

Almost sure central limit theorem for self-normalized partial sums and maxima

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Abstract Let X, X_1, X_2, \dots be a sequence of independent and identically distributed random variables with zero mean and finite second moment. A universal result in almost sure central limit theorem for the self-normalized partial sums S_n/V_n and maxima M_n is established, where $S_n = \sum_{i=1}^n X_i$, $V_n^2 = \sum_{i=1}^n X_i^2$, and $M_n = \max_{1 \leq i \leq n} X_i$.

Keywords Almost sure central limit theorem · Self-normalized partial sums · Maxima

Mathematics Subject Classification 60F15

1 Introduction

Throughout this paper we assume $\{X, X_n\}_{n \in \mathbb{N}}$ is a sequence of independent and identically distributed (i.i.d.) random variables with zero mean and finite second moment. For each $1 \leq k \leq n$, let $S_n = \sum_{i=1}^n X_i$, $S_{k,n} = \sum_{i=k+1}^n X_i$, $V_n^2 = \sum_{i=1}^n X_i^2$, $M_n = \max_{1 \leq i \leq n} X_i$, $M_{k,n} = \max_{k+1 \leq i \leq n} X_i$. The symbols S_n/V_n and M_n denote self-normalized partial sums and maxima respectively. Random sequence $\{X_n\}_{n \in \mathbb{N}}$ is said to satisfy the central limit theorem (CLT), or random variable X is said to belong to the domain of attraction of the normal law, if there exist constants $a_n > 0$, $b_n \in \mathbb{R}$ such that $(S_n - b_n)/a_n \xrightarrow{d} \mathcal{N}$, where

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\mathcal{N} is the standard normal random variable and \xrightarrow{d} denotes the convergence in distribution. It is known that CLT is equivalent to

$$\lim_{x \rightarrow \infty} \frac{x^2 \mathbb{P}(|X| > x)}{\mathbb{E}X^2 I(|X| \leq x)} = 0. \tag{1.1}$$

Giné et al. [7] considered the limiting properties of self-normalized partial sums and obtained the following self-normalized version of the central limit theorem: $S_n/V_n \xrightarrow{d} \mathcal{N}$ as $n \rightarrow \infty$ if and only if (1.1) holds.

Brosamer [3] and Schatte [15] obtained the almost sure central limit theorem (ASCLT) for partial sums. Some improved and generalized ASCLT results for partial sums were obtained by Miao [12], Berkes and Csáki [1], Hörmann [8], and Wu [16,17]. Huang and Pang [10], Wu [18], Wu and Chen [21] and Zhang and Yang [23] obtained ASCLT results for self-normalized version. Lin [11] and Cao and Peng [4] obtained ASCLT results for maxima. Further, Zang [22] and Peng et al. [14] obtained the ASCLT result for partial sums and maxima. Peng et al. [14] obtained the following ASCLT result for partial sums and maxima: Let $\{X, X_n\}_{n \in \mathbb{N}}$ be i.i.d. random variables with $EX = 0$ and $EX^2 = 1$. Suppose there exist constants $a_n > 0, b_n \in \mathbb{R}$ and a nondegenerate distribution $G(y)$ such that

$$\lim_{n \rightarrow \infty} P\left(\frac{M_n - b_n}{a_n} \leq y\right) = G(y), \quad -\infty < y < \infty. \tag{1.2}$$

Then

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k I\left(\frac{S_k}{\sqrt{k}} \leq x, \frac{M_k - b_k}{a_k} \leq y\right) = \Phi(x)G(y) \quad \text{a.s. for all } x, y \in \mathbb{R}, \tag{1.3}$$

with $d_k = 1/k$ and $D_n = \sum_{k=1}^n d_k$, where I denotes an indicator function, and $\Phi(x)$ is the standard normal distribution function.

However, ASCLT result for self-normalized partial sums and maxima has not been reported. Because the denominator of self-normalized partial sums contains random variables, so ASCLT for self-normalized partial sums and maxima is more difficult to study.

The difference between CLT and ASCLT lies in the weight in ASCLT. By a classical theorem of Hardy (see e.g. [5]: p. 35), if (1.3) holds with a weight sequence $\{d_k; k \geq 1\}$, then, under certain regularity conditions, it will also hold for all smaller weight sequences. Therefore, one should also expect to get stronger results if we use larger weights. Schatte [15] pointed out that ASCLT fails for weight $d_k = 1$. It would be of considerable interest to determine the optimal weights.

The purpose of this paper is to study and establish the ASCLT for self-normalized partial sums and maxima of i.i.d. random variables, we will show that the ASCLT holds under a fairly general growth condition on $d_k = k^{-1} \exp(\ln^\alpha k), 0 \leq \alpha < 1$.

Our theorem is formulated as follows.

Theorem 1.1 *Let $\{X, X_n\}_{n \in \mathbb{N}}$ be a sequence of i.i.d. random variables with $EX = 0$ and $EX^2 = 1$. Set*

$$d_k = \frac{\exp(\ln^\alpha k)}{k}, \quad D_n = \sum_{k=1}^n d_k, \quad \text{for } 0 \leq \alpha < 1. \tag{1.4}$$

Suppose that (1.2) holds. Then

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k I \left(\frac{S_k}{V_k} \leq x, \frac{M_k - b_k}{a_k} \leq y \right) = \Phi(x)G(y) \quad \text{a.s. for any } x, y \in \mathbb{R}. \tag{1.5}$$

Remark 1.2 By the terminology of summation procedures, Theorem 1.1 remains valid if we replace the weight sequence $\{d_k\}_{k \in \mathbb{N}}$ by any $\{d_k^*\}_{k \in \mathbb{N}}$ such that $0 \leq d_k^* \leq d_k, \sum_{k=1}^{\infty} d_k^* = \infty$.

Remark 1.3 Our results not only extend the ASCLT for partial sums and maxima obtained by Peng et al. [14] to the case of self-normalized partial sums and maxima but also give substantial improvements for weight sequence in Corollary 2.2 in Peng et al. [14].

Remark 1.4 By the Theorem 1 of Schatte [15], for $\alpha = 1$, i.e., $d_k = 1$, ASCLT does not hold. Therefore, in a sense, our Theorem 1.1 has been reached optimal form.

Remark 1.5 Obviously, $\mathbb{E}X^2 = 1 < \infty$ implies that (1.1) holds, so X is in the domain of attraction of the normal law.

2 Proofs

In the following, $a_n \sim b_n$ denotes $\lim_{n \rightarrow \infty} a_n/b_n = 1$ and $a_n \ll b_n$ denotes that there exists a constant $c > 0$ such that $a_n \leq cb_n$ for sufficiently large n . The symbol c stands for a generic positive constant which may differ from one place to another.

To prove Theorem 1.1, the following three lemmas play important role, the Lemma 2.1 is due to Csörgő et al. [6]. Proof of Lemmas 2.2 and 2.3 is very difficult and the proof is given in Appendix.

Lemma 2.1 *Let X be a random variable, and denote $l(x) = \mathbb{E}X^2 I\{|X| \leq x\}$. The following statements are equivalent:*

- (i) X is in the domain of attraction of the normal law.
- (ii) $x^2 \mathbb{P}(|X| > x) = o(l(x))$.
- (iii) $x \mathbb{E}(|X| I(|X| > x)) = o(l(x))$.
- (iv) $\mathbb{E}(|X|^\beta I(|X| \leq x)) = o(x^{\beta-2} l(x))$ for $\beta > 2$.

Lemma 2.2 *Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables, and let $\xi_{k,j} := f(X_{k+1}, \dots, X_j)$ and $\xi_j := \xi_{0,j} = g(X_1, \dots, X_j)$ be two functions which they are only related to X_{k+1}, \dots, X_j and X_1, \dots, X_j , respectively. If there exist constants $c > 0$ and $\delta > 0$ such that*

$$|\xi_{k,j}| \leq c, \quad \text{for any } 0 \leq k < j, \tag{2.1}$$

and

$$|\mathbb{E}\xi_k \xi_j| \ll \left(\frac{k}{j}\right)^\delta, \quad \mathbb{E}|\xi_j - \xi_{k,j}| \ll \left(\frac{k}{j}\right)^\delta, \quad \text{for } 1 \leq k < j, \tag{2.2}$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k \xi_k = 0 \quad \text{a.s.}, \tag{2.3}$$

where d_k and D_n are defined by (1.4).

Let $l(x) = \mathbb{E}X^2 I\{|X| \leq x\}$, $b = \inf\{x \geq 1; l(x) > 0\}$ and

$$\eta_j = \inf \left\{ s; s \geq b + 1, \frac{l(s)}{s^2} \leq \frac{1}{j} \right\} \quad \text{for } j \geq 1.$$

By the definition of η_j , we have $jl(\eta_j) \leq \eta_j^2$ and $jl(\eta_j - \varepsilon) > (\eta_j - \varepsilon)^2$ for any $\varepsilon > 0$. It implies that

$$nl(\eta_n) \sim \eta_n^2, \quad \text{as } n \rightarrow \infty. \tag{2.4}$$

Let

$$\bar{X}_{ni} = X_i I(|X_i| \leq \eta_n), \quad \bar{S}_n = \sum_{i=1}^n \bar{X}_{ni}, \quad \bar{V}_n^2 = \sum_{i=1}^n \bar{X}_{ni}^2, \quad \text{for } 1 \leq i \leq n,$$

$$\bar{S}_{k,n} = \sum_{i=k+1}^n \bar{X}_{ni}, \quad \bar{V}_{k,n}^2 = \sum_{i=k+1}^n \bar{X}_{ni}^2, \quad \text{for } 0 \leq k < n.$$

Lemma 2.3 *Suppose that the assumptions of Theorem 1.1 hold. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k I \left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\sqrt{kl(\eta_k)}} \leq x, \frac{M_k - b_k}{a_k} \leq y \right) = \Phi(x)G(y) \quad \text{a.s. for any } x, y \in \mathbb{R}, \tag{2.5}$$

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k \left(I \left(\bigcup_{i=1}^k (|X_i| > \eta_k) \right) - \mathbb{E} I \left(\bigcup_{i=1}^k (|X_i| > \eta_k) \right) \right) = 0 \quad \text{a.s.}, \tag{2.6}$$

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k \left(f \left(\frac{\bar{V}_k^2}{kl(\eta_k)} \right) - \mathbb{E} f \left(\frac{\bar{V}_k^2}{kl(\eta_k)} \right) \right) = 0 \quad \text{a.s.}, \tag{2.7}$$

where d_k and D_n are defined by (1.4) and f is a non-negative, bounded Lipschitz function.

Proof of Theorem 1.1. For any given $0 < \varepsilon < 1$, note that

$$\begin{aligned} I \left(\frac{S_k}{V_k} \leq x, \frac{M_k - b_k}{a_k} \leq y \right) &\leq I \left(\frac{\bar{S}_k}{\sqrt{(1 + \varepsilon)kl(\eta_k)}} \leq x, \frac{M_k - b_k}{a_k} \leq y \right) \\ &+ I(\bar{V}_k^2 > (1 + \varepsilon)kl(\eta_k)) + I \left(\bigcup_{i=1}^k (|X_i| > \eta_k) \right), \quad \text{for } x \geq 0, \end{aligned}$$

$$\begin{aligned} I \left(\frac{S_k}{V_k} \leq x, \frac{M_k - b_k}{a_k} \leq y \right) &\leq I \left(\frac{\bar{S}_k}{\sqrt{(1 - \varepsilon)kl(\eta_k)}} \leq x, \frac{M_k - b_k}{a_k} \leq y \right) \\ &+ I(\bar{V}_k^2 < (1 - \varepsilon)kl(\eta_k)) + I \left(\bigcup_{i=1}^k (|X_i| > \eta_k) \right), \quad \text{for } x < 0, \end{aligned}$$

and

$$\begin{aligned} I \left(\frac{S_k}{V_k} \leq x, \frac{M_k - b_k}{a_k} \leq y \right) &\geq I \left(\frac{\bar{S}_k}{\sqrt{(1 - \varepsilon)kl(\eta_k)}} \leq x, \frac{M_k - b_k}{a_k} \leq y \right) \\ &- I(\bar{V}_k^2 < (1 - \varepsilon)kl(\eta_k)) - I \left(\bigcup_{i=1}^k (|X_i| > \eta_k) \right), \quad \text{for } x \geq 0, \end{aligned}$$

$$I\left(\frac{S_k}{V_k} \leq x, \frac{M_k - b_k}{a_k} \leq y\right) \geq I\left(\frac{\bar{S}_k}{\sqrt{(1+\varepsilon)kl(\eta_k)}} \leq x, \frac{M_k - b_k}{a_k} \leq y\right) - I(\bar{V}_k^2 > (1+\varepsilon)kl(\eta_k)) - I\left(\bigcup_{i=1}^k (|X_i| > \eta_k)\right), \text{ for } x < 0.$$

Hence, to prove (1.5), it suffices to prove

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k I\left(\frac{\bar{S}_k}{\sqrt{kl(\eta_k)}} \leq x\sqrt{1 \pm \varepsilon}, \frac{M_k - b_k}{a_k} \leq y\right) = \Phi(x\sqrt{1 \pm \varepsilon})G(y) \text{ a.s.}, \tag{2.8}$$

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k I\left(\bigcup_{i=1}^k (|X_i| > \eta_k)\right) = 0 \text{ a.s.}, \tag{2.9}$$

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k I(\bar{V}_k^2 > (1+\varepsilon)kl(\eta_k)) = 0 \text{ a.s.}, \tag{2.10}$$

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k I(\bar{V}_k^2 < (1-\varepsilon)kl(\eta_k)) = 0 \text{ a.s.} \tag{2.11}$$

by the arbitrariness of $\varepsilon > 0$.

We first prove (2.8). Let $0 < \beta < 1/2$ and $h(\cdot, \cdot)$ be a real function, such that for any given $x, y \in \mathbb{R}$,

$$I\left(\chi \leq \sqrt{1 \pm \varepsilon}x - \beta, \frac{M_k - b_k}{a_k} \leq y\right) \leq h(\chi, y) \leq I\left(\chi \leq \sqrt{1 \pm \varepsilon}x + \beta, \frac{M_k - b_k}{a_k} \leq y\right). \tag{2.12}$$

Obviously, $\mathbb{E}X^2 = 1 < \infty$ implies that (1.1) holds, so X is in the domain of attraction of the normal law. By $\mathbb{E}X = 0$, Lemma 2.1 (iii) and (2.4), we have

$$\begin{aligned} |\mathbb{E}\bar{S}_k| &= |k\mathbb{E}XI(|X| \leq \eta_k)| = |k\mathbb{E}XI(|X| > \eta_k)| \leq k\mathbb{E}|X|I(|X| > \eta_k) \\ &= o(\sqrt{kl(\eta_k)}), \text{ as } k \rightarrow \infty. \end{aligned}$$

This, combining with (2.5), (2.12) and the arbitrariness of β in (2.12), (2.8) holds.

By Lemma 2.1 (ii) and (2.4), we get

$$\mathbb{P}(|X| > \eta_j) = o(1)\frac{l(\eta_j)}{\eta_j^2} = \frac{o(1)}{j}, \text{ as } j \rightarrow \infty. \tag{2.13}$$

This, combining with (2.6) and the Toeplitz lemma,

$$\begin{aligned} 0 &\leq \frac{1}{D_n} \sum_{k=1}^n d_k I\left(\bigcup_{i=1}^k (|X_i| > \eta_k)\right) \sim \frac{1}{D_n} \sum_{k=1}^n d_k \mathbb{E}I\left(\bigcup_{i=1}^k (|X_i| > \eta_k)\right) \\ &\leq \frac{1}{D_n} \sum_{k=1}^n d_k k \mathbb{P}(|X| > \eta_k) \rightarrow 0 \text{ a.s.}, \text{ as } n \rightarrow \infty. \end{aligned}$$

That is (2.9) holds.

Now we prove (2.10). For any $\mu > 0$, let f be a non-negative, bounded Lipschitz function such that

$$I(x > 1 + \mu) \leq f(x) \leq I(x > 1 + \mu/2).$$

From $\mathbb{E}\bar{V}_k^2 = kl(\eta_k)$, \bar{X}_{ni} is i.i.d., Lemma 2.1 (iv), and (2.4),

$$\begin{aligned} \mathbb{P}\left(\bar{V}_k^2 > \left(1 + \frac{\mu}{2}\right)kl(\eta_k)\right) &= \mathbb{P}\left(\bar{V}_k^2 - \mathbb{E}\bar{V}_k^2 > \frac{\mu}{2}kl(\eta_k)\right) \\ &\ll \frac{\mathbb{E}(\bar{V}_k^2 - \mathbb{E}\bar{V}_k^2)^2}{k^2I^2(\eta_k)} \ll \frac{\mathbb{E}X^4I(|X| \leq \eta_k)}{kl^2(\eta_k)} \\ &= \frac{o(1)\eta_k^2}{kl(\eta_k)} = o(1) \rightarrow 0, \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Therefore, from (2.7) and the Toeplitz lemma,

$$\begin{aligned} 0 &\leq \frac{1}{D_n} \sum_{k=1}^n d_k I(\bar{V}_k^2 > (1 + \mu)kl(\eta_k)) \leq \frac{1}{D_n} \sum_{k=1}^n d_k f\left(\frac{\bar{V}_k^2}{kl(\eta_k)}\right) \\ &\sim \frac{1}{D_n} \sum_{k=1}^n d_k \mathbb{E}f\left(\frac{\bar{V}_k^2}{kl(\eta_k)}\right) \leq \frac{1}{D_n} \sum_{k=1}^n d_k \mathbb{E}I(\bar{V}_k^2 > (1 + \mu/2)kl(\eta_k)) \\ &= \frac{1}{D_n} \sum_{k=1}^n d_k \mathbb{P}(\bar{V}_k^2 > (1 + \mu/2)kl(\eta_k)) \\ &\rightarrow 0 \text{ a.s., as } n \rightarrow \infty. \end{aligned}$$

Hence, (2.10) holds. By similar methods used to prove (2.10), we can prove (2.11). This completes the proof of Theorem 1.1.

3 Appendix

Proof of Lemma 2.2 Without loss of generality, we can suppose that $\alpha > 0$. By the proof of Lemma 2.2 in Wu [19], we know that (2.1) and (2.2) imply the following formula.

$$\mathbb{E}\left|\sum_{k=1}^n d_k \xi_k\right|^p \ll \left(\sum_{1 \leq k \leq j \leq n} d_k d_j \left(\frac{k}{j}\right)^\delta\right)^{p/2} \quad \text{for any } p > 0. \tag{3.1}$$

Note that

$$\begin{aligned} \sum_{1 \leq k \leq j \leq n} d_k d_j \left(\frac{k}{j}\right)^\delta &\leq \sum_{1 \leq k \leq j \leq n, k/j \leq (\ln D_n)^{-2/\delta}} d_k d_j \left(\frac{k}{j}\right)^\delta + \sum_{1 \leq k \leq j \leq n, k/j > (\ln D_n)^{-2/\delta}} d_k d_j \\ &:= T_{n_1} + T_{n_2}. \end{aligned} \tag{3.2}$$

$$T_{n_1} \leq \sum_{1 \leq k \leq j \leq n, k/j \leq (\ln D_n)^{-2/\delta}} d_k d_j \frac{1}{\ln^2 D_n} \leq \frac{D_n^2}{\ln^2 D_n}. \tag{3.3}$$

By (2.10) in Wu [20],

$$D_n \sim \frac{1}{\alpha} \ln^{1-\alpha} n \exp(\ln^\alpha n), \quad \ln D_n \sim \ln^\alpha n, \quad \exp(\ln^\alpha n) \sim \frac{\alpha D_n}{(\ln D_n)^{\frac{1-\alpha}{\alpha}}}, \quad \ln \ln D_n \sim \alpha \ln \ln n. \quad (3.4)$$

Hence,

$$T_{n_2} \leq \sum_{k=1}^n d_k \sum_{k \leq j < k(\ln D_n)^{2/\delta}} \frac{1}{j} \exp(\ln^\alpha n) \ll \frac{D_n}{(\ln D_n)^{\frac{1-\alpha}{\alpha}}} \sum_{k=1}^n d_k \ln \ln D_n \ll \frac{D_n^2 \ln \ln D_n}{(\ln D_n)^{\frac{1-\alpha}{\alpha}}}. \quad (3.5)$$

Thus, let $\alpha_1 = \min(2, (1 - \alpha)/\alpha) > 0$, by (3.2), (3.3) and (3.5), we get

$$\sum_{1 \leq k \leq l \leq n} d_k d_l \left(\frac{k}{l}\right)^\delta \ll \frac{D_n^2 \ln \ln D_n}{(\ln D_n)^{\alpha_1}}. \quad (3.6)$$

Let $p > 2(3\alpha + 1)/(\alpha_1\alpha)$, i.e., $\alpha_1 p/2 - 1 > 2$, by the Markov inequality, $(\ln \ln D_n)^{p/2} = o(\ln D_n)$ for any $p > 0$, (3.1) and (3.6), for sufficiently large n , we have

$$\begin{aligned} \mathbb{P}\left(\left|\frac{1}{D_n} \sum_{k=1}^n d_k \xi_k\right| > \varepsilon\right) &\ll \frac{1}{D_n^p} \mathbb{E}\left|\sum_{k=1}^n d_k \xi_k\right|^p \ll \frac{1}{D_n^p} \left(\sum_{1 \leq k \leq l \leq n} d_k d_l \left(\frac{k}{l}\right)^\delta\right)^{p/2} \\ &\ll \frac{1}{D_n^p} \left(\frac{D_n^2 \ln \ln D_n}{(\ln D_n)^{\alpha_1}}\right)^{p/2} \ll \frac{1}{(\ln D_n)^{\alpha_1 p/2 - 1}} \\ &\leq \frac{1}{\ln^2 D_n}. \end{aligned}$$

By (3.4), we have $D_{n+1} \sim D_n$. Let $n_k = \inf\{n; D_n \geq \exp(k^{2/3})\}$, then $D_{n_k} \geq \exp(k^{2/3})$, $D_{n_k-1} < \exp(k^{2/3})$. Therefore

$$1 \leq \frac{D_{n_k}}{\exp(k^{2/3})} \sim \frac{D_{n_k-1}}{\exp(k^{2/3})} < 1, \quad k \rightarrow \infty,$$

i.e.,

$$D_{n_k} \sim \exp(k^{2/3}).$$

Therefore, let $T_n := \frac{1}{D_n} \sum_{i=1}^n d_i \xi_i$, we have

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\left|\frac{1}{D_{n_k}} \sum_{i=1}^{n_k} d_i \xi_i\right| > \varepsilon\right) \ll \sum_{k=1}^{\infty} \frac{1}{k^{4/3}} < \infty,$$

i.e.,

$$T_{n_k} \rightarrow 0 \quad \text{a.s.}$$

For $n_k \leq n < n_{k+1}$, by (2.1)

$$|T_n| \leq |T_{n_k}| + \frac{2c}{D_{n_k}} (D_{n_{k+1}} - D_{n_k}) \rightarrow 0 \quad \text{a.s.}$$

from $D_{n_{k+1}}/D_{n_k} = \exp((k+1)^{2/3} - k^{2/3}) = \exp(k^{2/3}[(1+1/k)^{2/3} - 1]) \sim \exp(2k^{-1/3}/3) \rightarrow 1$. Therefore, (2.3) holds. This completes the proof of Lemma 2.2.

Proof of Lemma 2.3 By the central limit theorem for i.i.d. random variables and $\text{Var}\bar{S}_n \sim nl(\eta_n)$ as $n \rightarrow \infty$ from $\mathbb{E}X = 0$, Lemma 2.1 (iii), and (2.4), it follows that

$$\frac{\bar{S}_n - \mathbb{E}\bar{S}_n}{\sqrt{nl(\eta_n)}} \xrightarrow{d} \mathcal{N}, \text{ as } n \rightarrow \infty,$$

where \mathcal{N} denotes the standard normal random variable. By Theorem 1.1 in Hsing [9], we get

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{\bar{S}_n - \mathbb{E}\bar{S}_n}{\sqrt{nl(\eta_n)}} \leq x, \frac{M_n - b_n}{a_n} \leq y \right) = \Phi(x)G(y) \text{ for any } x, y \in \mathbb{R}.$$

This implies that for any $g(x, y)$ which is a non-negative, bounded Lipschitz function

$$\lim_{n \rightarrow \infty} \mathbb{E}g \left(\frac{\bar{S}_n - \mathbb{E}\bar{S}_n}{\sqrt{nl(\eta_n)}}, \frac{M_n - b_n}{a_n} \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\Phi(dx)G(dy).$$

Hence, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k \mathbb{E}g \left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\sqrt{kl(\eta_k)}}, \frac{M_k - b_k}{a_k} \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\Phi(dx)G(dy)$$

from the Toeplitz lemma.

On the other hand, note that (2.5) is equivalent to

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k g \left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\sqrt{kl(\eta_k)}}, \frac{M_k - b_k}{a_k} \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\Phi(dx)G(dy) \text{ a.s.}$$

from Theorem 7.1 of Billingsley [2] and Section 2 of Peligrad and Shao [13]. Hence, to prove (2.5), it suffices to prove

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k \left(g \left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\sqrt{kl(\eta_k)}}, \frac{M_k - b_k}{a_k} \right) - \mathbb{E}g \left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\sqrt{kl(\eta_k)}}, \frac{M_k - b_k}{a_k} \right) \right) = 0 \text{ a.s.}, \tag{3.7}$$

for any $g(x, y)$ which is a non-negative, bounded Lipschitz function.

For any $1 \leq k < j$, let

$$\begin{aligned} \xi_k &:= g \left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\sqrt{kl(\eta_k)}}, \frac{M_k - b_k}{a_k} \right) - \mathbb{E}g \left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\sqrt{kl(\eta_k)}}, \frac{M_k - b_k}{a_k} \right), \\ \xi_{k,j} &:= g \left(\frac{\bar{S}_{k,j} - \mathbb{E}\bar{S}_{k,j}}{\sqrt{jl(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right) - \mathbb{E}g \left(\frac{\bar{S}_{k,j} - \mathbb{E}\bar{S}_{k,j}}{\sqrt{jl(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right). \end{aligned}$$

For any $1 \leq k < j$, noting that $g \left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\sqrt{kl(\eta_k)}}, \frac{M_k - b_k}{a_k} \right)$ and $g \left(\frac{\bar{S}_{k,j} - \mathbb{E}\bar{S}_{k,j}}{\sqrt{jl(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right)$ are independent, and the fact that $g(x, y)$ is a non-negative, bounded Lipschitz function, it is easy to see that

$$\begin{aligned}
|\mathbb{E}\xi_k\xi_j| &= \left| \text{Cov} \left(g \left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\sqrt{kl(\eta_k)}}, \frac{M_k - b_k}{a_k} \right), g \left(\frac{\bar{S}_j - \mathbb{E}\bar{S}_j}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_j - b_j}{a_j} \right) \right) \right| \\
&\leq \left| \text{Cov} \left(g \left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\sqrt{kl(\eta_k)}}, \frac{M_k - b_k}{a_k} \right), g \left(\frac{\bar{S}_j - \mathbb{E}\bar{S}_j}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_j - b_j}{a_j} \right) \right. \right. \\
&\quad \left. \left. - g \left(\frac{\bar{S}_j - \mathbb{E}\bar{S}_j}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right) \right) \right| \\
&\quad + \left| \text{Cov} \left(g \left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\sqrt{kl(\eta_k)}}, \frac{M_k - b_k}{a_k} \right), g \left(\frac{\bar{S}_j - \mathbb{E}\bar{S}_j}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right) \right. \right. \\
&\quad \left. \left. - g \left(\frac{\bar{S}_{k,j} - \mathbb{E}\bar{S}_{k,j}}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right) \right) \right| \\
&\quad + \left| \text{Cov} \left(g \left(\frac{\bar{S}_k - \mathbb{E}\bar{S}_k}{\sqrt{kl(\eta_k)}}, \frac{M_k - b_k}{a_k} \right), g \left(\frac{\bar{S}_{k,j} - \mathbb{E}\bar{S}_{k,j}}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right) \right) \right| \\
&\ll \mathbb{E} \left| g \left(\frac{\bar{S}_j - \mathbb{E}\bar{S}_j}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_j - b_j}{a_j} \right) - g \left(\frac{\bar{S}_j - \mathbb{E}\bar{S}_j}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right) \right| \\
&\quad + \mathbb{E} \left| g \left(\frac{\bar{S}_j - \mathbb{E}\bar{S}_j}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right) - g \left(\frac{\bar{S}_{k,j} - \mathbb{E}\bar{S}_{k,j}}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right) \right| \\
&:= H_1 + H_2. \tag{3.8}
\end{aligned}$$

From the fact that $g(x, y)$ is a non-negative, bounded Lipschitz function, it follows that

$$H_1 \ll \mathbb{E} \left(\min \left(\frac{M_j - M_{k,j}}{a_j}, 2 \right) \right) \ll \mathbb{P}(M_j \neq M_{k,j}) = \mathbb{P}(M_j > M_{k,j}) \ll \frac{k}{j}. \tag{3.9}$$

By the definition of η_j and Cauchy–Schwarz inequality, we get

$$H_2 \ll \mathbb{E} \left| \frac{\bar{S}_j - \bar{S}_{k,j} - \mathbb{E}(\bar{S}_j - \bar{S}_{k,j})}{\sqrt{j\bar{l}(\eta_j)}} \right| \ll \frac{\sqrt{k\mathbb{E}X^2I(|X| \leq \eta_j)}}{\sqrt{j\bar{l}(\eta_j)}} = \left(\frac{k}{j} \right)^{1/2}. \tag{3.10}$$

On the other hand, by (3.9) and (3.10),

$$\begin{aligned}
|\mathbb{E}\xi_j - \xi_{k,j}| &\ll \mathbb{E} \left| g \left(\frac{\bar{S}_j - \mathbb{E}\bar{S}_j}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_j - b_j}{a_j} \right) - g \left(\frac{\bar{S}_{k,j} - \mathbb{E}\bar{S}_{k,j}}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right) \right| \\
&\leq \mathbb{E} \left| g \left(\frac{\bar{S}_j - \mathbb{E}\bar{S}_j}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_j - b_j}{a_j} \right) - g \left(\frac{\bar{S}_j - \mathbb{E}\bar{S}_j}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right) \right| \\
&\quad + \mathbb{E} \left| g \left(\frac{\bar{S}_j - \mathbb{E}\bar{S}_j}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right) - g \left(\frac{\bar{S}_{k,j} - \mathbb{E}\bar{S}_{k,j}}{\sqrt{j\bar{l}(\eta_j)}}, \frac{M_{k,j} - b_j}{a_j} \right) \right| \\
&= H_1 + H_2 \ll \left(\frac{k}{j} \right)^{1/2}. \tag{3.11}
\end{aligned}$$

By Lemma 2.2, (3.7) holds from (3.8)–(3.11), i.e., (2.5) holds.

In a similar way, we prove (2.6). For any $1 \leq k < j$, let

$$Z_k := I\left(\bigcup_{i=1}^k (|X_i| > \eta_k)\right) - \mathbb{E}I\left(\bigcup_{i=1}^k (|X_i| > \eta_k)\right),$$

and

$$Z_{k,j} := I\left(\bigcup_{i=k+1}^j (|X_i| > \eta_j)\right) - \mathbb{E}I\left(\bigcup_{i=k+1}^j (|X_i| > \eta_j)\right).$$

It is known that $I(A \cup B) - I(B) \leq I(A)$ for any sets A and B , then for $1 \leq k < j$, by (2.13),

$$\begin{aligned} |\mathbb{E}Z_k Z_j| &= \left| \text{Cov}\left(I\left(\bigcup_{i=1}^k (|X_i| > \eta_k)\right), I\left(\bigcup_{i=1}^j (|X_i| > \eta_j)\right)\right) \right| \\ &= \left| \text{Cov}\left(I\left(\bigcup_{i=1}^k (|X_i| > \eta_k)\right), I\left(\bigcup_{i=1}^j (|X_i| > \eta_j)\right) - I\left(\bigcup_{i=k+1}^j (|X_i| > \eta_j)\right)\right) \right| \\ &\leq \mathbb{E}\left| I\left(\bigcup_{i=1}^j (|X_i| > \eta_j)\right) - I\left(\bigcup_{i=k+1}^j (|X_i| > \eta_j)\right) \right| \\ &\leq \mathbb{E}I\left(\bigcup_{i=1}^k (|X_i| > \eta_j)\right) \leq k\mathbb{P}(|X| > \eta_j) \\ &\leq \frac{k}{j}, \end{aligned}$$

and

$$\mathbb{E}|Z_j - Z_{k,j}| \ll \mathbb{E}I\left(\bigcup_{i=1}^k (|X_i| > \eta_j)\right) \leq \frac{k}{j}.$$

By Lemma 2.2, (2.6) holds.

Finally, we prove (2.7). For any $1 \leq k < j$, let

$$\zeta_k := f\left(\frac{\bar{V}_k^2}{kl(\eta_k)}\right) - \mathbb{E}f\left(\frac{\bar{V}_k^2}{kl(\eta_k)}\right),$$

and

$$\zeta_{k,j} := f\left(\frac{\bar{V}_{k,j}^2}{jl(\eta_j)}\right) - \mathbb{E}f\left(\frac{\bar{V}_{k,j}^2}{jl(\eta_j)}\right).$$

For $1 \leq k < j$, noting that $f\left(\frac{\bar{V}_k^2}{\sqrt{kl(\eta_k)}}\right)$ and $f\left(\frac{\bar{V}_{k,j}^2}{\sqrt{jl(\eta_j)}}\right)$ are independent, we get

$$\begin{aligned} \mathbb{E}|\zeta_k \zeta_j| &= \left| \text{Cov} \left(f \left(\frac{\bar{V}_k^2}{kl(\eta_k)} \right), f \left(\frac{\bar{V}_j^2}{jl(\eta_j)} \right) \right) \right| \\ &= \left| \text{Cov} \left(f \left(\frac{\bar{V}_k^2}{kl(\eta_k)} \right), f \left(\frac{\bar{V}_j^2}{jl(\eta_j)} \right) - f \left(\frac{\bar{V}_{k,j}^2}{jl(\eta_j)} \right) \right) \right| \\ &\ll \frac{\mathbb{E} \left(\sum_{i=1}^k X_i^2 I(|X_i| \leq \eta_j) \right)}{jl(\eta_j)} = \frac{k \mathbb{E} X^2 I(|X| \leq \eta_j)}{jl(\eta_j)} = \frac{kl(\eta_j)}{jl(\eta_j)} \\ &= \frac{k}{j}, \end{aligned}$$

and

$$\mathbb{E}|\zeta_j - \zeta_{k,j}| \ll \mathbb{E} \left| f \left(\frac{\bar{V}_j^2}{jl(\eta_j)} \right) - f \left(\frac{\bar{V}_{k,j}^2}{jl(\eta_j)} \right) \right| \ll \frac{\mathbb{E} \left(\sum_{i=1}^k X_i^2 I(|X_i| \leq \eta_j) \right)}{jl(\eta_j)} = \frac{k}{j},$$

By Lemma 2.2, (2.7) holds. This completes the proof of Lemma 2.3.

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