RACSAM (2016) 110:667–679
DOI 10.1007/s13398-015-0256-0

ORIGINAL PAPER

Mackey topology on locally convex spaces and on locally quasi-convex groups. Similarities and historical remarks

Elena Martín-Peinador[1] · Vaja Tarieladze[2]

Received: 4 April 2015 / Accepted: 10 October 2015 / Published online: 2 November 2015

Abstract A counterpart of the Mackey–Arens Theorem for the class of locally quasi-convex topological Abelian groups (LQC-groups) was initiated in Chasco et al. (Stud Math 132(3):257–284, 1999). Several authors have been interested in the problems posed there and have done clarifying contributions, although the main question of that source remains open. Some differences between the Mackey Theory for locally convex spaces and for locally quasi-convex groups, stem from the following fact: *The supremum of all compatible locally quasi-convex topologies for a topological abelian group G may not coincide with the topology of uniform convergence on the weak quasi-convex compact subsets of the dual group $G^\wedge$.* Thus, a substantial part of the classical Mackey–Arens Theorem cannot be generalized to LQC-groups. Furthermore, the mentioned fact gives rise to a grading in the property of "being a Mackey group", as defined and thoroughly studied in Díaz Nieto and Martín-Peinador (Proceedings in Mathematics and Statistics 80:119–144, 2014). At present it is not known—and this is the main open question—if the supremum of all the compatible locally quasi-convex topologies on a topological group is in fact a compatible topology. In the present paper we do a sort of historical review on the Mackey Theory, and we compare it in the two settings of locally convex spaces and of locally quasi-convex groups. We point out some general questions which are still open, under the name of Problems.

The authors thank the IMI for supporting a visit to U.C.M. of the second author along the last term of 2014.
E. Martín-Peinador was partially supported by the Spanish Ministerio de Economía y Competitividad. Project: MTM 2013-42486-P.
V. Tarieladze was supported by Shota Rustaveli National Science Foundation Grant no. FR/539/5-100/13.

✉ Elena Martín-Peinador
em_peinador@mat.ucm.es

Vaja Tarieladze
vajatarieladze@yahoo.com

1 Instituto de Matemática Interdisciplinar y Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

2 Niko Muskhelishvili Institute of Computational Mathematics of the Georgian Technical University, 0171 Tbilisi, Georgia

Keywords Locally convex space · Locally quasi-convex group · Precompact group · Pre-locally compact group · Compatible topology · Mackey topology

Mathematics Subject Classification Primary 54C40 · 14E20; Secondary 46E25 · 20C20

1 Introduction

The Mackey–Arens Theorem asserts that for a given real (or complex) vector space duality (E, F) there always exists on E the finest *locally convex* vector space topology *compatible* with (E, F). J. Kakol was the first to realize that local convexity is an essential requirement in the Theorem of Mackey–Arens; in other words, for a given real vector space duality (E, F) there may not exist on E the finest vector space topology compatible with (E, F) [21].

The general group dualities were introduced by Varopoulos in [36], where an attempt to obtain an analogue of the Mackey–Arens Theorem is done. The author dealt with compatible pre-locally compact topologies and their projective limits. After 25 years of [36] appeared [11] in which the study of general group dualities was renewed and for them the investigation of the wider class of compatible locally quasi-convex topologies was fully developed.

In the present paper we mainly give a survey of the duality of groups from the very beginning. To this end, we first describe some features of its predecessor: the duality of vector spaces, giving a few historical data about the development of the Mackey topology for spaces. Then, starting from the first definitions of duality of groups done in [36], we clarify some points of our root paper [11], and give an account of the progress since then. We present the state of the art, stressing the differences and the similarities which appear in these two settings: the Mackey topology for topological vector spaces and the Mackey topology for topological abelian groups. In particular, we see how Propositions 2.2, 2.4 and 2.5 develop in this new context of groups. We also point out as Problems some questions which are still open.

2 Vector space dualities

In what follows we consider vector spaces over a complete non-discrete valued field $\mathbb{K}$, which in many (but not all) cases will be either the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers.

For a vector space E we denote by E^a the algebraic dual (or the conjugate) vector space of E consisting of all linear functionals $f: E \to \mathbb{K}$.

For a topologized vector space E the symbol E' will stand for the (topological) dual (or the conjugate) vector space of E consisting of all continuous linear functionals $x': E \to \mathbb{K}$.

For an infinite-dimensional Hausdorff topological vector space E it may happen that $E' = \{0\}$. Such a space and the corresponding topology sometimes is called *dual-less*. A topological vector space E for which E' separates points of E is called *dually separated* or *dual-separated*; a commonly accepted term does not exist. The Hahn–Banach theorem implies that every Hausdorff locally convex topological vector space over $\mathbb{R}$ or $\mathbb{C}$ is dually separated.

A vector space duality is a pair (E, F), where E is a vector space (without a topology) and F is a vector subspace of E^a. A vector space duality (E, F) is called *separating* if F separates point of E. Two canonical topologies are associated to a vector space duality (E, F). Namely

$\sigma(E, F)$ defined as the coarsest topology in E which makes continuous all members of F, and $\sigma(F, E)$ defined as the coarsest topology in F which makes continuous the functions $x' \mapsto l_x(x') := x'(x)$, $x \in E$ (in other words, $\sigma(F, E)$ is the topology of point-wise convergence in F). The pair $(E, \sigma(E, F))$ is a (locally convex if $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) topological vector space which is Hausdorff iff (E, F) is a separating duality, while $(F, \sigma(F, E))$ is always a Hausdorff (locally convex if $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) topological vector space. A topology $\mathcal{T}$ in E is called *compatible with the duality* (E, F) if F coincides with the set of all $\mathcal{T}$-continuous linear functionals $x' \colon E \to \mathbb{K}$. In a similar way, a topology $\mathcal{T}'$ in F is called *compatible with the duality* (E, F) if the set $\{l_x : x \in E\}$ coincides with the set of all $\mathcal{T}'$-continuous linear functionals $l \colon F \to \mathbb{K}$. A topology $\mathcal{T}$ in a vector space E is called a *vector space topology* if the pair $(E, \mathcal{T})$ is a topological vector space. For a vector space duality (E, F) the topologies $\sigma(E, F)$ and $\sigma(F, E)$ are (locally convex if $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) vector space topologies on E and F respectively, which are compatible with the duality (E, F). As a rule, for a given vector space duality (E, F) there can be other topologies on E and F distinct from $\sigma(E, F)$ and $\sigma(F, E)$ which are also compatible with (E, F). Clearly, $\sigma(E, F)$ and $\sigma(F, E)$ are minimum with this condition. These (not very deep but important) statements, as well as the notion of a vector space duality, can be found already in [14].

For a vector space duality (E, F) over $\mathbb{R}$ or $\mathbb{C}$ let $\tau(E, F)$ be the least upper bound (in the lattice of all topologies on E) of the set of *all locally convex vector space topologies* on E which are compatible with (E, F). The topology $\tau(F, E)$ on F is defined in a similar way.

Theorem 2.1 (The Mackey–Arens Theorem) *Let (E, F) be a vector space duality over $\mathbb{R}$ or $\mathbb{C}$. Then:*

(a) *The topology $\tau(E, F)$ is compatible with the duality (E, F).*
(b) *The topology $\tau(E, F)$ coincides with the topology of uniform convergence on all $\sigma(F, E)$-compact absolutely convex subsets of F.*

If (E, F) is a separating duality, then similar statements hold for the topology $\tau(F, E)$ as well.

Theorem 2.1(a) was announced by Mackey [29] and it was proved by the same author in [30]. Independently and in a different way, Theorem 2.1 was also proved by Arens [1].

For a vector space duality (E, F) the notation $\tau(E, F)$ and the above given formulation of the Mackey–Arens Theorem first appeared in the seminal paper [15, pp. 64–65]; while the topology $\tau(E, F)$ was named as *the Mackey topology* in the text of Bourbaki [10]. In the Literature the symbol $\tau(E, F)$ for the Mackey topology is not standard; it is called $\mu(E, F)$ in the book of Jarchow [19] and it is denoted by $m(E, F)$ in the book by Pérez Carreras and Bonet [31].

The following statement expresses a remarkable property of the Mackey topology.

Proposition 2.2 [19, Corollary 8.6.5 (p. 161)] *Let (E_1, F_1) and (E_2, F_2) be vector space dualities over $\mathbb{R}$ or $\mathbb{C}$ and $u \colon E_1 \to E_2$ a linear mapping, which is continuous with respect to the weak topologies $\sigma(E_1, F_1)$ and $\sigma(E_2, F_2)$. Then u is continuous with respect to the the Mackey topologies $\tau(E_1, F_1)$ and $\tau(E_2, F_2)$.*

To formulate a variant of Theorem 2.1 we recall that a topology $\mathcal{T}$ on a vector space E over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is *a topological vector group topology*, for short, *a* TVG-*topology*, if the pair $(E, \mathcal{T})$ is a topological vector space over $\mathbb{K}$ endowed with the discrete topology.

For a vector space duality (E, F) over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ let $\kappa(E, F)$ be the least upper bound (in the lattice of all topologies on E) of the set of *all locally convex* TVG-*topologies* on E which are compatible with (E, F).

Theorem 2.3 [27, Theorem 4] *Let (E, F) be a separating vector space duality over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then:*

(a) *The topology $\kappa(E, F)$ is compatible with the duality (E, F).*
(b) *The topology $\kappa(E, F)$ coincides with the topology of uniform convergence on all $\sigma(F, E)$-complete absolutely convex subsets of F.*

Note that for a separating vector space duality (E, F) over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we have $\tau(E, F) \subset \kappa(E, F)$; it would be interesting to characterize the dualities for which this inclusion is strict.

A topological vector space E gives rise in a natural way to a duality (E, E'), where E' is the dual space of E. This duality in general may not be separating. For a topological vector space E the topology $\sigma(E, E')$ is called *the weak* (*or weakened*) *topology of* E and for a topological vector space E over $\mathbb{R}$ or $\mathbb{C}$ the topology $\tau(E, E')$ is called *the Mackey topology of* E. These topologies are Hausdorff iff E is dually separated. For a topological vector space E over $\mathbb{R}$ or $\mathbb{C}$ we have $\sigma(E, E') \subset \tau(E, E')$ and this inclusion is strict if E is an infinite-dimensional normed space.

A (locally convex) topological vector space E over $\mathbb{R}$ or $\mathbb{C}$ is a *Mackey space* if its original topology coincides with $\tau(E, E')$. It seems that the term "a Mackey space" first appeared in [26, Chapter 5, Section 18]. It is also used by Schaefer [32], Valdivia [34] and other classical texts.

This concept is also presented in the above cited papers of Mackey under the name *a relatively strong space.*

From Mackey–Arens theorem we get the following characterization of Mackey spaces.

Proposition 2.4 *For a locally convex topological vector space E over $\mathbb{R}$ or $\mathbb{C}$. TFAE:*

(i) *E is a Mackey space.*
(ii) *Every $\sigma(E', E)$-compact convex subset of E' is equicontinuous.*

We also have:

Proposition 2.5 [26, Corollary 22.3] *Every metrizable locally convex topological vector space is a Mackey space.*

The class of Mackey spaces has good permanence properties, for instance:

(1) a quotient of a Mackey space with respect to a closed vector subspace is Mackey,
(2) a space having a dense Mackey vector subspace is a Mackey space,
(3) a complemented closed vector subspace of a Mackey space is Mackey (this may not be true for an arbitrary closed subspace of a barrelled space),
(4) a topological product of an arbitrary family of Mackey spaces is Mackey,
(5) an arbitrary direct sum of Mackey spaces is Mackey (a 1-codimensional (non-closed) vector subspace of a Mackey space may not be a Mackey space).

In Mackey's papers a locally convex topological vector space E such that $\sigma(E, E') = \tau(E, E')$ is called *relatively weak*. As mentioned above an infinite-dimensional normed space over $\mathbb{R}$ or $\mathbb{C}$ is not relatively weak. The space $E = \mathbb{R}^{\mathbb{N}}$ endowed with the product topology is an example of an infinite-dimensional complete separable metrizable topological vector space which is relatively weak.

Let us note that in all existing proofs of Theorem 2.1 the following variant of Hahn–Banach theorem plays an important role: if E is a locally convex topological vector space over $\mathbb{R}$ or $\mathbb{C}$, E_0 is a vector subspace of E and $f_0 \colon E_0 \to \mathbb{K}$ is a continuous linear functional, then there exists a continuous linear functional $f : E \to \mathbb{K}$ such that $f(x) = f_0(x),\ \forall x \in E_0$. It is known that this statement may not be true for a complete separable metrizable dually separated non-locally convex topological vector space [20, Theorem 4.8]. In this context the following question arises.

Question 2.6 Let (E, F) be a separating vector space duality over $\mathbb{R}$ or $\mathbb{C}$ and let $\tilde{\tau}(E, F)$ be the least upper bound (in the lattice of all topologies in E) of the set of all vector space topologies in E which are compatible with (E, F). Is then $\tilde{\tau}(E, F)$ compatible with the duality (E, F)?

In [25] it was shown that there are vector space dualities for which Question 2.6 has a negative answer. A characterization of vector space dualities (E, F) for which Question 2.6 has a positive answer seems to be unknown.

Let again (E, F) be a separating vector space duality and $\mathcal{R}$ a class of topological vector spaces; denote now by $\tilde{\tau}_{\mathcal{R}}(E, F)$ the least upper bound (in the lattice of all topologies in E) of the set of all vector space topologies τ in E which are compatible with (E, F) *and* $(E, \tau) \in \mathcal{R}$.

The Mackey–Arens theorem tells us that if $\mathcal{R}$ is the class of locally convex topological vector spaces (over $\mathbb{R}$ or $\mathbb{C}$), then $\tilde{\tau}_{\mathcal{R}}(E, F)$ *is compatible* with the duality (E, F).

In this terms, the result of [25] can be formulated as follows: if (E, F) is a vector space duality and $\mathcal{TVS}$ is the class of all topological vector spaces (over $\mathbb{R}$ or $\mathbb{C}$), then $\tilde{\tau}_{\mathcal{TVS}}(E, F)$ *may not be compatible* with the duality (E, F). We have the following more impressive statement for some natural proper subclasses of $\mathcal{TVS}$:

Proposition 2.7 [22, Proposition] *Let (E, F) be a vector space duality over $\mathbb{R}$ or $\mathbb{C}$ and let $\mathcal{R}$ be either the class of all locally r-convex ($0 < r < 1$) topological vector spaces or the class of all locally pseudo-convex topological vector spaces. Then $\tilde{\tau}_{\mathcal{R}}(E, F)$ is compatible with the duality (E, F) if and only if $F = E^a$.*

Let $\mathbb{K}$ be a complete non-trivially non-Archimedean valued field. For a vector space E over $\mathbb{K}$, the notions of a convex subset of E and of a locally convex vector space topology in E can also be defined. Thus, for a vector space duality (E, F) let $\tau(E, F)$ be the least upper bound (in the lattice of all topologies in E) of the set of all locally convex vector space topologies in E which are compatible with (E, F). In [35, Theorem 4.17] it was shown that if $\mathbb{K}$ is *spherically complete*, then $\tau(E, F)$ *is compatible* with the duality (E, F). In [23] (see also [24,25]) J. Kakol proved that the spherical completeness of $\mathbb{K}$ *is a necessary condition* for the validity of this statement.

3 Group dualities

In this section $\mathbb{T}$ will stand for the multiplicative group of complex numbers with modulus one endowed with the topology induced from $\mathbb{C}$.

For an Abelian group G a group homomorphisms $\chi : G \to \mathbb{T}$ is called *a character* and $\mathrm{Hom}(H, \mathbb{T})$ will denote the (multiplicative Abelian) group of all characters.

For a topologized Abelian group G we denote by $G^\wedge$ the dual (or character group) of G, which is the subgroup of $\mathrm{Hom}(H, \mathbb{T})$ consisting of all continuous characters.

For an infinite Hausdorff topological Abelian group G it may happen that $G^\wedge = \{1\}$. Such a topological Abelian group and the corresponding topology is called *minimally almost periodic*. A topological Abelian group G for which $G^\wedge$ separates points of G is called *maximally* *almost periodic (MAP)* or *dually separated*. The Peter–Weyl–van Kampen theorem implies that every Hausdorff (pre-)locally compact topological Abelian group is MAP.

For a topological Abelian group G its dual $G^\wedge$ endowed with the compact-open topology is often called *the character group* (or the Pontryagin dual group) of G. The topology in $G^\wedge$ induced from $\mathbb{T}^G$, that is the point-wise convergence topology, will play an important role in the sequel.

The theory of group dualities analogous to that of vector space dualities has its origin in [36]. The main reference for the latter is the above-mentioned Bourbaki's text. Below we will expose Varopoulos' results up to the following notational change: instead of our letter H, in [36] the letter G' is used.

Definition 3.1 A group duality is a pair (G, H), where G is an Abelian group (without a topology) and H is a subgroup of $\mathrm{Hom}(G, \mathbb{T})$. A group duality (G, H) is called *separating* if H separates point of G.

In the sequel, we stick to [36, p. 479], where only separated group dualities are considered. If G is a *finite* Abelian group and (G, H) is a separating group duality, then, necessarily, $H = \mathrm{Hom}(G, \mathbb{T})$; if G is an *infinite* Abelian group, then for some *proper* subgroup H of $\mathrm{Hom}(G, \mathbb{T})$ the pair (G, H) can also be a separating group duality.

To a group duality (G, H) there are two topologies associated, $\sigma(G, H)$ and $\sigma(H, G)$. The topology $\sigma(G, H)$ is the coarsest topology in G which makes continuous all members of H, while $\sigma(H, G)$ is the coarsest topology in H which makes continuous the functions $\chi \mapsto \alpha_x(\chi) := \chi(x)$, $x \in G$ (in other words, $\sigma(H, G)$ is the topology of point-wise convergence in H). The pairs $(G, \sigma(G, H))$ and $(H, \sigma(H, G))$ are Hausdorff precompact topological Abelian groups [36, Remark (p. 480)].[1]

A topology $\mathcal{T}$ in G is called *compatible with the duality* (G, H) if H coincides with the set of all $\mathcal{T}$-continuous characters $\chi : G \to \mathbb{T}$. In a similar way, a topology $\mathcal{T}'$ in H is called *compatible with the duality* (G, H) if the set $\{\alpha_x : x \in G\}$ coincides with the set of all $\mathcal{T}'$-continuous characters $\varphi : H \to \mathbb{T}$.

A topology $\mathcal{T}$ in a group G is called a *group topology* if the pair $(G, \mathcal{T})$ is a topological group.

A characterization of the compatible pre-locally compact topologies $\mathcal{T}$ in G for a group duality (G, H) is obtained in [36, Proposition 1 (p. 480)]. From it, the following fact is derived: the topologies $\sigma(G, H)$ and $\sigma(H, G)$ are precompact group topologies (respectively in G and in H) compatible with (G, H) [36, Corollary (p. 481)]. This statement is deeper than the corresponding statement for vector space dualities; it was obtained independently in [12]. For a group duality (G, H) let $\upsilon(G, H)$ be the least upper bound (in the lattice of all topologies in G) of the set of all pre-locally compact compatible group topologies in G. (In [36, Definition 4 (p. 483)] this topology is denoted by $\tau(G, H)$; we will clarify below why we are avoiding this notation for group dualities). The main result of [36] about compatible topologies can be summarized as follows.

Theorem 3.2
[36, Proposition 5 (p. 483); Corollary 1 (p. 484)] *Let (G, H) be a group duality. Then:*

(a) *The topology $\upsilon(G, H)$ is compatible with the duality (G, H).*

[1] A subset of a topological Abelian group is called precompact in [36, Definition 3 (p. 469)] if it can be covered by a finite number of translates of each neighborhood of the neutral element (in [12] such sets are called bounded). A topological Abelian group is called precompact if it is a precompact subset of itself. A topological Abelian group (as well as its topology) is called pre-locally compact if it has a precompact neighborhood of the neutral element.

(b) *The topology* $\upsilon(G, H)$ *is a projective limit of a family of pre-locally compact group topologies.*
(c) *If* $\mathcal{T}$ *is a group topology in G which is a projective limit of a family of pre-locally compact group topologies, then* $\mathcal{T}$ *is compatible with the duality if and only if* $\sigma(G, H) \subseteq \mathcal{T} \subseteq \upsilon(G, H)$.
(d) *Suppose that* $\{s_n : n = 1, 2, \dots\}$ *is a sequence in G which converges to* s_0 *in* $\sigma(G, H)$. *Then, it also converges to* s_0 *in* $\upsilon(G, H)$.

A topological Abelian group G gives rise in a natural way to a group duality $(G, G^\wedge)$, where $G^\wedge$ is the dual group of G. For a topological Abelian group G, the topology $\sigma(G, G^\wedge)$ is called *the Bohr topology of* G. It is Hausdorff iff G is MAP. If G is a topological Abelian non-precompact MAP-group, then its Bohr topology is strictly coarser than the original topology of G. It does not follow directly from Theorem 3.2 that for a LCA group G its original topology coincides with $\upsilon(G, G^\wedge)$. Nevertheless, this statement is obtained in [36, Corollary 2 (p. 488)] together with the following result: *for a given group duality there may exist at most one locally compact group topology compatible with the duality* [36, Theorem (p. 485)]; a bit earlier, a similar result (not in terms of group dualities) was obtained in [17].

An attempt to characterize the group dualities admitting a locally compact compatible group topology was done in [4]. If G is the underlying additive group of an infinite-dimensional real Banach space, then there is no locally compact group topology in G compatible with the group duality $(G, G^\wedge)$, as proved in [11, Remark 12 (p. 279)].

For the proof of Theorem 3.2, among other tools, the following fact plays the most important role: every subgroup G_0 of a pre-locally compact group G is dually embedded (a subgroup G_0 of a topological abelian group is *dually embedded* if for every continuous character $\chi_0 : G_0 \to \mathbb{T}$ there exists a continuous character $\chi : G \to \mathbb{T}$ such that $\chi(x) = \chi_0(x), \ \forall x \in G_0$). It is known that this statement may not be true for non-pre-locally compact topological Abelian groups.

As the reader may already have noted, we have not called the topology $\upsilon(G, H)$ the 'Mackey topology', for a given group duality (G, H). Neither have we called 'a Mackey group' to a MAP group G for which the topology $\upsilon(G, G^\wedge)$ coincides with the original topology of G. Now we want to clarify a reason for this.

Denote by $p : \mathbb{R} \to \mathbb{T}$ the exponential mapping. It plays an important role in order to compare space dualities and group dualities. Let (E, F) be a duality of real vector spaces. Since E is an Abelian group as well (with respect to the vector space addition), for a fixed $x' \in F$ the mapping $p \circ x' : E \to \mathbb{T}$ can be considered as a character defined on this Abelian group. Write: $p \circ F := \{p \circ x' : x' \in F\}$. Clearly now the pair $(E, p \circ F)$ can be treated as a group duality. It is easy to see that if (E, F) is a separating real vector space duality, then $(E, p \circ F)$ is a separating group duality as well. Therefore to every (separating) real vector space duality there corresponds in a rather natural way a (separating) group duality $(E, p \circ F)$.

Let now E be a real topological vector space with dual space E'. Since E is in particular a topological Abelian group, it has a dual group $E^\wedge$. It is known that $E^\wedge = p \circ E'$ (Smith [33], Hewitt and Zuckermann [18]).

From the above consideration we can conclude that in a real topological vector space E, besides its original topology, we have four other topologies: $\sigma(E, E')$, $\tau(E, E')$, $\sigma(E, E^\wedge)$ and $\upsilon(E, E^\wedge)$. It is clear that $\sigma(E, E^\wedge) \subset \sigma(E, E')$ and this inclusion is strict provided $E \neq \{0\}$ and the duality (E, E') is separating (because $(E, \sigma(E, E^\wedge))$ is a precompact Hausdorff topological Abelian group, while $(E, \sigma(E, E'))$ is a Hausdorff topological vector space and a non-trivial Hausdorff topological vector space cannot be precompact). Consequently *for* *every non-trivial dually separated topological vector space its weak topology and the Bohr topology of its underlying topological group are distinct.*

Let $E = l_2$ with its usual norm topology. As we have noted above, its original topology coincides with $\tau(E, E')$. On the other hand, there exists a sequence (s_n) in E which tends to zero weakly, but does not tend to zero in the original topology of E. From this and Theorem 3.2(d) it follows that for $E = l_2$ we have that $\tau(E, E') \neq \upsilon(E, E^\wedge)$.

The example just considered gives a reason to claim that for an arbitrary group duality (G, H) the topology $\upsilon(G, H)$ cannot be viewed as a satisfactory group version of the Mackey topology for the vector space dualities. Another reason is contained in the following assertion.

Proposition 3.3 [11, Proposition 5.5 (p. 282)] *Let E be a real dually separating topological vector space. Then $\upsilon(E, E^\wedge) = \sigma(E, E')$.*

Consequently, if E is a real infinite-dimensional Banach space, then $\upsilon(E, E^\wedge) \neq \tau(E, E')$.

A further study of group dualities and wider classes of compatible topologies for them was made in [11]. First of all in [11] it was made an attempt to answer the following natural question (which was not posed in [36]).

Question 3.4 Let (G, H) be a separating group duality and let $\tilde{\upsilon}(G, H)$ be the least upper bound (in the lattice of all topologies in G) of the set of all group topologies in G which are compatible with (G, H). Is then $\tilde{\upsilon}(G, H)$ compatible with the duality (G, H)?

Proposition 3.5 [11, Proposition 2.2] *Let E be an infinite-dimensional metrizable locally convex topological vector space. Then the topology $\tilde{\upsilon}(E, E^\wedge)$ is not compatible with the group duality $(E, E^\wedge)$.*

Consequently Question 3.4 has a negative answer in general.

Problem 3.6 Characterize the group dualities (G, H) for which Question 3.4 has a positive answer.

Proposition 3.5 shows that some restriction on the class of compatible group topologies should be made to guarantee the compatibility of their least upper bound. In this spirit, the class of *locally quasi-convex group topologies* was chosen in [11]. A subset A of a topological Abelian group G is called *quasi-convex* if for every $x \in G \setminus A$ there is a continuous character $\chi \in G^\wedge$ such that $\mathrm{Re}\chi(x) < 0$ and $\mathrm{Re}\chi(a) \geq 0$ for each $a \in A$. Note that while the concept of a convex subset of a real (or complex) vector space E does not require a topology on E, the quasi-convexity is defined only for a subset of a *topological* Abelian group. A topological Abelian group G, as well as its topology, is called locally quasi-convex if it has a fundamental system of neighborhoods of the neutral element consisting of quasi-convex subsets of G [37]. In [36] the locally quasi-convex groups are not mentioned at all.

The class of locally quasi-convex group topologies is rather wide, it includes in particular all pre-locally compact group topologies and the topologies of real (or complex) locally convex topological vector spaces [5]. It follows that for a group duality (G, H) the precompact topological groups $(G, \sigma(G, H))$ and $(H, \sigma(H, G))$ are locally quasi-convex; as we have already noted, the topologies $\sigma(G, H)$ and $\sigma(H, G)$ are compatible with (G, H).

We introduce for the sequel the following:

Notation 3.7

(1) LQC *will stand for the class of all locally quasi-convex Hausdorff topological Abelian groups;*
(2) *for a group duality* (G, H), $\tau_g(G, H)$ *will stand for the least upper bound (in the lattice of all topologies in* G*) of the set of all locally quasi-convex compatible group topologies on* G*, and* $\tau_{qc}(G, H)$ *will stand for the topology on* G *of uniform convergence on all subsets of* H *which are quasi-convex and compact in* $(H, \sigma(H, G))$.

In [11] it is shown that for a group duality (G, H) the topologies $\tau_g(G, H)$ and $\tau_{qc}(G, H)$ are locally quasi-convex and $\tau_g(G, H) \subset \tau_{qc}(G, H)$.

In order to formulate the next theorem we recall the notion of g-barrelled group as introduced in [11].

Definition 3.8 A topological Abelian group G is g-barrelled if any $\sigma(G^\wedge, G)$-compact subset of $G^\wedge$ is equicontinuous with respect to the topology of G.

Every topological vector space over $\mathbb{R}$ which is barrelled, is in particular a g-barrelled group, as proved in [11, (5.3)]. Some big classes of topological Abelian groups are also enclosed in that of g-barrelled groups, for instance the locally compact Abelian groups, or the complete metrizable topological Abelian groups (see [11, Corollary (1.6)]).

Theorem 3.9 [11, Theorem 4.1 (p. 277); Proposition 5.4 (p. 282)] *Let* (G, H) *be a group duality for which there exists on* G *at least one compatible* g*-barrelled group topology. Then:*

(a) *The topology* $\tau_{qc}(G, H)$ *is compatible with the duality* (G, H).
(b) $\tau_g(G, H) = \tau_{qc}(G, H)$ *(and hence,* $\tau_g(G, H)$ *is compatible with the duality* (G, H) *as well).*
(c) *(cf. Proposition 3.3). If* E *is a* g*-barrelled real topological vector space, then* $\tau(E, E') = \tau_g(E, E^\wedge) = \tau_{qc}(E, E^\wedge)$.

The main question left open in [11] was precisely whether for an arbitrary group duality (G, H) the topologies $\tau_{qc}(G, H)$ and $\tau_g(G, H)$ are compatible. Later, in [9] it was provided an example of a group duality (G, H) for which the topology $\tau_g(G, H)$ is compatible whilst $\tau_{qc}(G, H)$ is non-compatible. We reproduce it next, (without details), since it was the first clarifying result in this direction:

Example Let G be the direct sum of countably many copies of the cyclic group $\mathbb{Z}_5$, say $G := \mathbb{Z}_5^{(\mathbb{N})}$, endowed with the topology induced from the product $\mathbb{Z}_5^{\mathbb{N}}$. Its dual group $G^\wedge$ coincides with the dual of $\mathbb{Z}_5^{\mathbb{N}}$, which is isomorphic to $\mathbb{Z}_5^{(\mathbb{N})}$. Loosely speaking, G is autodual. As proved in [9, Examples (4.2), (4.4)], the only locally quasi-convex compatible topology in G is precisely $\sigma(G, G^\wedge)$. Therefore, the Mackey topology on G coincides with $\sigma(G, G^\wedge)$.

However, the topology $\tau_{qc}(G, G^\wedge)$ is shown to be discrete (Example (4.5) in [9]). Thus, $(\mathbb{Z}_5^{(\mathbb{N})}, \tau_{qc})^\wedge = \mathbb{Z}_5^{\mathbb{N}}$ and $\tau_{qc}(G, G^\wedge)$ is non-compatible.

It remains open so far the following:

Problem 3.10 For an arbitrary group duality (G, H),

(1) Is the topology $\tau_g(G, H)$ compatible?
(2) If $\mathcal{T}_1, \mathcal{T}_2$ are locally quasi-convex group topologies in G compatible with (G, H), is then $\sup(\mathcal{T}_1, \mathcal{T}_2)$ also a compatible topology with (G, H)?

Both assertions (1) and (2) are equivalent as proved in [11, (3.11)]. The difficulty of the problem lies on the fact that the subgroups of a locally quasi-convex topological Abelian group are not necessarily dually embedded.

In [11] for a group duality (G, H) none of the topologies were called "'the Mackey topology"' and the term "'a Mackey group"' was not introduced. A further study of locally quasi-convex sets and locally quasi-convex compatible topologies was made in [28]. According to [28] for a topological Abelian group G the topology $\tau_g(G, G^\wedge)$ is its *Mackey topology* provided it is compatible with the duality $(G, G^\wedge)$ and a topological Abelian group G is *a Mackey group* if its original topology coincides with $\tau_g(G, G^\wedge)$.

An analogue of Proposition 2.4 remains true only partially:

Proposition 3.11 *For a topological Abelian group $G \in$ LQC consider the statements:*

(i) *G is a Mackey group.*
(ii) *Every $\sigma(G^\wedge, G)$-compact quasi-convex subset of $G^\wedge$ is equicontinuous.*

Then (ii) $\Longrightarrow$ (i).

The implication (i) $\Longrightarrow$ (ii) may fail, see the above example and [28, Theorem 8.61].

By Proposition 3.11 and the definition of g-barrelled group it is easily seen the following:

Corollary 3.12 *If G is a locally quasi-convex g-barrelled group, then G is a Mackey group. In particular, locally compact Abelian groups are Mackey groups.*

Observe that LQC g-barrelled groups are Mackey groups in the best possible way: a parallel version of the Mackey–Arens Theorem holds for them. In fact, if G is such a group, $\tau_g(G, G^\wedge) = \tau_{qc}(G, G^\wedge)$ which is further the original topology of G.

Although being g-barrelled is a sufficient condition for an LQC group to be a Mackey group, it is not necessary. Several intermediate topologies between $\tau_g(G, G^\wedge)$ and $\tau_{qc}(G, G^\wedge)$ were defined and studied in [13], which led to a grading in the property of "being a Mackey group", based upon the following comments and facts.

Facts Let (G, τ) be a topological Abelian group.

(1) For every equicontinuous subset $S \subset G^\wedge$ there exists another equicontinuous subset with respect to τ, which is further $\sigma(G^\wedge, G)$-compact and quasi-convex and contains S.
(2) All the topologies on G compatible with τ give rise to the same family of $\sigma(G^\wedge, G)$-compact and quasi-convex subsets. Call it $\mathcal{Q}(G^\wedge, G)$, or simply $\mathcal{Q}$ since it is an invariant of the duality.
(3) Considering (1) and (2) we shall denote by $\mathcal{Q}(\lambda)$ the family of $\sigma(G^\wedge, G)$-compact, quasi-convex subsets of $G^\wedge$ that are equicontinuous with respect to a topology λ which is compatible with τ. By its definition, $\mathcal{Q}(\lambda) \subseteq \mathcal{Q}$.

Fact (1) is proved in [11], and the second one is an evident consequence of having the same dual group. Concerning (3), we observe that distinct locally quasi convex topologies λ_1 and λ_2 give rise to different families $\mathcal{Q}(\lambda_1) \neq \mathcal{Q}(\lambda_2)$. The interesting point is that from the family $\mathcal{Q}(\lambda)$ the topology λ can be recovered uniquely if we require λ to be locally quasi-convex. Namely, λ is the topology of uniform convergence on the elements of $\mathcal{Q}(\lambda)$.

Since for any locally quasi-convex compatible topology on G, the family $\mathcal{Q}(\lambda) \subseteq \mathcal{Q}$, we can think now from the opposite point of view: recognize a locally quasi-convex topology compatible with a duality $(G, G^\wedge)$ from the analysis of certain subfamilies of $\mathcal{Q}$. It may happen that the whole family $\mathcal{Q}$ gives rise to an LQC topology which is not compatible (this is precisely what happens in the Example above mentioned). Clearly if (G, τ) is g-barrelled and locally quasi-convex, then $\mathcal{Q}(\tau) = \mathcal{Q}$.

This is the lead taken in [13] to define the mentioned grading of being Mackey. The name $(\mathcal{Q})$-Mackey is assigned to a Mackey group (G, τ) such that $\mathcal{Q}(\tau) = \mathcal{Q}$. Other names like $(\mathcal{M})$-Mackey in that reference are given to groups which are Mackey in a stronger sense than the required on the plane definition of a Mackey group.

In [16] the following local versions of these concepts were introduced:

Definition 3.13 Let $\mathcal{G}$ be a class of topological Abelian groups. A topological group (G, μ) with $H := (G, \mu)^\wedge$ is called a *Mackey group* in $\mathcal{G}$ or *a $\mathcal{G}$-Mackey group* if $(G, \mu) \in \mathcal{G}$ and if ν is a compatible group topology with (G, H) *and* $(G, \nu) \in \mathcal{G}$, then $\nu \leq \mu$.

Definition 3.14 Let $\mathcal{G}$ be a class of topological Abelian groups and $(G, \nu) \in \mathcal{G}$ with $H := (G, \nu)^\wedge$. If there exists a group topology μ in G compatible with (G, H) such that $(G, \mu) \in \mathcal{G}$ *and* (G, μ) is a Mackey group in $\mathcal{G}$, then μ is called the $\mathcal{G}$-Mackey topology in G associated with ν (or with the group duality (G, H)).

Let $\mathcal{B}$ be the class of all Hausdorff topological Abelian groups which are topologically isomorphic to a subgroup of a product of locally compact Abelian groups. It is shown in [6] that for every $(G, \nu) \in \mathcal{B}$ there exists the $\mathcal{B}$-Mackey topology associated with ν.

Clearly, a topological Abelian group $(G, \mu) \in$ LQC is LQC-Mackey group iff $\mu = \tau_g(G, G^\wedge)$.

Let LCS be the class of Hausdorff topological Abelian groups which admit a structure of *locally convex* topological vector space over $\mathbb{R}$. Loosely speaking, LCS is the class of real Hausdorff locally convex topological vector spaces considered just as additive groups. We can state that a topological Abelian group $(G, \mu) \in$ LCS is an LCS-Mackey group iff $\mu = \tau(G, G^{'})$.

As LCS $\subset$ LQC, the following question arises in a natural way:

Problem 3.15 Let $(G, \mu) \in$ LCS be a LCS-Mackey group. Is then (G, μ) a LQC-Mackey group as well?

From [11, Proposition 5.2] it can be concluded that Problem 3.15 has a positive answer provided the space $(G', \sigma(G', G))$ has *the convex compactness property.*

A quasi-convex version of Proposition 2.5 fails too: a metrizable topological Abelian group $G \in$ LQC may not be a LQC-Mackey group [2,16]. The most impressive result so far obtained in this direction is that the group of rational numbers endowed with the topology induced from the real line is not a Mackey group (proved independently in [8,13]). Infinite torsion subgroups of $\mathbb{T}$ (endowed with the topology induced by $\mathbb{T}$) are not Mackey groups either. However, metrizable LQC topologies on torsion groups of *bounded exponent* are Mackey (this is the main theorem in [3]).

We do not know whether the quasi-convex version of Proposition 2.2 remains true; in other words the following question for $\mathcal{G} =$ LQC remains unanswered:

Problem 3.16 Let $\mathcal{G}$ be a class of topological Abelian groups, G_1, G_2 be $\mathcal{G}$-Mackey groups and $u\colon G_1 \to G_2$ be a group homomorphism which is continuous with respect to the topologies $\sigma(G_1, G_1^\wedge)$ and $\sigma(G_2, G_2^\wedge)$. Is then u is continuous?

A characterization of classes $\mathcal{G}$ of topological Abelian groups, for which the Problem 3.16 has a positive answer was obtained in categorical terms in [7].

Permanence properties for quotients of the class of LQC-Mackey groups were thoroughly studied in [13]. We do not know if permanence properties hold for products, even in the finite case.

References

1. Arens, R.: Duality in linear spaces. Duke Math. J. **14**, 787–794 (1947)
2. Außenhofer, L., de la Barrera Mayora, D.: Linear topologies on $\mathbb{Z}$ are not Mackey topologies. J. Pure Appl. Algebra **216**(6), 1340–1347 (2012)
3. Außenhofer, L., de la Barrera Mayoral, D., Dikranjan, D., Martín-Peinador, E.: Varopoulos paradigm: Mackey property vs. metrizability in topological groups. Submitted
4. Baker, J.W.: A note on the duality of locally compact groups. Glasg. Math. J. **9**, 87–91 (1968)
5. Banaszczyk, W.: Additive subgroups of topological vector spaces. In: Lecture Notes in Mathematics, vol. 1466. Springer, Berlin (1991)
6. Barr, M.: On duality of topological abelian groups. Preprint
7. Barr, M., Kleisly, H.: On Mackey topologies in topological abelian groups. Theory Appl. Categories **8**(4), 54–62 (2001)
8. de la Barrera Mayoral, D.: $\mathbb{Q}$ is not a Mackey group. Topol. Appl. **178**, 265–275 (2014)
9. Bonales, F.G., Trigos-Arrieta, F.J., Vera Mendoza, R.: A Mackey–Arens theorem for topological abelian groups. Bol. Soc. Mat. Mex. (3) **9**(1), 79–88 (2003)
10. Bourbaki, N.: Elements de mathematique. XVIII. Premiere partie: Les structures fondamentales de l'analyse. Livre V: Espaces vectoriels topologiques. In: Actualits Sci. Ind., vol. 1229. Hermann & Cie, Paris (1955)
11. Chasco, M.J., Martín-Peinador, E., Tarieladze, V.: On Mackey topology for groups. Stud. Math. **132**(3), 257–284 (1999)
12. Comfort, W.W., Ross, K.A.: Topologies induced by groups of characters. Fundam. Math. **55**, 283–291 (1964)
13. Díaz Nieto, J.M., Martín-Peinador, E.: Characteristics of the Mackey topology for abelian topological groups. In: Ferrando, J.C., Pellicer, M.L. (eds.) Descriptive Topology and Functional Analysis, Chapter 7. Proceedings in Mathematics and Statistics, vol. 80, pp. 119-144. Springer, Berlin (2014)
14. Dieudonne, J.: La dualité dans les espaces vectoriels topologiques. (French) Ann. Sci. Ecole Norm. Sup. (3) **59**, 107–139 (1942)
15. Dieudonnee, J., Schwartz, L.: La dualité dans les espaces $\mathcal{F}$ et $\mathcal{LF}$. Ann. Inst. Fourier Grenoble **1**, 61–101 (1949) (1950)
16. Dikranjan, D., Martín-Peinador, E., Tarieladze, V.: Group valued null sequences and metrizable non-Mackey groups. Forum Math. **26**, 723–757 (2014)
17. Glicksberg, I.: Uniform boundedness for groups. Can. J. Math. **14**, 269–276 (1962)
18. Hewitt, E., Zuckerman, H.: A group-theoretic method in approximation theory. Ann. Math. **52**, 557–567 (1950)
19. Jarchow, H.: Locally convex spaces. Mathematische Leitfäden. (Mathematical Textbooks) B. G. Teubner, Stuttgart (1981)
20. Kalton, N.J., Peck, N.T., Roberts, J.W.: An F-space sampler. In: LNS, vol. 89. Cambridge University Press, London (1984)
21. Kakol, J.: Note on compatible vector topologies. Proc. Am. Math. Soc. **99**(4), 690–692 (1987)
22. Kakol, J.: The Mackey–Arens Theorem for non-locally convex spaces. Collect. Math. **41**(2), 129–132 (1990)
23. Kakol, J.: Remarks on spherical completeness of non-Archimedean valued fields. Indag. Math. N.S. **5**, 321–323 (1994)
24. Kakol, J.: The Mackey–Arens Property for spaces over valued fields. Bull. Pol. Acad. Sci. **42**(2), 97–101 (1994)
25. Kakol, J.: The Mackey–Arens and Hahn–Banach theorems for spaces over valued fields. Ann. Math. Blaise Pascal **2**(1), 147–153 (1995)
26. Kelley, J.L., Namioka, I.: Linear topological spaces. In: The University Series in Higher Mathematics D. Van Nostrand Co., Inc, Princeton, (1963)
27. Kenderov, P.S.: On topological vector groups. Mat. Sb. (N.S.), 1970, Volume 81(123), Number 4, 580–599
28. de Leo, L.: Weak and strong topologies in topological abelian groups. Diss. Universidad Complutense de Madrid, Servicio de Publicaciones (2009)
29. Mackey, G.W.: On convex topological linear spaces. Proc. Nat. Acad. Sci. U. S. A. **29**, 315–319 (1943). (Erratum 30, 24)
30. Mackey, G.W.: On convex topological linear spaces. Trans. Am. Math. Soc. **60**, 519–537 (1946)
31. Pérez Carreras, P., Bonet, J.: Barrelled locally convex spaces. In: North-Holland Mathematics Studies, vol. 131. Amsterdam (1987)
32. Schaefer, H.: Topological Vector Spaces. The Macmillan Co., New York (1966)
33. Smith, M.: The Pontrjagin duality theorem in linear spaces. Ann. Math. **56**, 248–253 (1952)

34. Valdivia, M.: Topics in Locally Convex Spaces. North Holland Publishing Company, Amsterdam (1982)
35. Van Tiel, J.: Espaces localment K-convexes. Indag. Math. **27**, 249–289 (1965)
36. Varopoulos, N.T.: Studies in harmonic analysis. Proc. Camb. Phil. Soc. **60**, 467–516 (1964)
37. Vilenkin, N.Ya.: The theory of characters of topological Abelian groups with boundedness given. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 439–162 (1951)