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Complete spacelike hypersurfaces in generalized Robertson–Walker and the null convergence condition: Calabi–Bernstein problems

Juan A. Aledo[1] · Rafael M. Rubio[2] · Juan J. Salamanca[2]

Abstract We study constant mean curvature spacelike hypersurfaces in generalized Robertson–Walker spacetimes $\overline{M} = I \times_f F$ which are spatially parabolic (i.e. its fiber F is a (non-compact) complete Riemannian parabolic manifold) and satisfy the null convergence condition. In particular, we provide several rigidity results under appropriate mathematical and physical assumptions. We pay special attention to the case where the GRW spacetime is Einstein. As an application, some Calabi–Bernstein type results are given.

1 Introduction

For a Generalized Robertson–Walker (GRW) spacetime we mean a product manifold $I \times F$ of an open interval I of the real line $\mathbb{R}$ endowed with the metric dt^2 and an $n(\geq 2)$-dimensional (connected) Riemannian manifold (F, g_F), furnished with the Lorentzian metric

✉ Rafael M. Rubio
rmrubio@uco.es

Juan A. Aledo
juanangel.aledo@uclm.es

Juan J. Salamanca
jjsalamanca@uco.es

1 Departamento de Matemáticas, E.S.I. Informática, Universidad de Castilla-La Mancha, 02071 Albacete, Spain

2 Departamento de Matemáticas, Campus de Rabanales, Universidad de Córdoba, 14071 Córdoba, Spain

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2 \pi_F^*(g_F),$$

where π_I and π_F denote the projections onto I and F, respectively, and f is a positive smooth function on I [8]. We will denote this $(n + 1)$-dimensional Lorentzian manifold by $\overline{M} = I \times_f F$. So defined, $\overline{M}$ is a warped product in the sense of [22, Chap. 7], with base $(I, -dt^2)$, fiber (F, g_F) and warping function f. Observe that the family of GRW spacetimes includes the classical *Robertson–Walker* (RW) spacetimes. Recall that in a RW spacetime the fiber is 3-dimensional and of constant sectional curvature, and the warping function (sometimes called *scale-factor*) can be thought, when the curvature sectional of the fiber is positive, as the radius of the spatial universe $\{t\} \times F$.

Note that a RW spacetime obeys the *cosmological principle*, i.e., it is spatially homogeneous and spatially isotropic, at least locally. Thus, GRW spacetimes widely extend to RW spacetimes and include, for instance, the Lorentz-Minkowski spacetime, the Einstein-de Sitter spacetime, the Friedmann cosmological models, the static Einstein spacetime and the de Sitter spacetime. GRW spacetimes are useful to analyze if a property of a RW spacetime $\overline{M}$ is *stable*, i.e., if it remains true for spacetimes close to $\overline{M}$ in a certain topology defined on a suitable family of spacetimes [19]. Moreover, a conformal change of the metric of a GRW spacetime with a conformal factor which only depends on t, produces a new GRW spacetime.

Observe that a GRW spacetime is not necessarily spatially homogeneous. Recall that spatial homogeneity seems appropriate just as a rough approach to consider the universe in the large. However, this assumption could not be realistic when the universe is considered in a more accurate scale. Thus, these warped Lorentzian manifolds become suitable spacetimes to model universes with inhomogeneous spacelike geometries [23]. A GRW spacetime such that f is constant will be called *static*. Indeed, a static GRW spacetime is in fact a Lorentzian product. On the other hand, if the warping function f is non-locally constant (i.e. there is no open subinterval $J(\neq \emptyset)$ of I such that $f_{|J}$ is constant) then the GRW spacetime $\overline{M}$ is said to be *proper*. This assumption means that there is no (nonempty) open subset of $\overline{M}$ such that the sectional curvature in $\overline{M}$ of any plane tangent to a *spacelike slice* $\{t\} \times F$ equals to the sectional curvature of that plane in the inner geometry of the slice.

If the fiber of a GRW spacetime is compact, then it is called *spatially closed*. Classically, the subfamily of spatially closed GRW spacetimes has been very useful to get closed cosmological models. On the other hand, a number of observational and theoretical arguments on the total mass balance of the universe [16] suggests the convenience of adopting open cosmological models.

In this work we are interested in the class of *spatially parabolic* GRW spacetimes. This notion was introduced and motivated in [25] as a natural counterpart of the spatially closed GRW spacetimes. Spatially parabolic GRW spacetimes have a parabolic Riemannian manifold as fiber, what provides a significant wealth from a geometric-analytic point of view. Recall that a complete Riemannian manifold is parabolic if its only positive superharmonic functions are the constants.

The importance in General Relativity of maximal and constant mean curvature spacelike hypersurfaces in spacetimes is well-known; a summary of several reasons justifying it can be found in [21]. Classical papers dealing with uniqueness problems for such kind of hypersurfaces are [11,17,21], although a previous relevant result in this direction was the proof of the Calabi–Bernstein conjecture [13] for maximal hypersurfaces in the n-dimensional Lorent-Minkowski spacetime given by Cheng and Yau [15]. In [11], Brill and Flaherty replaced the Lorent-Minkowski spacetime by a spatial closed universe, and proved uniqueness in the large by assuming $\mathrm{Ric}(z, z) > 0$ for all timelike vectors z. In [21], this energy condition was relaxed

by Marsden and Tipler to include, for instance, non-flat vacuum spacetimes. More recently, Bartnik proved in [10] very general existence theorems and consequently, he claimed that it would be useful to find new satisfactory uniqueness results. Still more recently, in [8] Alías, Romero and Sánchez gave new uniqueness results in the class of spatially closed GRW spacetimes under the Temporal Convergence Condition (TCC). In [12] several known uniqueness results for compact CMC spacelike hypersurfaces in GRW spacetimes were widely extended by means of new techniques to the case of compact CMC spacelike hypersurfaces in spacetimes with a timelike gradient conformal vector field. Finally, in [25] Romero, Rubio and Salamanca, obtained uniqueness results in the maximal case for spatially parabolic GRW spacetimes under a convexity property of the warping function.

Our main aim in this paper is to give new uniqueness results for (non-compact) complete CMC hypersurfaces in spatially parabolic GRW spacetimes which obey the Null Convergence Condition (NCC). As known, the TCC is violated in inflationary spacetimes and so it is natural to study uniqueness problems under the NCC, since some inflationary scenarios can be modeled by spacetimes obeying this energy condition. Moreover, certain class of GRW spacetimes obeying the NCC arise as physically realistic cosmological models since they satisfy the weak energy condition (see Sect. 5). Some recent papers dealing with uniqueness problems in GRW spacetimes obeying the NCC under hypothesis relative to the curvatures of the spacelike hypersurfaces are [2,5,6,14,20]

The paper is organized as follows. In Sect. 2 we revise some notions regarding spacelike hypersurfaces in GRW spacetimes. In Sect. 3 we provide several rigidity results for CMC hypersurfaces in spatially parabolic GRW spacetimes satisfying the NCC. We pay special attention to the case when the GRW spacetime is Einstein, so completing the characterization of compact CMC spacelike hypersurfaces in spatially closed Einstein GRW spacetimes partially developed in some previous papers (see [9,12]), and extending this study to complete CMC spacelike hypersurfaces in spatially parabolic Einstein GRW spacetimes. Section 4 is devoted to provide several Calabi–Bernstein results which follow from the former parametric study. Finally, in Sect. 5 we justify the adequacy of GRW spacetimes which satisfy the NCC condition to model some physically realistic cosmological universes.

2 Preliminaries

Let (F, g_F) be an n-dimensional ($n \geq 2$) connected Riemannian manifold and $I \subseteq \mathbb{R}$ an open interval in $\mathbb{R}$ endowed with the metric $-dt^2$. The warped product $\overline{M} = I \times_f F$ endowed with the Lorentzian metric

$$\bar{g} = -\pi_I^* \left(dt^2 \right) + f \left(\pi_I \right)^2 \pi_F^*(g_F) \tag{1}$$

where $f > 0$ is a smooth function on I, and π_I and π_F denote the projections onto I and F respectively, is said to be a *Generalized Robertson–Walker (GRW) spacetime* with *fiber* (F, g_F), *base* $(I, -dt^2)$ and *warping function* f (see [8]).

The coordinate vector field $\partial_t := \partial/\partial t$ globally defined on $\overline{M}$ is (unitary) timelike, and so $\overline{M}$ is time-orientable. We will also consider on $\overline{M}$ the conformal closed timelike vector field $K := f(\pi_I) \partial_t$. From the relationship between the Levi-Civita connections of $\overline{M}$ and those of the base and the fiber [22, Corollary 7.35], it follows that

$$\overline{\nabla}_X K = f'(\pi_I) X \tag{2}$$

for any $X \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of the Lorentzian metric (1).

We will denote by $\overline{\mathrm{Ric}}$ the Ricci tensor of $\overline{M}$. From [22, Corollary 7.43] it follows that

$$\overline{\mathrm{Ric}}(X, Y) = \mathrm{Ric}^F\left(X^F, Y^F\right) + \left(\frac{f''}{f} + (n-1)\frac{f'^2}{f^2}\right)\overline{g}\left(X^F, Y^F\right) - n\frac{f''}{f}\overline{g}(X, \partial_t)\overline{g}(Y, \partial_t) \tag{3}$$

for $X, Y \in \mathfrak{X}(\overline{M})$, where Ric^F stands for the Ricci tensor of F. Here X^F denotes the lift of the projection of the vector field X onto F, that is,

$$X = X^F - \overline{g}(X, \partial_t)\partial_t.$$

Regarding the scalar curvature $\overline{S}$ of $\overline{M}$, we get from (3) that

$$\overline{S} = \mathrm{trace}(\overline{\mathrm{Ric}}) = \frac{S^F}{f^2} + 2n\frac{f''}{f} + n(n-1)\frac{f'^2}{f^2}, \tag{4}$$

where S^F stands for the scalar curvature of F.

Recall that a Lorentzian manifold $\overline{M}$ obeys the *Null Convergence Condition (NCC)* if its Ricci tensor $\overline{\mathrm{Ric}}$ satisfies $\overline{\mathrm{Ric}}(X, X) \geq 0$ for all null vector $X \in \mathfrak{X}(\overline{M})$. In the case when $\overline{M} = I \times_f F$ is a GRW spacetime, it can be checked (see [4]) that $\overline{M}$ obeys the NCC if and only if

$$Ric^F - (n-1)f^2(\log f)'' \geq 0, \tag{5}$$

where Ric^F stands for the Ricci curvature of (F, g_F). Recall that the Ricci curvature at each point $p \in F$ in the direction $X(p) \in T_p F$, $X \in \mathfrak{X}(F)$, is defined as

$$Ric^F(X(p)) = \frac{\mathrm{Ric}^F(X(p), X(p))}{g_F(X(p), X(p))} = \mathrm{Ric}^F\left(\frac{X(p)}{|X(p)|_F}, \frac{X(p)}{|X(p)|_F}\right).$$

On the other hand, we will say that a spacetime $\overline{M}$ verifies the *NCC with strict inequality* if its Ricci tensor $\overline{\mathrm{Ric}}$ satisfies $\overline{\mathrm{Ric}}(X, X) > 0$ for all null vector $X \in \mathfrak{X}(\overline{M})$. Now, a GRW spacetime $\overline{M} = I \times_f F$ obeys the NCC with strict inequality if and only if $Ric^F - (n-1) f^2(\log f)'' > 0$.

A smooth immersion $\psi : M^n \longrightarrow \overline{M}$ of an n-dimensional (connected) manifold M is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric g on M.

Since $\overline{M}$ is time-orientable we can take, for each spacelike hypersurface M in $\overline{M}$, a unique unitary timelike vector field $N \in \mathfrak{X}^\perp(M)$ globally defined on M with the same time-orientation as ∂_t, i.e., such that $\overline{g}(N, \partial_t) < 0$. From the wrong-way Cauchy-Schwarz inequality (see [22, Proposition 5.30], for instance), we have $\overline{g}(N, \partial_t) \leq -1$, and the equality holds at a point $p \in M$ if and only if $N = \partial_t$ at p. The *hyperbolic angle* φ, at any point of M, between the unit timelike vectors N and ∂_t, is given by $\overline{g}(N, \partial_t) = -\cosh\varphi$. This angle has a reasonable physical interpretation. In fact, in a GRW spacetime $\overline{M}$ the integral curves of ∂_t are called *comoving observers* [26, p. 43]. If p is a point of a spacelike hypersurface M in $\overline{M}$, among the instantaneous observers at p, $\partial_t(p)$ and N_p appear naturally. In this sense, observe that the energy $e(p)$ and the speed $v(p)$ that $\partial_t(p)$ measures for N_p are given, respectively, by $e(p) = \cosh\varphi(p)$ and $|v(p)|^2 = \tanh^2\varphi(p)$ [26, pp. 45–67].

We will denote by A and $H := -(1/n)\mathrm{tr}(A)$ the *shape operator* and the *mean curvature function* associated to N. A spacelike hypersurface with $H = 0$ is called a *maximal hypersurface*. The reason for this terminology is that the mean curvature is zero if and only if the spacelike hypersurface is a local maximum of the n-dimensional area functional for compactly supported normal variations.

In any GRW spacetime $\overline{M}$ there is a remarkable family of spacelike hypersurfaces, namely its spacelike *slices* $\{t_0\} \times F$, $t_0 \in I$. The spacelike slices constitute for each value t_0 the restspace of the distinguished observers in ∂_t. A spacelike hypersurface in $\overline{M}$ is a (piece of) spacelike slice if and only if the function $\tau := \pi_I \circ \psi$ is constant. Furthermore, a spacelike hypersurface in $\overline{M}$ is a (piece of) spacelike slice if and only if the hyperbolic angle φ vanishes identically. The shape operator of the spacelike slice $\tau = t_0$ is given by $A = -f'(t_0)/f(t_0) I$, where I denotes the identity transformation, and so its (constant) mean curvature is $H = f'(t_0)/f(t_0)$. Thus, a spacelike slice is maximal if and only if $f'(t_0) = 0$ (and hence, totally geodesic). We will say that the spacelike hypersurface is contained in a *slab*, if it is contained between two spacelike slices.

If we put $\partial_t^T = \partial_t + \overline{g}(\partial_t, N)N$ the tangential part of ∂_t and $N^F = N + \overline{g}(N, \partial_t)\partial_t$, it follows from $\overline{g}(N, N) = -1 = \overline{g}(\partial_t, \partial_t)$ that

$$\left|\partial_t^T\right|^2 = \left|N^F\right|^2 = \sinh^2 \varphi. \tag{6}$$

Hence, a spacelike hypersurface in $\overline{M}$ is a (piece of) spacelike slice if and only if $\left|\partial_t^T\right|^2 = \left|N^F\right|^2$ vanishes identically on M.

To finish this section, let us briefly revise some important notions on parabolicity in GRW spacetimes. Recall that a GRW spacetime $\overline{M} = I \times_f F$ is said to be *spatially parabolic* [25] if its fiber is *parabolic*; i.e., it is a non-compact complete Riemannian manifold such that the only superharmonic functions on it which are bounded from below are the constants. GRW spacetimes which admit a complete parabolic spacelike hypersurface have been studied in [25], where the following result is proved:

Lemma 1 *Let M be a complete spacelike hypersurface in a spatially parabolic GRW space-time $\overline{M} = I \times_f F$. If the hyperbolic angle of M is bounded and the restriction $f(\tau)$ on M of the warping function f satisfies:*

(i) sup $f(\tau) < \infty$, and
(ii) inf $f(\tau) > 0$,

then, M is parabolic.

This result will be used in Sect. 3.

3 Parametric type results

Let $\psi : M \to \overline{M}$ be a spacelike hypersurface in a GRW spacetime $\overline{M} = I \times_f F$. It is easy to check that the gradient of $\tau = \pi_I \circ \psi$ on M is given by

$$\nabla \tau = -\partial_t^T \tag{7}$$

and its Laplacian by

$$\Delta \tau = -\frac{f'(\tau)}{f(\tau)} \left\{ n + |\nabla \tau|^2 \right\} - nH \, \overline{g}(N, \partial_t). \tag{8}$$

Let us take $G : I \longrightarrow \mathbb{R}$ such that $G' = f$. Using (7) we have that the gradient of $G(\tau)$ on M is given by

$$\nabla G(\tau) = G'(\tau)\nabla \tau = -f(\tau)\partial_t^T = -K^T, \tag{9}$$

where $K^T = K + \overline{g}(K, N)N$ is the tangential component of K along ψ, and so its Laplacian on M (see [8, Eq. 6]) yields

$$\Delta G(\tau) = \operatorname{div}(\nabla G(\tau)) = -nf'(\tau) - nH\overline{g}(K, N). \tag{10}$$

A direct computation from (2) gives

$$\nabla \overline{g}(K, N) = -AK^T,$$

where we have also used (7), and so the Laplacian of $\overline{g}(K, N)$ on M becomes (see [8, Eq. 8])

$$\Delta \overline{g}(K, N) = \operatorname{div}(\nabla \overline{g}(K, N)) = \overline{\operatorname{Ric}}(K^T, N) + n\overline{g}(\nabla H, K) + nf'(\tau)H + \overline{g}(K, N)\operatorname{tr}(A^2). \tag{11}$$

On the other hand, from (3) we have

$$\begin{aligned}
\overline{\operatorname{Ric}}\left(K^T, N\right) &= \overline{g}(K, N)\,\overline{\operatorname{Ric}}\left(N^F, N^F\right) - \overline{g}(K, N)\left|\partial_t^T\right|^2 \overline{\operatorname{Ric}}(\partial_t, \partial_t) \\
&= \overline{g}(K, N)\left(\operatorname{Ric}^F\left(N^F, N^F\right) - (n-1)\left|N^F\right|^2 (\log f)''(\tau)\right) \\
&= \overline{g}(K, N)\left|N^F\right|_F^2 \left(Ric^F\left(N^F\right) - (n-1)f^2(\tau)(\log f)''(\tau)\right), \quad (12)
\end{aligned}$$

where $\left|N^F\right|_F = g_F(N^F, N^F)^{1/2}$. In particular, observe that if $\overline{M}$ obeys the NCC then $\overline{\operatorname{Ric}}(K^T, N) \leq 0$. Furthermore, if $\overline{M}$ obeys the NCC with strict inequality, then $\overline{\operatorname{Ric}}(K^T, N) \equiv 0$ if and only if M is a (piece of) spacelike slice [see (6)].

Let $\overline{M} = I \times_f F$ be a spatially parabolic GRW spacetime obeying the NCC. From the study developed above, next we will provide several rigidity results for CMC complete spacelike hypersurfaces in $\overline{M}$. In some of these results, in order to derive the parabolicity of the spacelike hypersurface it is used that the assumptions $\inf f(\tau) > 0$ and $\sup f(\tau) < \infty$ are automatically satisfied if the hypersurface is contained in a slab.

Theorem 2 *Let $\overline{M} = I \times_f F$ be a spatially parabolic GRW spacetime obeying the NCC and $\psi : M \to \overline{M}$ a complete CMC spacelike hypersurface which is contained in a slab and whose hyperbolic angle is bounded. Then M is totally umbilical. Moreover, if the spacetime obeys NCC spacetime obeying the NCC with strict inequality, then M must be a spacelike slice.*

Proof Consider us the distinguished function $(HG(\tau) + \overline{g}(K, N))$, defined on M. Then, from (10)–(12), we get

$$\begin{aligned}
\Delta(HG(\tau) + \overline{g}(K, N)) = -\overline{g}(K, N)\Big\{nH^2 - \operatorname{tr}\left(A^2\right) \\
- \left|N^F\right|_F^2 \left(Ric^F\left(N^F\right) - (n-1)f^2(\tau)(\log f)''(\tau)\right)\Big\}.
\end{aligned}$$

In particular, if $\overline{M}$ obeys the NCC then $\Delta(HG(\tau) + \overline{g}(K, N)) \leq 0$.

Observe that, since M is contained between two spacelike slices, both $G(\tau)$ and $f(\tau)$ are bounded, being also $\inf f(\tau) > 0$. As said in Sect. 2, under the assumptions above it follows that M is parabolic. Then, since $HG(\tau) + \overline{g}(K, N)$ is a bounded function on M whose Laplacian is non positive, we conclude that such Laplacian must vanish identically and consequently $nH^2 - \operatorname{tr}(A^2) \equiv 0$ on M, i.e., M is totally umbilical.

Finaly, note that, under this additional assumption, it must be $\left|N^F\right|^2 \equiv 0$ on M, which implies (see 6) that M is a spacelike slice.   □

For the particular case when M is maximal, we have

Corollary 3 *Let $\overline{M} = I \times_f F$ be a spatially parabolic GRW spacetime obeying the NCC and $\psi : M \to \overline{M}$ a complete maximal spacelike hypersurface which is contained in a slab and whose hyperbolic angle is bounded. Then M is totally geodesic. Moreover, if the spacetime obeys NCC spacetime obeying the NCC with strict inequality, then M is a totally geodesic spacelike slice.*

Recall that a GRW spacetime is said to be *proper* if the warping function f is non-locally constant, i.e., there is no open subinterval $J(\neq\emptyset)$ of I such that $f_{|J}$ is constant. Next we characterize the spacelike slices of a proper spatially parabolic GRW spacetime obeying the NCC by means of a pinching condition for its (constant) mean curvature H.

Theorem 4 *Let $\overline{M} = I \times_f F$ be a proper spatially parabolic GRW spacetime obeying the NCC and $\psi : M \to \overline{M}$ a complete CMC spacelike hypersurface whose hyperbolic angle is bounded. If the mean curvature function of M satisfies that $H^2 \geq \frac{f'(\tau)^2}{f(\tau)^2}$ and the restriction $f(\tau)$ of the warping function f on M is such that $\inf f(\tau) > 0$ and $\sup f(\tau) < \infty$, then M is a spacelike slice ($\tau = t_0$) with $H^2 = \frac{f'(t_0)^2}{f(t_0)^2}$.*

Proof Since the hyperbolic angle of M is bounded and $f(\tau)$ satisfies that $\inf f(\tau) > 0$ and $\sup f(\tau) < \infty$, we conclude that M is parabolic (see Sect. 2).

From the assumption on the mean curvature of M we have that

$$|H| \geq \frac{|f'(\tau)|}{f(\tau)},$$

and so

$$\operatorname{tr}\left(A^2\right) \geq nH^2 \geq \frac{n}{f(\tau)}|f'(\tau)H|.$$

Then

$$nf'(\tau)H + \overline{g}(K, N)\operatorname{tr}\left(A^2\right) \leq 0,$$

which implies that the Laplacian of $\overline{g}(K, N)$ (11) is non positive and consequently constant.

Moreover

$$|nf'(\tau)H| = |\overline{g}(N, K)|\operatorname{tr}\left(A^2\right) \geq f \operatorname{tr}\left(A^2\right) \geq |nf'(\tau)H|,$$

and therefore $f = |\overline{g}(N, K)| = f(\tau)\cosh\varphi$. Consequently φ vanishes identically on M, which means that M is a spacelike slice. □

Remark 5 The inequality $H^2 \geq \frac{f'(\tau)^2}{f(\tau)^2}$ can be geometrically interpreted as follows: the mean curvature of the spacelike hypersurface, at any point is, in absolute value, greater or equal than the mean curvature of the spacelike slice at that point.

As commented in the introduction, a GRW spacetime is spatially closed if its fiber F is compact [8, Proposition 3.2]. Since on a compact Riemannian manifold the only functions with signed Laplacian are the constants, reasoning as in Theorem 4 it can be proved the following

Theorem 6 *Let $\overline{M} = I \times_f F$ be a proper spatially closed GRW spacetime obeying the NCC and $\psi : M \to \overline{M}$ a compact CMC spacelike hypersurface whose mean curvature satisfies that $H^2 \geq \frac{f'(\tau)^2}{f(\tau)^2}$. Then M is a spacelike slice ($\tau = t_0$) with $H^2 = \frac{f'(t_0)^2}{f(t_0)^2}$.*

A relevant example of proper spatially closed GRW spacetime obeying the NCC is the de Sitter spacetime which, in its intrinsic version is given as the Robertson–Walker spacetime $\mathbb{S}_1^{n+1} = \mathbb{R} \times_{\cosh t} \mathbb{S}^n$. In [3, Theorem 1] the authors established a sufficient condition for a compact spacelike in $\mathbb{S}_1^{n+1}$ (considered as an hyperquadric of the $(n+2)$-dimensional Lorent-Minkowski spacetime) to be totally umbilical, in terms of a lower bound for the squared of its mean curvature. As a consequence of Theorem 6, we obtain the following intrinsic approach of the previously cited result:

Corollary 7 *Let $\psi : M \to \mathbb{S}_1^{n+1}$ be a spacelike hypersurface in the de Sitter spacetime whose constant mean curvature satisfies that $H^2 \geq \tanh^2(\tau)$. Then M is a spacelike slice with $H^2 = \tanh^2(\tau)$.*

Notice that in $\mathbb{S}_1^{n+1}$ there exists an only maximal slice and, for any $t \neq 0$, exactly two spacelike slices with $H^2 = \tanh^2(t)$.

Next, we provide another uniqueness result under the hypothesis of monotony of the warping function.

Theorem 8 *Let $\overline{M} = I \times_f F$ be a spatially parabolic GRW spacetime obeying the NCC, and let $\psi : M \to \overline{M}$ be a complete CMC spacelike hypersurface whose hyperbolic angle is bounded and such that $\sup f(\tau) < \infty$ and $\inf f(\tau) > 0$.*

If the restriction of f to $\tau(M)$ is non-increasing (resp. non decreasing) and $H \geq 0$ (resp. $H \leq 0$), then M is totally geodesic.

Proof From (11) we have that $\overline{g}(K, N)$ is subharmonic on the parabolic manifold (M, g). Since moreover that function is bounded, it must be constant. Finally, using again (11) it follows that $\text{tr}(A^2)$ vanishes identically and therefore M is totally geodesic. $\square$

In the above theorem, if we ask $\overline{M} = I \times_f F$ to obey the NCC with strict inequality, then we conclude that M is a totally geodesic spacelike slice.

Next we provide another rigidity result (Theorem 10) for complete CMC spacelike hypersurfaces in GRW spacetimes whose fiber has its sectional curvature bounded from below and whose warping function f satisfies that $(\log f)'' \leq 0$. Note that the NCC will be not required in this theorem. In order to do that, we will need the following result which extends [4, Lemma 13]. In fact, note that in such Lemma the fiber is asked to have non-negative sectional curvature, whereas in the following result this assumption changes to have sectional curvature bounded from below.

Lemma 9 *Let $\psi : M \to \overline{M}$ be a complete CMC spacelike hypersurface in a GRW spacetime $\overline{M} = I \times_f F$ whose warping function satisfies $(\log f)'' \leq 0$ and whose fiber has its sectional curvature bounded from below. Then the Ricci curvature of M is bounded from below.*

Proof Given $Y \in \mathfrak{X}(M)$ such that $g(Y, Y) = 1$, let us write

$$Y = -\overline{g}(\partial_t, Y)\partial_t + Y^F.$$

From the Schwarz inequality, we get using (7) and (6) that

$$\overline{g}(\partial_t, Y)^2 = g(\nabla \tau, Y)^2 \leq |\nabla \tau|^2 = \sinh^2 \varphi.$$

As a consequence, $|Y^F|^2 = 1 + \overline{g}(\partial_t, Y)^2$ is bounded.

Given $p \in M$, let us take a local orthonormal frame $\{U_1, \ldots, U_n\}$ around p. From the Gauss equation

$$\langle R(X, Z)V, W \rangle = \langle \overline{R}(X, Z)V, W \rangle + \langle AZ, W \rangle \langle AX, V \rangle$$
$$- \langle AZ, V \rangle \langle AX, W \rangle, \quad X, Z, V, W \in \mathfrak{X}(M)$$

where $\overline{R}$ and R denote the curvature tensors of $\overline{M}$ and M respectively, and A is the shape operator of ψ, we get that the Ricci curvature of M, Ric^M, satisfies

$$\mathrm{Ric}^M(Y, Y) \geq \sum_k \overline{g}(\overline{R}(Y, U_k)Y, U_k) - \frac{n^2}{4}H^2|Y|^2, \quad Y \in \mathfrak{X}(M), \ g(Y, Y) = 1.$$

Now, from [22, Proposition 7.42] we have

$$\sum_{k=1}^{n} \overline{g}(\overline{R}(Y, U_k)Y, U_k) = \sum_{k=1}^{n} g_F(R^F(Y^F, U_k^F)Y^F, U_k^F) + (n-1)\frac{f'^2}{f^2}$$
$$- (n-2)(\log f)'' g(Y, \nabla \tau)^2 - (\log f)''|\nabla \tau|^2,$$

where R^F denotes the curvature tensor of the fiber F. Since the sectional curvature of F is bounded from below, there exists a constant C such that $\sum_{k=1}^{n} \overline{g}(\overline{R}(Y, U_k)Y, U_k) \geq C$. Therefore

$$\mathrm{Ric}^M(Y, Y) \geq C - \frac{n^2}{4}H^2,$$

namely, the Ricci curvature of M is bounded from below as we wanted to prove. $\qquad \square$

To demonstrate Theorem 10 we will use [4, Lemma 12]. To facilitate the understanding of its proof, observe that in the paper [4] the hypersurface $\psi : M \to \overline{M}$ was oriented by choosing the Gauss map N such that $\overline{g}(N, \partial_t) > 0$. This change of orientation means that, according to the orientation chosen in the present article, the thesis of [4, Lemma 12] becomes $H = f'(\tau)/f(\tau)$.

Theorem 10 *Let $\overline{M} = I \times_f F$ be a spatially parabolic GRW spacetime whose warping function satisfies $(\log f)'' \leq 0$ and whose fiber has its sectional curvature bounded from below. Let $\psi : M \to \overline{M}$ be a complete CMC spacelike hypersurface which is contained in a slab and whose hyperbolic angle is bounded. Then M is a spacelike slice.*

Proof From the assumptions it follows using Lemma 9 and [4, Lemma 12] that

$$H = \frac{f'(\tau)}{f(\tau)}.$$

Now, using (10) we obtain

$$\Delta G(\tau) = -nf(\tau)(-H + H \cosh \varphi) \leq 0.$$

Taking into account the boundedness of the function $G(\tau)$ and the parabolicity of M, we have that $G(\tau)$ must be constant and $\nabla G(\tau) = -f(\tau)\partial_t^T = 0$, namely M is a spacelike slice. $\qquad \square$

Remark 11 Observe that Theorem 10 widely improves [4, Theorem 14] in many aspects:

- In [4, Theorem 14] the dimension of M is restricted to $n \leq 4$, whereas in Theorem 10 this dimension is arbitrary.

- In [4, Theorem 14] the fiber is asked to have non-negative sectional curvature, whereas in Theorem 10 this assumption changes to have sectional curvature bounded from below.
- In [4, Theorem 14] the warped function f is asked to satisfy $f''(\tau) \leq 0$, whereas in Theorem 10 this assumption changes to the weaker one $(\log f)''(\tau) \leq 0$.
- Finally, in contrast to [4, Theorem 14], in Theorem 10 the maximal case is included.

In [1, Sect. 4], Albujer and Alías introduced the notion of *steady state type spacetimes*, as the warped products with fiber an n-dimensional Riemannian manifold (F, g_F), base $(\mathbb{R}, -dt^2)$ and warping function $f(t) = e^t$. This family contains, for instance, the De Sitter cusp [18]. In particular, these GRW spacetimes obey the NCC provided that the fiber F has non-negative Ricci curvature. As a consequence of our Theorem 10, we can enunciate

Corollary 12 *Let $\overline{M} = \mathbb{R} \times_{e^t} F$ be a spatially parabolic stedy state type spacetime, whose fiber has non-negative Ricci curvature. Let $\psi : M \to \overline{M}$ be a complete CMC spacelike hypersurface which is contained in a slab and whose hyperbolic angle is bounded. Then M is a spacelike slice.*

This result extends [1, Theorem 8] to arbitrary dimension. In fact, in [1, Theorem 8] the authors obtain the same rigidity result when the fiber has dimension 2 using that a complete 2-dimensional Riemannian manifold whose Gaussian curvature is non-negative is parabolic.

3.1 Einstein GRW spacetimes

Recall that a spacetime $(\overline{M}, \overline{g})$ is called *Einstein* if its Ricci tensor $\overline{\mathrm{Ric}}$ is proportional to the metric $\overline{g}$. When $\overline{M} = I \times_f F$ is a GRW spacetime, it is well-known that $\overline{M}$ is Einstein with $\overline{\mathrm{Ric}} = \overline{c}\,\overline{g}, \overline{c} \in \mathbb{R}$, if and only if the fiber (F, g_F) has constant Ricci curvature c and the warping function f satisfies the differential equations

$$\frac{f''}{f} = \frac{\overline{c}}{n} \quad \text{and} \quad \frac{\overline{c}(n-1)}{n} = \frac{c + (n-1)(f')^2}{f^2}, \tag{13}$$

which, in particular, imply that $(n-1)(\log f)'' = \frac{c}{f^2}$ (see [12, Sect. 6]). Obviously, every Einstein spacetime obeys the NCC.

All the positive solutions to (13) were collected in [9]. For the sake of completeness, we show such classification in Table 1.

In [12, Theorem 6.1], the authors proved that the spacelike slices are the only compact CMC spacelike hypersurfaces in an Einstein GRW spacetime whose fiber has Ricci curvature $c \leq 0$. This result covers the cases 2–6 in Table 1. However, the techniques used there cannot

Table 1 Warping functions for Einstein GRW spacetimes

1	$\overline{c} > 0$	$c > 0$	$f(t) = a\,e^{bt} + \frac{cn}{4a\overline{c}(n-1)}\,e^{-bt}, \quad a > 0, \quad b = \sqrt{\overline{c}/n}$
2	$\overline{c} > 0$	$c = 0$	$f(t) = a\,e^{\varepsilon bt}, \quad a > 0, \quad \varepsilon = \pm 1, \quad b = \sqrt{\overline{c}/n}$
3	$\overline{c} > 0$	$c < 0$	$f(t) = a\,e^{bt} + \frac{cn}{4a\overline{c}(n-1)}\,e^{-bt}, \quad a \neq 0, \quad b = \sqrt{\overline{c}/n}$
4	$\overline{c} = 0$	$c = 0$	$f(t) = a, \quad a > 0$
5	$\overline{c} = 0$	$c < 0$	$f(t) = \varepsilon\sqrt{\frac{-c}{n-1}}\,t + a, \quad \varepsilon = \pm 1$
6	$\overline{c} < 0$	$c < 0$	$f(t) = a_1 \cos(bt) + a_2 \sin(bt), \quad a_1^2 + a_2^2 = \frac{cn}{\overline{c}(n-1)}, \quad b = \sqrt{-\overline{c}/n}$

be applied to study the first case ($\overline{c} > 0$ and $c > 0$). For these values, from the Bonnet-Myers Theorem we have that the fiber F is compact, and so the GRW spacetime is spatially closed.

Since on a compact Riemannian manifold the only functions with signed Laplacian are the constants, as a direct consequence of the proof of Theorem 2 we conclude that

Corollary 13 *Every compact CMC spacelike hypersurface in an Einstein GRW spacetime whose fiber has positive Ricci curvature $c > 0$ is totally umbilical.*

Actually, this is the best possible result. In fact, recall that the de Sitter spacetime has a realization as the GRW spacetime $\mathbb{S}_1^{n+1} = \mathbb{R} \times_{\cosh t} \mathbb{S}^n$. In particular, $\mathbb{S}_1^{n+1}$ is included in the case 1 of Table 1 and, as is well-known, it contains compact CMC spacelike hypersurfaces which are not spacelike slices.

Also observe that Theorem 2 allows to extend the previous study from the compact case to the one of complete CMC spacelike hypersurfaces in a spatially parabolic Einstein GRW spacetime, being able to consider jointly the six cases mentioned above. Specifically, we have the following corollary which widely extend [12, Theorem 6.1] and the rigidity results in [9]

Corollary 14 *Let $\overline{M} = I \times_f F$ be a spatially parabolic Einstein GRW spacetime and $\psi : M \rightarrow \overline{M}$ a complete CMC spacelike hypersurface which is contained in a slab and whose hyperbolic angle is bounded. Then M is totally umbilical.*

Anyway, we are able to go further in the cases 2–6. In fact, note that in these cases the warping function f satisfies that $(\log f)'' \leq 0$. Then, if additionally we ask the fiber F to have its sectional curvature bounded from below we have

Corollary 15 *Let $\overline{M} = I \times_f F$ be a spatially parabolic Einstein GRW spacetime whose fiber has Ricci curvature $c \leq 0$ (cases 2–6 in Table 1) and whose sectional curvature is bounded from below. Let $\psi : M \rightarrow \overline{M}$ be a complete CMC spacelike hypersurface which is contained in a slab and whose hyperbolic angle is bounded. Then M is a spacelike slice.*

4 Calabi–Bernstein type problems

Let (F, g_F) be a (non-compact) n-dimensional Riemannian manifold and $f : I \longrightarrow \mathbb{R}$ a positive smooth function. For each $u \in C^\infty(F)$ such that $u(F) \subseteq I$, we can consider its graph $\Sigma_u = \{(u(p), p) : p \in F\}$ in the Lorentzian warped product $(\overline{M} = I \times_f F, \overline{g})$. The graph inherits from $\overline{M}$ a metric, represented on F by

$$g_u = -du^2 + f(u)^2 g_F.$$

This metric is Riemannian (i.e. positive definite) if and only if u satisfies $|Du| < f(u)$ everywhere on F, where Du denotes the gradient of u in (F, g_F) and $|Du|^2 = g_F(Du, Du)$. Note that $\tau(u(p), p) = u(p)$ for any $p \in F$, and so τ and u may be naturally identified on Σ_u.

When Σ_u is spacelike, the unitary normal vector field on Σ_u satisfying $\overline{g}(N, \partial_t) < 0$ is

$$N = \frac{1}{f(u)\sqrt{f(u)^2 - |Du|^2}} \left(f(u)^2 \partial_t + Du \right).$$

Then the hyperbolic angle φ, at any point of M, between the unit timelike vectors N and ∂_t, is given by

$$\cosh \varphi = \frac{f(u)}{\sqrt{f(u)^2 - |Du|^2}} \tag{14}$$

and the corresponding mean curvature function is

$$H(u) = \text{div}\left(\frac{Du}{nf(u)\sqrt{f(u)^2 - |Du|^2}}\right) + \frac{f'(u)}{n\sqrt{f(u)^2 - |Du|^2}}\left(n + \frac{|Du|^2}{f(u)^2}\right).$$

In this section, our aim is to derive non-parametric uniqueness results from the parametric ones provided in Sect. 4. To do that, we need the induced metric g_u to be complete. Observe that, in general, the induced metric on a closed spacelike hypersurface in a complete Lorentzian manifold could be non-complete (see, for instance, [7]). In our setting, we can derive the completeness of Σ_u as follows [4, Lemma 17]

Lemma 16 *Let $\overline{M} = I \times_f F$ be a GRW spacetime whose fiber is a (non-compact) complete Riemannian manifold. Consider a function $u \in C^\infty(F)$, with $\text{Im}(u) \subseteq I$, such that the entire graph $\Sigma_u = \{(u(p), p) : p \in F\} \subset \overline{M}$ endowed with the metric $g_u = -du^2 + f(u)^2 g_F$ is spacelike. If the hyperbolic angle of Σ_u is bounded and $\inf f(u) > 0$, then the graph (Σ_u, g_{Σ_u}) is complete, or equivalently the Riemannian surface (F, g_u) is complete.*

As a consequence of Theorem 2, we have

Theorem 17 *Let (F, g) be a simply connected parabolic Riemannian n-manifold, $I \subseteq \mathbb{R}$ an open interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}^+$ a positive continuous function satisfying that $Ric^F - (n - 1) f^2 (\log f)'' > 0$. Then the only bounded entire solutions $u \in C^\infty(F)$, with $\text{Im}(u) \subseteq I$, to the uniformly elliptic non-linear differential equation*

$$H(u) = cte$$

$$|Du| < \lambda f(u), \quad 0 < \lambda < 1 \tag{15}$$

are the constant functions $u = u_0$ with $H = \frac{f'(u_0)}{f(u_0)}$.

For the particular case when $H(u) = 0$, then the only bounded entire solutions are the constant functions $u = u_0$ with $f'(u_0) = 0$.

Proof First observe that, from (14), the constraint condition (15) can be written as

$$\cosh \varphi < \frac{1}{\sqrt{1 - \lambda^2}}. \tag{16}$$

Hence, (15) holds if and only if Σ_u has bounded hyperbolic angle. Moreover, (15) also implies that the metric g_u is spacelike, and furthermore it is complete from Lemma 16. Finally, the thesis follows from Theorem 2 and Corollary 3. $\qquad\square$

Remark 18 Note that the restriction (16) makes $H(u)$ into a uniformly elliptic operator.

As a consequence of Theorem 10, we obtain (compare with [12, Theorem 7.1]),

Theorem 19 *Let (F, g) be a simply connected parabolic Riemannian n-manifold whose sectional curvature is bounded from below, $I \subseteq \mathbb{R}$ an open interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}^+$ a positive smooth function satisfying that $(\log f)'' \leq 0$. Then the only bounded entire solutions $u \in C^\infty(F)$, with $\text{Im}(u) \subseteq I$, to the uniformly elliptic non-linear differential equation*

$$H(u) = cte$$

$$|Du| < \lambda f(u), \quad 0 < \lambda < 1$$

are the constant functions $u = u_0$ with $H = \frac{f'(u_0)}{f(u_0)}$.

5 Additional comments

As is known, in an exact solution to the Einstein's field equation the NCC follows from the weak energy condition, even if there is a cosmological constant.

Conversely, consider a GRW spacetime $\overline{M}$ obeying the NCC and Z a timelike vector field on $\overline{M}$. Then from (3) and (4) we can compute the Einstein's tensor $G = \overline{\text{Ric}} - \frac{1}{2}\overline{S}\overline{g}$ evaluated at Z, so obtaining

$$G(Z, Z) = \text{Ric}^F\left(Z^F, Z^F\right) - (n-1)f^2(\log f)'' g_F\left(Z^F, Z^F\right) - \frac{S^F}{2f^2}\overline{g}(Z, Z)$$

$$-\frac{n(n-1)}{2}\frac{f'^2}{f^2}\overline{g}(Z, Z).$$

Hence, $G(Z, Z) \geq 0$ when the scalar curvature of the fiber satisfies $S^F + n(n-1)f'^2 \geq 0$ or equivalently $S^F \geq -n(n-1)\inf_I f'^2$. In particular, it holds when S^F is non-negative. Therefore, under this assumption on the scalar curvature of the fiber a GRW spacetime obeying the NCC satisfies the weak energy condition. Of course, the weak energy condition will also be satisfied if the Einstein's tensor includes the additional term with non-negative cosmological constant.

Recall that the weak energy condition is a natural physical assumption for normal matter. Thus, taking all of this into account, we conclude that GRW spacetimes obeying the NCC and whose fiber has non-negative scalar curvature can be suitable models for realistic universes.

On the other hand, in a GRW spacetime there is a privileged family of observer, that is the observers in the unitary timelike vector field ∂_t, which moreover are proper time synchronizable.

For each $p \in F$ the curve $\gamma_p(t) = (t, p)$ is the *worldline* or *galaxy* of the corresponding observer in ∂_t. Taking t as a constant, we get the hypersurface

$$M(t) = \{(t, p) : p \in F\},$$

which represents the physical space of the observer at the instant t. Then, the distance between two galaxies γ_p and γ_q in $M(t)$ is $f(t)d(p, q)$, where d is the Riemannian distance in the fiber F. In particular, when f has positive (resp. negative) derivative, the spaces $M(t)$ are expanding (resp. contracting). Furthermore, if $f' > 0$ and $f'' > 0$ (resp. $f'' < 0$) the GRW spacetime describes universes in accelerated (resp. decelerated) expansion.

Recall that in a GRW spacetime the timelike energy condition (TCC), which is stronger than the NCC, implies that $f'' \leq 0$. Therefore GRW spacetimes obeying the TCC are not suitable models for accelerated expanding universes. On the contrary, certain GRW spacetimes obeying the NCC can be appropriate models for describing such universes.

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