilde{\Sigma},\tilde{\eta}).$$

The same procedure can be done with $L^\infty(\Omega,\Sigma,\eta)$. This allows the identification of $(L^1(m))\tilde{}$ with a function space by forming the ultraproducts of the inclusions in (5), namely

$$L^\infty(\tilde{\Omega},\tilde{\Sigma},\tilde{\eta}) \subseteq (L^1(m))\tilde{} \subseteq L^1(\tilde{\Omega},\tilde{\Sigma},\tilde{\eta}),$$

with both inclusions being continuous.

The σ-algebra $\tilde{\Sigma}$ is isomorphic to the Boolean ring $\{[\chi_{A_n}] : A_n \in \Sigma\}$ formed in the quotient space $L^1(\Omega,\Sigma,\lambda)_{\mathcal{U}}$. Thus, every measurable set $\tilde{A} \in \tilde{\Sigma}$ can be identified with a sequence of sets $\{A_n\}$ with each $A_n \in \Sigma$, where two sequences of measurable sets $\{A_n\}$ and $\{B_n\}$ are identified if $\lim_{\mathcal{U}} \eta(A_n \triangle B_n) = 0$. Here $A \triangle B$ denotes the symmetric difference of two sets A and B. The measure $\tilde{\eta}$ is then defined via

$$\tilde{A} = \{A_n\} \in \tilde{\Sigma} \mapsto \tilde{\eta}(\tilde{A}) := \lim_{\mathcal{U}} \eta(A_n) \in \mathbb{R}^+.$$

A function $\tilde{f}$ in $L^1(\tilde{\Omega},\tilde{\Sigma},\tilde{\eta})$ is an element $[f_n]$ in $L^1(\Omega,\Sigma,\eta)_{\mathcal{U}}$, and the integral of $\tilde{f}$ over measurable sets with respect to $\tilde{\eta}$ is defined as

$$\int_{\tilde{A}} \tilde{f}\, d\tilde{\eta} = \lim_{\mathcal{U}} \int_{A_n} f_n\, d\eta, \quad \tilde{A} = \{A_n\} \in \tilde{\Sigma}.$$

For further details, see [15, §5].

We define a vector measure $\tilde{m}$ by

$$\tilde{A} = \{A_n\} \in \tilde{\Sigma} \longmapsto \tilde{m}(\tilde{A}) = [m(A_n)] \in X_{\mathcal{U}}.$$

As $m(\Sigma)$ is a bounded subset of X it is clear that $\tilde{m}$ is well defined. Moreover, $\tilde{m}$ is finitely additive [10, p. 322]. To verify its σ-additivity, let $\varepsilon > 0$. As m is absolutely continuous with respect to η, there exists $\delta > 0$ such that if $\eta(A) < \delta$, then $\|m\|(A) < \varepsilon$. Let $\tilde{A} = \{A_n\} \in \tilde{\Sigma}$ satisfy $\tilde{\eta}(\tilde{A}) < \delta$, that is, $\lim_{\mathcal{U}} \eta(A_n) < \delta$. Then there exists $V \in \mathcal{U}$ such that for every $n \in V$ we have $\eta(A_n) < \delta$. Thus, for every $n \in V$ it follows that $\|m(A_n)\| \leq \|m\|(A_n) < \varepsilon$. So, $\|\tilde{m}(\tilde{A})\|_{\mathcal{U}} < \varepsilon$. Hence, $\tilde{m}$ is absolutely continuous with respect to $\tilde{\eta}$ from which we deduce that $\tilde{m}$ is σ-additive.

By hypothesis, the range $m(\Sigma)$ of the measure m has uniformly a.c. norm in X. In order to show that the measure $\tilde{m}$ actually takes its values in $\tilde{X} \subseteq X_{\mathcal{U}}$ we use [26, 1.5 Proposition] which asserts that if $\{f_n\}$ has uniformly a.c. norm in X, then $[f_n] \in \tilde{X}$. Let $\tilde{A} = \{A_n\} \in \tilde{\Sigma}$. Then $\tilde{m}(\tilde{A}) = [m(A_n)] \in X_{\mathcal{U}}$. But, $\{m(A_n)\} \subseteq m(\Sigma)$ which has uniformly a.c. norm. Hence, $\tilde{m}(\tilde{A}) = [m(A_n)] \in \tilde{X}$.

Next, we prove that $(L^1(m))^{\sim}$ is contained in $L^1(\tilde{m})$. To this aim, it suffices to show that each $\tilde{f} \in (L^1(m))^{\sim}$ is scalarly $\tilde{m}$-integrable. The reason for this is two-fold. On the one hand, $\tilde{X}$ does not contain a copy of c_0 since X satisfies the subsequence splitting property; see (iii) above. On the other hand, for vector measures with values in a Banach space not containing c_0, integrability and scalar integrability coincide [18, Theorem 5.1].

Since X and X^* satisfy the subsequence splitting property, we have $(\tilde{X})^* = (X^*)^{\sim}$ and the norms in both spaces coincide [26, Corollary 2.7]. Hence, the elements of $(\tilde{X})^*$ are of the form $\tilde{g}^* = [g_n^*]$ for $\{g_n^*\}$ a bounded sequence in X^*.

Fix $\tilde{g}^* \in (\tilde{X})^*$. The scalar measure $\tilde{g}^*\tilde{m} : \tilde{\Sigma} \to \mathbb{R}$ is absolutely continuous with respect to $\tilde{\eta}$ (since $\tilde{m}$ is absolutely continuous with respect to $\tilde{\eta}$). Thus, $\tilde{g}^*\tilde{m}$ has a Radon–Nikodym derivative with respect to $\tilde{\eta}$. We denote it by $h_{\tilde{g}^*}$; it belongs to $L^1(\tilde{\eta})$.

Let $\tilde{A} = \{A_n\} \in \tilde{\Sigma}$. Then,

$$\begin{aligned} \left\langle \tilde{g}^*, \tilde{m}(\tilde{A}) \right\rangle &= \left\langle [g_n^*], [m(A_n)] \right\rangle = \lim_{\mathcal{U}} \left\langle g_n^*, m(A_n) \right\rangle \\ &= \lim_{\mathcal{U}} \int_{A_n} 1\, d(g_n^* m) = \lim_{\mathcal{U}} \int_{A_n} h_{g_n^*}\, d\eta = \int_{\tilde{A}} \tilde{h}\, d\tilde{\eta}, \end{aligned}$$

where $\tilde{h} := [h_{g_n^*}]$ and $h_{g_n^*} \in L^1(\eta)$ is the Radon–Nikodym derivative of the measure $g_n^* m$ with respect to η, for each $n \in \mathbb{N}$. Hence, $h_{\tilde{g}^*} = [h_{g_n^*}]$.

Let now $\tilde{f} \in (L^1(m))^{\sim}$. Then $\tilde{f} = [f_n]$ for a bounded sequence $\{f_n\}$ in $L^1(m)$, with $\|\tilde{f}\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|f_n\|_{L^1(m)}$. Accordingly,

$$\begin{aligned} \int |\tilde{f}|\, d|\tilde{g}^*\tilde{m}| &= \int |\tilde{f}| \cdot |\tilde{h}_{\tilde{g}^*}|\, d\,\tilde{\eta} = \lim_{\mathcal{U}} \int |f_n| \cdot |h_{g_n^*}|\, d\eta \\ &= \lim_{\mathcal{U}} \int |f_n|\, d|g_n^* m| \leq \lim_{\mathcal{U}} \|f_n\|_{L^1(m)} \cdot \|g_n^*\|_{X^*} \\ &= \|\tilde{f}\|_{\mathcal{U}} \cdot \|\tilde{g}^*\|_{\mathcal{U}}. \end{aligned}$$

Hence, $\tilde{f}$ is integrable with respect to $\tilde{g}^*\tilde{m}$. It follows that $\tilde{f}$ is scalarly $\tilde{m}$-integrable and hence, integrable with respect to the vector measure $\tilde{m}$. We also deduce from the previous inequality that

$$\|\tilde{f}\|_{L^1(\tilde{m})} \le \|\tilde{f}\|_{\mathcal{U}}, \quad \tilde{f} \in (L^1(m))\tilde{}\,.$$

Let $\varepsilon > 0$. By using the equivalent norm $|||\cdot|||$ in $L^1(m)$, we can select for every $n \in \mathbb{N}$, a measurable set A_n such that

$$\left\| \int_{A_n} f_n \, dm \right\|_X \ge \frac{1-\varepsilon}{2} \|f_n\|_{L^1(m)}.$$

Set $\tilde{A} := \{A_n\}$ in $\tilde{\Sigma}$. Then

$$\begin{aligned} \|\tilde{f}\|_{L^1(\tilde{m})} &\ge \left\| \int_{\tilde{A}} \tilde{f} d\tilde{m} \right\|_{\tilde{X}} = \lim_{\mathcal{U}} \left\| \int_{A_n} f_n \, dm \right\|_X \\ &\ge \frac{1-\varepsilon}{2} \lim_{\mathcal{U}} \|f_n\|_{L^1(m)} = \frac{1-\varepsilon}{2} \|\tilde{f}\|_{\mathcal{U}}. \end{aligned}$$

Thus, the norm of $(L^1(m))\tilde{}$ and the norm of $L^1(\tilde{m})$ are equivalent on $L^1(m)\tilde{}$. Hence, $(L^1(m))\tilde{}$ is order isomorphic to a subspace of $L^1(\tilde{m})$ which completes the proof. □

Well known examples of B.f.s.' satisfying the subsequence splitting property include those Orlicz spaces satisfying the Δ_2 condition, q-concave B.f.s.' for $q < \infty$, and r.i. spaces not containing c_0 [26].

4 The weak Banach–Saks property for $L^1(m)$

In the following result we require the vector measure $m : \Sigma \to X$ to be *separable*. In analogy to the scalar case, this means that the associated pseudometric space (Σ, d_m) is separable, that is, it contains a countable dense subset. The pseudometric d_m is given by

$$d_m(A, B) := \|m\|(A \triangle B), \quad A, B \in \Sigma,$$

where $\|m\|(\cdot)$ is the semivariation of m. For η a Rybakov control measure for m (see Sect. 2), due to the mutual absolute continuity between $\eta(\cdot)$ and $\|m\|(\cdot)$, this it is equivalent to the pseudometric space (Σ, d_η) being separable, where $d_\eta(A, B) := \eta(A \triangle B)$, for $A, B \in \Sigma$. We point out that m is separable precisely when the B.f.s. $L^1(m)$ is separable [24].

Theorem 5 *Let X be a B.f.s. and $m : \Sigma \to X$ be a σ-additive vector measure. The following conditions are assumed to hold.*

(a) *X has the weak Banach–Saks property.*
(b) *The measure m is separable and positive, i.e., $m(A) \in X^+$ for $A \in \Sigma$.*
(c) *$L^1(m)$ has the subsequence splitting property.*

Then, the B.f.s. $L^1(m)$ has the weak Banach–Saks property.

Proof We need to verify, for a given weakly null sequence $\{f_n\} \subseteq L^1(m)$, that there exists a subsequence $\{f_n'''\} \subseteq \{f_n\}$ whose arithmetic means converge to zero in the norm of $L^1(m)$, that is,

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} f_k''' \right\|_{L^1(m)} = 0.$$

The proof will be carried out in several steps.

Step 1 An important observation, which Szlenk credits to Pełczyński [25, Remarque 1], is that the weak Banach–Saks property for a Banach space Z is equivalent to the following (a priori stronger) property: for every weakly null sequence $\{z_n\} \subseteq Z$ there exists a subsequence $\{z'_n\} \subseteq \{z_n\}$ satisfying

$$\lim_{m\to\infty} \sup_{n_1<n_2<\cdots<n_m} \left\| \frac{1}{m} \sum_{k=1}^{m} z'_{n_k} \right\|_Z = 0. \tag{6}$$

It is to be remarked that this condition is a technical improvement: any further subsequence extracted from $\{z'_n\}$ again satisfies (6), for that new subsequence.

Step 2 Let $f_n \to 0$ weakly in $L^1(m)$. Then, $\{f_n\}$ is a bounded sequence in $L^1(m)$. Since $L^1(m)$ has the subsequence splitting property, there is a subsequence $\{f'_n\} \subseteq \{f_n\}$ and sequences $\{g_n\}$ and $\{h_n\}$ in $L^1(m)$ such that

(a) $f'_n = g_n + h_n$, with g_n and h_n having disjoint support, $n \in \mathbb{N}$.
(b) $\{g_n\}$ has uniformly a.c. norm in $L^1(m)$.
(c) $\{h_n\}$ have pairwise disjoint support.

Since $f_n \to 0$ weakly in $L^1(m)$, also $f'_n \to 0$ weakly in $L^1(m)$. The claim is that (a), (b), (c) imply that both $g_n \to 0$ weakly in $L^1(m)$ and $h_n \to 0$ weakly in $L^1(m)$.

To establish this claim, recall that sets of functions having uniformly a.c. norm are relatively weakly compact (see Sect. 2). Thus, from (b), the set $\{g_n : n \in \mathbb{N}\}$ is a relatively weakly compact set in $L^1(m)$. By the Eberlein–Šmulian Theorem, there is a subsequence $\{g_{n_k}\}$ and $g \in L^1(m)$ such that $g_{n_k} \to g$ weakly in $L^1(m)$. Since $f_{n_k} \to 0$ weakly in $L^1(m)$, it follows that $h_{n_k} \to (-g)$ weakly in $L^1(m)$. Let D_k denote the support of h_{n_k}; from (c) the sets D_k, $k \in \mathbb{N}$, are pairwise disjoint. Set $E := \cup_1^\infty D_k$ and $E_j := \cup_1^j D_k$. Since $L^\infty \subseteq L^1(m)^*$, we have $\chi_A \in L^1(m)^*$ for every $A \in \Sigma$. Let $A \in \Sigma$ with $A \subseteq E^c$. Then, $\langle \chi_A, h_{n_k} \rangle \to \langle \chi_A, (-g) \rangle$. But, $\langle \chi_A, h_{n_k} \rangle = 0$ for all $k \geq 1$ and so $g = 0$ a.e. on E^c. Fix $j \in \mathbb{N}$. For any $A \in \Sigma$ with $A \subseteq E_j$ we have $\langle \chi_A, h_{n_k} \rangle \to \langle \chi_A, (-g) \rangle$. But, $\langle \chi_A, h_{n_k} \rangle = 0$ for all $k > j$ and so $g = 0$ a.e. on E_j. Since this occurs for all $j \in \mathbb{N}$, it follows that $g = 0$ a.e. on E. Consequently, $g = 0$ a.e. and so $g_{n_k} \to 0$ weakly. This argument shows that the sequence $\{g_n\}$ has the property that, for each of its subsequences, there is a further subsequence which converges weakly to zero. This implies that the original sequence $g_n \to 0$ weakly. Consequently, also $h_n \to 0$ weakly.

Step 3 Consider the functions $\{h_n\} \subseteq L^1(m)$ from Step 2. They have pairwise disjoint support. Let B_n be the support of h_n, for $n \in \mathbb{N}$, and B be the complement of $\cup_n B_n$. Define

$$F := \chi_B + \sum_{n=1}^{\infty} \operatorname{sign}(h_n) \chi_{B_n},$$

where $\operatorname{sign}(h_n) = h_n/|h_n|$ on B_n. The function F is measurable and satisfies $|F| \equiv 1$. The operator $f \in L^1(m) \mapsto fF \in L^1(m)$ of multiplication by F is a linear isometric isomorphism on $L^1(m)$. Since $h_n \to 0$ weakly in $L^1(m)$ and $h_n F = |h_n|$, for $n \in \mathbb{N}$, it follows that $|h_n| \to 0$ weakly in $L^1(m)$.

Due to the continuity of the integration operator, it follows that $\int_\Omega |h_n|\, dm \to 0$ weakly in X. Since X has the weak Banach–Saks property, there exists a subsequence $\{h'_n\} \subseteq \{h_n\}$ such that

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} \int_\Omega |h'_k|\, dm \right\|_X = 0. \tag{7}$$

Due to the fact that the vector measure m is positive we have

$$\|f\|_{L^1(m)} = \|\,|f|\,\|_{L^1(m)} = \left\| \int_\Omega |f|\,dm \right\|_X, \quad f \in L^1(m),$$

[23, Theorem 3.13]. This, together with the fact (due to the supports of the functions h'_n, $n \in \mathbb{N}$, being disjoint) that $\sum_{k=1}^n |h'_k| = |\sum_{k=1}^n h'_k|$ for $n \in \mathbb{N}$ implies, from (7), that

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^n h'_k \, dm \right\|_{L^1(m)} = \lim_{n\to\infty} \left\| \int_\Omega \left(\frac{1}{n} \sum_{k=1}^n |h'_k| \right) dm \right\|_X = 0. \tag{8}$$

Note, in view of Step 1, that the above conclusion still holds if we replace $\{h'_n\}$ by any subsequence $\{h''_n\} \subset \{h'_n\}$.

Step 4 Consider now the functions $\{g_n\} \subseteq L^1(m)$ from Step 2. Let $\{g'_n\}$ be the subsequence of $\{g_n\}$ corresponding to the subsequence $\{h'_n\} \subseteq \{h_n\}$ from Step 3. Since $g_n \to 0$ weakly in $L^1(m)$, also $g'_n \to 0$ weakly in $L^1(m)$.

Let η be a Rybakov control measure for m. Since $L^1(m) \subseteq L^1(\eta)$ continuously and $g'_n \to 0$ weakly in $L^1(m)$, we have that $g'_n \to 0$ weakly in $L^1(\eta)$. Due to the well known Komlós theorem [17, Theorem 1a], applied in $L^1(\eta)$ to the norm bounded sequence $\{g'_n\}$, there exists a subsequence $\{g''_n\} \subseteq \{g'_n\}$ and a function $g_0 \in L^1(\eta)$ such that, for every further subsequence $\{g'''_n\} \subseteq \{g''_n\}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n g'''_k(x) \to g_0(x), \quad \eta\text{-}a.e.$$

Since $g'_n \to 0$ weakly in $L^1(\eta)$, also $g''_n \to 0$ weakly in $L^1(\eta)$ and so its averages $\frac{1}{n}\sum_{k=1}^n g''_k \to 0$ weakly in $L^1(\eta)$. Set $\xi_n := \frac{1}{n}\sum_{k=1}^n g''_k \in L^1(\eta)$. Then $\xi_n \to 0$ weakly in $L^1(\eta)$ and $\xi_n \to g_0$ η-a.e. Combining the Egorov theorem and the Dunford-Pettis criterion for relative weak compactness in $L^1(\eta)$ [1, Theorem 5.2.9], we deduce that $\xi_n \to g_0$ for the norm in $L^1(\eta)$ and so $g_0 = 0$.

Consequently, we have selected a subsequence $\{g''_n\} \subseteq \{g'_n\}$ with the property that, for every subsequence $\{g'''_n\} \subseteq \{g''_n\}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n g'''_k(x) \to 0, \quad m\text{-}a.e. \tag{9}$$

Step 5 Due to the separability of the measure m, there exists a sequence $\{A_n\} \subset \Sigma$ which is dense in the pseudometric space (Σ, d_η).

We start a diagonalization process. For notational convenience, let

$$I_m(f, A) := \int_A f\,dm, \quad f \in L^1(m),\ A \in \Sigma.$$

Define $g_n^{(1)} := g''_n$, $n \in \mathbb{N}$, where $\{g''_n\}$ is the sequence obtained in Step 4. Since $g_n^{(1)} \to 0$ weakly in $L^1(m)$ and the operator of integration over A_1, namely

$$f \in L^1(m) \mapsto I_m(f, A_1) = \int_{A_1} f\,dm \in X$$

is continuous, it follows that $I_m(g_n^{(1)}, A_1) \to 0$ weakly in X. But, X has the weak Banach–Saks property and so there exists a subsequence of $\{I_m(g_n^{(1)}, A_1)\}$ satisfying the condition

(6) in X. We denote that subsequence by $\{I_m(g_n^{(2)}, A_1)\}$. In this way we have also selected a subsequence $\{g_n^{(2)}\} \subseteq \{g_n^{(1)}\}$.

Next we apply the same procedure to the subsequence $\{g_n^{(2)}\}$ and the set A_2 as follows. Since $g_n^{(2)} \to 0$ weakly in $L^1(m)$ and the operator of integration over A_2, i.e.,

$$f \in L^1(m) \mapsto I_m(f, A_2) = \int_{A_2} f\, dm \in X$$

is continuous, it follows that $I_m(g_n^{(2)}, A_2) \to 0$ weakly in X. But, X has the weak Banach–Saks property and so there exists a subsequence of $\{I_m(g_n^{(2)}, A_2)\}$ satisfying the condition (6) in X. We denote that subsequence by $\{I_m(g_n^{(3)}, A_2)\}$. In this way we have selected a subsequence $\{g_n^{(3)}\} \subseteq \{g_n^{(2)}\}$. Note, from Step 1, that $\{I_m(g_n^{(3)}, A_1)\}$ also satisfies the condition (6) in X.

For the general step, consider the subsequence $\{g_n^{(k)}\} \subseteq \{g_n^{(k-1)}\}$. Since $g_n^{(k)} \to 0$ weakly in $L^1(m)$ and the operator of integration over A_k, i.e.,

$$f \in L^1(m) \mapsto I_m(f, A_k) = \int_{A_k} f\, dm \in X$$

is continuous, it follows that $I_m(g_n^{(k)}, A_k) \to 0$ weakly in X. But, X has the weak Banach–Saks property and so there exists a subsequence of $\{I_m(g_n^{(k)}, A_k)\}$ satisfying the condition (6). We denote that subsequence by $\{I_m(g_n^{(k+1)}, A_k)\}$. In this way we have also selected a subsequence $\{g_n^{(k+1)}\} \subseteq \{g_n^{(k)}\}$. Note, from Step 1, that also $\{I_m(g_n^{(k+1)}, A_j)\}$ satisfies the condition (6) in X for all $1 \leq j \leq k$.

By defining $g_n''' := g_n^{(n)}$, $n \in \mathbb{N}$, we obtain a subsequence $\{g_n'''\} \subseteq \{g_n''\}$ satisfying

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} \int_{A_j} g_k'''\, dm \right\|_X = 0, \quad j = 1, 2, \ldots. \tag{10}$$

Set

$$F_n := \frac{1}{n} \sum_{k=1}^{n} g_k''', \quad n = 1, 2, \ldots.$$

Then, $\{F_n\} \subseteq L^1(m)$ and we can write (10) as

$$\lim_{n\to\infty} \left\| \int_{A_j} F_n\, dm \right\|_X = 0, \quad j = 1, 2, \ldots. \tag{11}$$

Step 6 Since the functions $\{g_n\}$ have uniformly a.c. norm in $L^1(m)$, also the functions $\{g_n'''\} \subseteq \{g_n\}$ have uniformly a.c. norm in $L^1(m)$. Recall that $L^1(m)$ is a B.f.s. over the finite measure space (Ω, Σ, η), where η is the Rybakov control measure in Step 4. The uniform a.c. of the norm of $\{g_n'''\}$ in $L^1(m)$ implies that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\eta(A) < \delta \Rightarrow \sup_n \left\| g_n''' \chi_A \right\|_{L^1(m)} < \varepsilon. \tag{12}$$

Our next objective is to extend the validity of (11) to an arbitrary measurable set $A \in \Sigma$. So, fix $A \in \Sigma$ and let $\epsilon > 0$. Select $\delta > 0$ to satisfy (12). Due to the separability of (Σ, d_η)

there exists $j \in \mathbb{N}$ such that $\eta(A \triangle A_j) < \delta$. Then,

$$\begin{aligned}\left\| \int_A F_n \, dm \right\|_X &\le \left\| \int_A F_n \, dm - \int_{A_j} F_n \, dm \right\|_X + \left\| \int_{A_j} F_n \, dm \right\|_X \\ &\le \frac{1}{n} \sum_{k=1}^{n} \left\| \int_A g_k''' \, dm - \int_{A_j} g_k''' \, dm \right\|_X + \left\| \int_{A_j} F_n \, dm \right\|_X \\ &\le \frac{1}{n} \sum_{k=1}^{n} \| g_k''' \chi_{A \triangle A_j} \|_{L^1(m)} + \left\| \int_{A_j} F_n \, dm \right\|_X \\ &\le \varepsilon + \left\| \int_{A_j} F_n \, dm \right\|_X ,\end{aligned}$$

where we have used $|\chi_{A \setminus A_j} g_k'''| \le |\chi_{A \triangle A_j} g_k'''|$ and $\| \int_\Omega g \, dm \|_X \le \|g\|_{L^1(m)} = \| \, |g| \, \|_{L^1(m)}$ for $g \in L^1(m)$. Due to (11), the last term can be made smaller than ε for $n \ge n_0$ and some $n_0 \in \mathbb{N}$. Hence,

$$\lim_{n \to \infty} \left\| \int_A F_n \, dm \right\|_X \to 0, \quad A \in \Sigma.$$

Note that $\{g_n'''\} \subseteq \{g_n''\}$ implies, from (9), that $F_n \to 0$ a.e. Consequently, we have a sequence $\{F_n\}$ in $L^1(m)$ such that $F_n \to 0$ a.e. and $\int_A F_n \, dm \to 0$ in X, for each $A \in \Sigma$. These two conditions, via Lemma 3, imply that $F_n \to 0$ in $L^1(m)$, that is

$$\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=1}^{n} g_k''' \right\|_{L^1(m)} = 0. \tag{13}$$

Step 7 Let $\{h_n'''\}$ be the subsequence of $\{h_n\}$ corresponding to the subsequence $\{g_n'''\}$ of $\{g_n\}$ from Step 6. For the subsequence $f_n''' = g_n''' + h_n'''$, $n \in \mathbb{N}$, of $\{f_n\}$ it follows from (8) and (13) that

$$\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=1}^{n} f_k''' \, dm \right\|_{L^1(m)} = 0.$$

This completes the proof. □

The combination of Theorems 4 and 5 renders the following result.

Theorem 6 *Let X be a B.f.s. and $m \colon \Sigma \to X$ be a σ-additive vector measure. The following conditions are assumed to hold.*

(a) *X has the weak Banach–Saks property.*
(b) *X and X^* have the subsequence splitting property.*
(c) *The measure m is separable and positive, i.e., $m(A) \in X^+$ for $A \in \Sigma$.*
(d) *The range $m(\Sigma)$ of m has uniformly a.c. norm in X.*

Then, the B.f.s. $L^1(m)$ has both the subsequence splitting property and the weak Banach–Saks property.

We now turn to the

Proof of Theorem 1 We will deduce Theorem 1 from Theorem 6. We first define the relevant vector measure m and verify that the conditions (c), (d) of Theorem 6 are satisfied. So, let

$$m\colon A \in \Sigma \mapsto m(A) := T(\chi_A) \in X.$$

It a well defined, finitely additive measure (as T is linear) with values in X^+ (as T is positive). For the σ-additivity of m, let $\{A_n\} \subseteq \Sigma$ be pairwise disjoint sets. Since $\chi_{\cup_1^n A_k} \uparrow \chi_{\cup_1^\infty A_k}$ and T is positive, it follows that $T(\chi_{\cup_1^n A_k}) \uparrow T(\chi_{\cup_1^\infty A_k})$ in X. Since X has a.c. norm (as it has the subsequence splitting property), this implies that $T(\chi_{\cup_1^n A_k})$ converges to $T(\chi_{\cup_1^\infty A_k})$ in the norm of X. Hence, $\sum_1^n m(A_k) \to \sum_1^\infty m(A_k)$ in the norm of X, i.e., m is σ-additive.

Next we verify condition (c) of Theorem 6. The vector measure m is absolutely continuous with respect to the underlying measure μ. Indeed, if $\mu(A) = 0$ for some $A \in \Sigma$, then $m(A) = T(\chi_A) = 0$ (as T is linear). Actually, for any $B \in \Sigma$ with $B \subseteq A$ we have $\mu(B) = 0$ and so $m(B) = 0$. This implies that $\|m\|(A) = 0$. It follows for any given $\varepsilon > 0$ that there is $\delta > 0$ such that $\mu(A) < \delta$ implies $\|m\|(A) < \varepsilon$. Since μ is separable, there exists a countable set $\{A_j\}$ which is dense in (Σ, d_μ). For any $A \in \Sigma$ and $\varepsilon > 0$, let $\delta > 0$ be chosen as above. The separability of (Σ, d_μ) ensures there is $j \in \mathbb{N}$ such that $\mu(A \triangle A_j) < \delta$ and so $\|m\|(A \triangle A_j) < \varepsilon$. Thus, $\{A_j\}$ is dense in (Σ, d_m). Hence, m is separable.

In order to verify condition (d) of Theorem 6 note, for every $A \in \Sigma$, that $0 \le T(\chi_A) \le T(\chi_{[0,1]})$. Then, for any $B \in \Sigma$, it follows that $0 \le \chi_B T(\chi_A) \le \chi_B T(\chi_{[0,1]})$ and so $\|\chi_B T(\chi_A)\|_X \le \|\chi_B T(\chi_{[0,1]})\|_X$. Since X has a.c. norm, the function $T(\chi_{[0,1]})$ has a.c. norm in X. So, given $\varepsilon > 0$ there is $\delta > 0$ such that $\mu(B) < \delta$ implies that $\|\chi_B T(\chi_{[0,1]})\|_X < \varepsilon$. Then also $\|\chi_B T(\chi_A)\|_X < \varepsilon$ for all $A \in \Sigma$, that is, the set $\{T(\chi_A) : A \in \Sigma\}$ has uniformly a.c. norm in X. From $m(\Sigma) = \{T(\chi_A) : A \in \Sigma\}$ it follows that $m(\Sigma)$ has uniformly a.c. norm in X.

Theorem 6 now implies that $L^1(m)$ has both the subsequence splitting property and the weak Banach–Saks property. It remains to establish the equality between $L^1(m)$ and the optimal domain $[T, X]$. This is a general fact for optimal domains of kernel operators on spaces with a.c. norm [6, Corollary 3.3]. □

5 Applications

We provide an application of Theorem 6 to function spaces arising from convolution operators on groups. The proof of Corollary 2 on functions spaces arising from kernel operators on [0, 1] is also presented.

Let G be a compact, metrizable, abelian group and λ denote normalized Haar measure on G. Let ν be any positive, finite Borel measure on G. Define a vector measure $m_\nu^{(p)} : \mathcal{B}(G) \to L^p(G)$, for each $1 < p < \infty$, by convolution with ν, i.e.,

$$m_\nu^{(p)}(A) := \chi_A * \nu, \quad A \in \mathcal{B}(G).$$

Note that the space $L^p(G)$ has a.c. norm, possesses the subsequence splitting property and has the weak Banach–Saks property. Moreover, its dual space $(L^p(G))^* = L^q(G)$, with $1/p + 1/q = 1$, also has the subsequence splitting property. In addition, the vector measure $m_\nu^{(p)}$ is clearly positive and separable (as the σ-algebra $\mathcal{B}(G)$ of Borel subsets of G is countably generated). Concerning the range of $m_\nu^{(p)}$ being uniformly a.c. in $L^p(G)$, it suffices for this range to be relatively compact in $L^p(G)$ (see Sect. 2). For $1 < p < \infty$, this is the case precisely when $\nu \in M_0(G)$, i.e., the Fourier–Stieltjes coefficients of ν vanish at infinity on the dual group of G [23, Proposition 7.58]. In particular, this is so whenever $\nu \in L^1(G)$, that is, whenever ν has an integrable density with respect to λ, i.e., $\nu = f\,d\lambda$ with $f \in L^1(G)$. So, Theorem 1 implies that each of the B.f.s.'

$$L^1(m_\nu^{(p)}) = \{f : \nu * |f| \in L^p(G)\}, \quad \nu \geq 0,\ \nu \in M_0(G),$$

[23, pp. 350–351], has the subsequence splitting property and the weak Banach–Saks property. It should be remarked in the event that the measure $\nu \notin L^p(G)$, then the B.f.s. $L^1(m_\nu^{(p)})$ described above is situated strictly between $L^p(G)$ and $L^1(G)$, i.e.,

$$L^p(G) \subsetneq L^1(m_\nu^{(p)}) \subsetneq L^1(G);$$

see [23, Proposition 7.83] and the discussion following that result. It is known that always $L^1(G) \subsetneq M_0(G)$ [23, p. 320].

We now turn to the

Proof of Corollary 2 We verify that the conditions of Theorem 1 are satisfied.

The Lebesgue measure space $([0,1], \mathcal{M}, \lambda)$ is separable. Moreover, the operator T_K defined by (4) is linear and positive (as the kernel $K \geq 0$). To verify that T_K maps L^∞ into X note, for each $f \in L^\infty$, that $|T(f)| \leq T(|f|) \leq \|f\|_\infty T(\chi_{[0,1]})$. As the function $T(\chi_{[0,1]})$ belongs to X by assumption, it follows that $T(f) \in L^\infty$. So, $T : L^\infty([0,1]) \to X$.

In the case when X is reflexive, neither X nor X^* can contain a subspace isomorphic to c_0. Accordingly, as both X and X^* are r.i., they have the subsequence splitting property [26, 2.6 Corollary]. □

Corollary 2 applies to many different situations, e.g., to the following kernels on [0, 1].

(i) The Volterra kernel, $K(x,y) := \chi_\Delta(x,y)$ with $\Delta := \{(x,y) \in [0,1]\times[0,1] : 0 \leq y \leq x\}$.
(ii) The Riemann–Liouville fractional kernel, $K(x,y) := |x-y|^{\alpha-1}$ for $0 < \alpha < 1$.
(iii) The Poisson semigroup kernel, $K(x,y) := \arctan(y/x)$ for $x \neq 0$ and $K(0,y) = \pi/2$.
(iv) The kernel associated with Sobolev's inequality, $K(x,y) := y^{(1/n)-1}\chi_{[x,1]}(y)$ for $n \geq 2$.
(v) The Cesàro kernel, $K(x,y) := (1/x)\chi_{[0,x]}(y)$.

All of these kernels K generate positive operators T_K on L^∞. The function $T_K(\chi_{[0,1]})$ belongs, in all cases, to L^∞ and hence, to all r.i. spaces on [0, 1]. In particular, the function belongs to all r.i. spaces with a.c. norm.

In relation to condition (d) of Theorem 6, let us comment on the range of the associated vector measures $m_K : A \mapsto m_K(A) = T_K(\chi_A)$. In the cases (i)–(iv), the range is, in fact, relatively compact in $C([0,1])$ and hence, also in any r.i. space X; see [23, Example 4.25] and the references given there. In the case (v), the range is relatively compact in any r.i. space $X \neq L^\infty$ [9, Theorem 2.1]. So, in all cases (i)–(v) the B.f.s. $[T_K, X] = L^1(m_K)$ has the subsequence splitting property and the weak Banach–Saks property, whenever the B.f.s. X satisfies the hypotheses in Corollary 2.

In conclusion, we point out for a Banach-space valued measure $m : \Sigma \to X$ that if the integration map $I_m : L^1(m) \to X$ is compact, then m has a finite variation measure $|m| : \Sigma \to [0,\infty)$ and $L^1(m) = L^1(|m|)$ is a classical L^1-space [22, Theorems 1 & 4]. Accordingly, $L^1(m)$ has the weak Banach–Saks property. Moreover, the compactness of I_m ensures that $m(\Sigma)$ is a relatively compact subset of X but, the converse is not true in general [22, Remark 3.3(ii)]. So, the condition (d) of Theorem 6 is typically weaker than the compactness of I_m. Indeed, for the convolution vector measures $m_\nu^{(p)}$ discussed above, it was noted that $m_\nu^{(p)}$ has relatively compact range if and only if $\nu \in M_0(G)$. On the other hand, $I_{m_\nu^{(p)}}$ is compact if and only if $\nu \in L^p(G) \subsetneq M_0(G)$ [23, Theorem 7.67]. Or, for the X-valued vector measure m_X corresponding to the Cesàro kernel in (v) above, with $X \neq L^\infty$ any r.i. space whose upper Boyd index $\overline{\alpha}_X < 1$, it is known that the integration map $I_{m_X} : L^1(m_X) \to X$ is never compact; see the discussion after Proposition 4.1 in [9]. On the other hand, it was noted above that m_X always has relatively compact range.

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