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ORIGINAL PAPER

Fixed point results in *b*-metric spaces approach to the existence of a solution for nonlinear integral equations

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Abstract The purpose of this work is to introduce new nonlinear mappings in setup of b-metric spaces and prove fixed point theorems for such mappings. Examples are provided in order to distinguish these results from the known ones. At the end of paper, we apply our fixed point result to prove the existence of a solution for the following nonlinear integral equation:

$$x(c) = \Omega(\phi(c), c) + K(c, c, \phi(c)) + \int_a^b K(c, r, x(r))dr, \tag{0.1}$$

where $a, b \in \mathbb{R}$ with $a < b$, $x \in C[a, b]$ (the set of all continuous real functions defined on $[a, b]$), $\phi : [a, b] \to \mathbb{R}$, $\Omega : \mathbb{R} \times [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are given mappings.

Keywords α-Admissible mappings · α-Regularity · b-metric spaces · Hölder inequality · Nonlinear integral equations

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1 Introduction and preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, non-negative real numbers and real numbers, respectively.

In the recent year, several mathematicians improved and extended the famous Banach contraction mappings principle by many directions as follows:

- How to find generalizations or other types of contractive conditions?

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- How to extend Banach contraction mappings principle in metric spaces to other spaces?
- How to extend Banach contraction mappings principle to multi-valued mappings?

One of the most interested generalization is extension of contractive condition in to case of weak contractive condition which was first introduced by Alber et al. [2] in the framework of Hilbert spaces. Afterward, Rhoades [22] showed that the result of Alber et al. [2] is also valid in complete metric spaces. Fixed point theorems involving mappings satisfying weak contractive type inequalities have been considered in [7,8,12,13,17,26] and references therein.

In 1984, Khan et al. [15] introduced the useful function called an altering distance function as follows:

Definition 1.1 ([15]) The function $\varphi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties hold:

1. φ is continuous and non-decreasing;
2. $\varphi(t) = 0$ if and only if $t = 0$.

Here, we give some examples of altering distance function.

Example 1.2 Let $\varphi_i : [0, \infty) \to [0, \infty)$, where $i \in \{1, 2, \dots, 5\}$, be defined by

(φ_1) $\varphi_1(t) = kt$, where $k > 0$,
(φ_2) $\varphi_2(t) = t^k$, where $k > 0$,
(φ_3) $\varphi_3(t) = \begin{cases} \frac{t}{3}, & t \in [0, 1], \\ t - \frac{2}{3}, & t \in (1, \infty), \end{cases}$
(φ_4) $\varphi_4(t) = \sinh^{-1} t$,
(φ_5) $\varphi_5(t) = \cosh(t) - 1$.

Then φ_i is altering distance function for all $i \in \{1, 2, \dots, 5\}$.

By using the concept of an altering distance, Choudhury et al. [9] generalized the concept of weak contraction mappings and proved fixed point theorem for such mappings.

On the other hand, in 1993, Czerwik [10] introduced the concept of a b-metric spaces as follows:

Definition 1.3 ([10]) Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that the mapping $d : X \times X \to \mathbb{R}_+$ satisfies the following conditions:

(B_1) $d(x, y) = 0$ if and only if $x = y$;
(B_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(B_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z, \in X$.

Then (X, d) is called a b-metric space with coefficient s.

Any metric space is a b-metric space with $s = 1$ and so the class of b-metric spaces is larger than the class of metric spaces. In general a b-metric space does not necessarily need to be a metric space. Some known examples of b-metric which show that b-metric space is real generalization of metric space are the following.

Example 1.4 Let (X, d) be a metric space and $\sigma_d : X \times X \to \mathbb{R}_+$ defined by

$$\sigma_d(x, y) = [d(x, y)]^p \quad \text{for all } x,\ y \in X,$$

where $p > 1$ is a fixed real number. Then σ_d is a b-metric with $s = 2^{p-1}$. Indeed, conditions (B_1) and (B_2) in Definition 1.3 are satisfied and thus we only to show that condition (B_3) holds for σ_d.

It is easy to see that if $1 < p < \infty$, then the convexity of the function $f(x) = x^p$, where $x \geq 0$, implies

$$\left(\frac{a+c}{2}\right)^p \leq \frac{1}{2}(a^p + c^p),$$

and hence

$$(a+c)^p \leq 2^{p-1}(a^p + c^p).$$

Therefore, for each $x, y, z \in X$, we get

$$\begin{aligned}\sigma_d(x,y) &= [d(x,y)]^p \\ &\leq [d(x,z) + d(z,y)]^p \\ &\leq 2^{p-1}([d(x,z))^p + (d(z,y)]^p) \\ &= 2^{p-1}[\sigma_d(x,z) + \sigma_d(z,y)].\end{aligned}$$

So condition (B_3) in Definition 1.3 holds and then σ_d is a b-metric coefficient $s = 2^{p-1} > 1$.

Example 1.5 The set $l_p(\mathbb{R})$ with $0 < p < 1$, where

$$l_p(\mathbb{R}) := \left\{ \{x_n\} \subseteq \mathbb{R} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

together with the mapping $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}_+$ defined by

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}$$

for each $x = \{x_n\}, y = \{y_n\} \in l_p(\mathbb{R})$, is a b-metric space with coefficient $s = 2^{\frac{1}{p}} > 1$. The above result also holds for the general case $l_p(X)$ with $0 < p < 1$, where X is a Banach space.

Example 1.6 Let p be a given real number in the interval $(0, 1)$. The space $L_p[0, 1]$ of all functions $x : [0, 1] \to \mathbb{R}$ such that $\int_0^1 |x(t)|^p dt < 1$, together with the mapping $d : L_p[0, 1] \times L_p[0, 1] \to \mathbb{R}_+$ defined by

$$d(x,y) := \left(\int_0^1 |x(t) - y(t)|^p dt \right)^{1/p}, \quad \text{for each } x,\ y \in L_p[0,1],$$

is a b-metric space with constant $s = 2^{\frac{1}{p}} > 1$.

Example 1.7 Let $X = \{0, 1, 2\}$ and the mapping $d : X \times X \to \mathbb{R}_+$ defined by

$$\begin{aligned}&d(0,0) = d(1,1) = d(2,2) = 0, \\ &d(0,1) = d(1,0) = d(1,2) = d(2,1) = 1\end{aligned}$$

and

$$d(2,0) = d(0,2) = m,$$

where m is given real number such that $m \geq 2$. It is easy to see that

$$d(x, y) \leq \frac{m}{2}[d(x, z) + d(z, y)],$$

for all $x, y, z \in X$. Therefore, (X, d) is a b-metric space with coefficient $s = m/2$. We obtain that the ordinary triangle inequality does not hold if $m > 2$ and then (X, d) is not a metric space.

Next, we give the concepts of convergence, Cauchy sequence, b-continuity and b-completeness in a b-metric space.

Definition 1.8 ([4]) Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:

1. b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.
2. A b-Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Proposition 1.9 ([4]) *In a b-metric space (X, d), the following assertions hold:*

(p_1) *A b-convergent sequence has a unique limit.*
(p_2) *Each b-convergent sequence is a b-Cauchy sequence.*
(p_3) *In general, a b-metric is not continuous.*

From the fact that in (p_3), we need the following lemma as regards b-convergent sequences in the proof of our results.

Lemma 1.10 ([1]) *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be b-convergent to points $x, y \in X$, respectively. Then we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x, z).$$

Definition 1.11 ([4]) Let (X, d_X) and (Y, d_Y) be two b-metric spaces.

1. The space (X, d_X) is b-complete if every b-Cauchy sequence in X b-converges.
2. A function $f : X \to Y$ is b-continuous at a point $x \in X$ if it is b-sequentially continuous at x, that is, whenever $\{x_n\}$ is b-convergent to x, $\{f x_n\}$ is b-convergent to $f x$.

Many researchers studied fixed point results in b-metric spaces (see also [5,11,16,19–21,24] and references therein).

Recently, Samet et al. [23] was first introduced the following popular concept.

Definition 1.12 ([23]) Let X be a nonempty set and $\alpha : X \times X \to [0, \infty)$ be a given mapping. A mapping $f : X \to X$ is said to be α-admissible if the following condition holds:

$$x, y \in X \text{ with } \alpha(x, y) \geq 1 \Longrightarrow \alpha(f x, f y) \geq 1.$$

Example 1.13 Let $X = [0, \infty)$. Define $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$f(x) = \begin{cases} \frac{x^2 + 2x + 2}{5}, & x \in [0, 1], \\ |\sin(x^2 + 1)|, & x \in (1, \infty) \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 1 + |\sin(x + y)|, & x, y \in [0, 1] \\ \frac{1}{1 + \max\{x, y\}}, & \text{otherwise.} \end{cases}$$

Then, f is α-admissible.

Samet et al. [23] established fixed point theorems for some type of generalized contraction mapping by using the concept of α-admissible mapping. Also, they applied these results to derive fixed point theorems in partially ordered metric spaces.

In recently, the author [25] gave the new concepts of weak α-admissible mappings as follows:

Definition 1.14 ([25]) Let X be a nonempty set and $\alpha : X \times X \to [0, \infty)$ be a given mapping. A mapping $f : X \to X$ is said to be weak α-admissible if the following condition holds:

$$x \in X \text{ with } \alpha(x, fx) \geq 1 \Longrightarrow \alpha(fx, ffx) \geq 1.$$

Unless otherwise specified, for fixed a nonempty set X and a mapping $\alpha : X \times X \to [0, \infty)$, we use $\mathcal{A}(X, \alpha)$ and $\mathcal{WA}(X, \alpha)$ stand for the collection of all α-admissible mappings on X and the collection of all weak α-admissible mappings on X, that is,

$$\mathcal{A}(X, \alpha) := \{f : X \to X | f \text{ is an } \alpha\text{-admissible mapping}\}$$

and

$$\mathcal{WA}(X, \alpha) := \{f : X \to X | f \text{ is a weak } \alpha\text{-admissible mapping}\}.$$

Remark 1.15 It is easy to see that α-admissibility implies weak α-admissibility, that is, $\mathcal{A}(X, \alpha) \subseteq \mathcal{WA}(X, \alpha)$.

By using the concept of weak α-admissibility, we prove some fixed point theorems satisfying generalized weak contractive condition by using altering distance function in setting of b-metric spaces. Examples are provided in order to distinguish these results from the known ones. Our main result extends and improves many well-known fixed point results in setup of metric spaces and b-metric spaces. We pointed out that many fixed point results in b-metric spaces endowed with partially ordered (or arbitrary binary relation or graph) and fixed point results for cyclic mappings can be concluded from our results. Also, fixed point results for nonlinear mappings satisfying some Lebesgue integral conditions can be obtained by our main results. At the last section, we apply our result to prove the existence of a solution for the nonlinear integral equation as follows:

$$x(c) = \Omega(\phi(c), c) + K(c, c, \phi(c)) + \int_a^b K(c, r, x(r))dr, \tag{1.1}$$

where $a, b \in \mathbb{R}$ with $a < b$, $x \in C[a, b]$ (the set of all continuous real functions defined on $[a, b]$), $\phi : [a, b] \to \mathbb{R}$, $\Omega : \mathbb{R} \times [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are given mappings.

2 Main results

Unless otherwise stated, $Fix(f)$ stands for the set of all fixed points of self mapping f on a nonempty set X, that is,

$$Fix(f) := \{x \in X | fx = x\}.$$

Let (X, d) be a b-metric space with coefficient $s \geq 1$. For each elements x and y, let

$$M_s(x, y) := \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s}\right\},$$

and

$$N(x, y) := \min\{d(x, fx), d(y, fx)\}.$$

We also write $M(x, y)$ instead $M_s(x, y)$ when $s = 1$, that is,

$$M(x, y) := \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\right\}.$$

Definition 2.1 Let (X, d) be a b-metric space with coefficient $s \geq 1$, $\alpha : X \times X \to [0, \infty)$ and $\psi, \varphi : [0, \infty) \to [0, \infty)$ be given mappings. We say that a mapping $f : X \to X$ is an almost generalized $(\alpha, \psi, \varphi)_s$-contractive mapping if there exists $L \geq 0$ such that the following condition holds:

$$\begin{aligned} x, y \in X \text{ with } \alpha(x, y) \geq 1 \quad &\Longrightarrow \quad \psi(s^3 d(fx, fy)) \\ &\leq \psi(M_s(x, y)) - \varphi(M_s(x, y)) + L\psi(N(x, y)). \end{aligned} \tag{2.1}$$

We denote with $\Xi_s(X, \alpha, \psi, \varphi)$ the collection of all almost generalized $(\alpha, \psi, \varphi)_s$-contractive mappings.

Theorem 2.2 *Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$, $\psi, \varphi : [0, \infty) \to [0, \infty)$ be altering distance functions and $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:*

- (AS_1) *$f \in \Xi_s(X, \alpha, \psi, \varphi) \cap \mathcal{WA}(X, \alpha)$;*
- (AS_2) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
- (AS_3) *α has transitive property, that is, for $x, y, z \in X$*

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, z) \geq 1 \quad \Longrightarrow \quad \alpha(x, z) \geq 1;$$

- (AS_4) *f is b-continuous.*

Then $Fix(f) \neq \emptyset$.

Proof Starting from a point $x_0 \in X$ in condition (AS_2), we get $\alpha(x_0, fx_0) \geq 1$. We will construct the Picard iterative sequence $\{x_n\}$ in X, that is,

$$x_{n+1} = fx_n$$

for all $n \in \mathbb{N} \cup \{0\}$. If $x_{\tilde{n}} = x_{\tilde{n}+1}$ for some $\tilde{n} \in \mathbb{N} \cup \{0\}$, then a point $x_{\tilde{n}}$ is a fixed point of f. So we have noting to proof. Now we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, that is, $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. First, we will show that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{2.2}$$

Since $f \in \mathcal{WA}(X, \alpha)$ and $\alpha(x_0, fx_0) \geq 1$, we have

$$\alpha(x_1, x_2) = \alpha(fx_0, ffx_0) \geq 1. \tag{2.3}$$

By continuous this process, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{2.4}$$

for all $n \in \mathbb{N} \cup \{0\}$. It follows from $f \in \Xi_s(X, \alpha, \psi, \varphi)$ that inequality (2.4) implies that

$$\begin{aligned}\psi(d(fx_n, fx_{n+1})) &\leq \psi(s^3 d(fx_n, fx_{n+1}))\\ &\leq \psi(M_s(x_n, x_{n+1})) - \varphi(M_s(x_n, x_{n+1})) + L\psi(N(x_n, x_{n+1}))\end{aligned} \tag{2.5}$$

for all $n \in \mathbb{N} \cup \{0\}$. Note that for each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}&M_s(x_n, x_{n+1})\\ &= \max\left\{d(x_n, x_{n+1}), d(x_n, fx_n), d(x_{n+1}, fx_{n+1}), \frac{d(x_n, fx_{n+1}) + d(x_{n+1}, fx_n)}{2s}\right\}\\ &= \max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2s}\right\}\\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}\end{aligned}$$

and

$$\begin{aligned}N(x_n, x_{n+1}) &= \min\{d(x_n, fx_n), d(x_{n+1}, fx_n)\}\\ &= \min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+1})\}\\ &= \min\{d(x_n, x_{n+1}), 0\}\\ &= 0.\end{aligned}$$

If $M_s(x_{n^*}, x_{n^*+1}) = d(x_{n^*+1}, x_{n^*+2})$ for some $n^* \in \mathbb{N} \cup \{0\}$, then inequality (2.5) implies that

$$\begin{aligned}\psi(d(fx_{n^*}, fx_{n^*+1})) &\leq \psi(d(x_{n^*+1}, x_{n^*+2})) - \varphi(d(x_{n^*+1}, x_{n^*+2})) + L\psi(0)\\ &< \psi(d(x_{n^*+1}, x_{n^*+2})),\end{aligned}$$

which is a contradiction. Therefore, $M_s(x_n, x_{n+1}) = d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. From (2.5), we have

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) &= \psi(d(fx_n, fx_{n+1}))\\ &\leq \psi(d(x_n, x_{n+1})) - \varphi(d(x_n, x_{n+1})) + L\psi(0)\\ &< \psi(d(x_n, x_{n+1})),\end{aligned} \tag{2.6}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since ψ is a non-decreasing mapping, we get $\{d(x_n, x_{n+1})\}$ is decreasing sequence in $\mathbb{R}$ and then there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Taking limit as $n \to \infty$ in (2.6), we get

$$\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r)$$

and thus $\varphi(r) = 0$. This implies that $r = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

This claims that (2.2) holds.

Next, we will prove that $\{x_n\}$ is a b-Cauchy sequence in X. Assume this to contrary that there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k) > m(k) \geq k$ and

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon \tag{2.8}$$

and $n(k)$ is the smallest number such that (2.8) holds. From (2.8), we have

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \tag{2.9}$$

By (B_3), (2.8) and (2.9), we get

$$\begin{aligned} \epsilon &\leq d(x_{m(k)}, x_{n(k)}) \\ &\leq s[d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})] \\ &< s[\epsilon + d(x_{n(k)-1}, x_{n(k)})]. \end{aligned} \tag{2.10}$$

Taking limit supremum as $k \to \infty$ in (2.10), by using (2.7) we get

$$\epsilon \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq s\epsilon. \tag{2.11}$$

Again, by using (B_3), we obtain that

$$d(x_{m(k)}, x_{n(k)}) \leq s[d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})] \tag{2.12}$$

and

$$d(x_{m(k)}, x_{n(k)+1}) \leq s[d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})]. \tag{2.13}$$

Taking limit supremum as $k \to \infty$ in (2.12) and (2.13), from (2.7) and (2.11), we get

$$\epsilon \leq s\left(\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1})\right). \tag{2.14}$$

and

$$\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq s^2\epsilon. \tag{2.15}$$

From (2.14) and (2.15), we have

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq s^2\epsilon. \tag{2.16}$$

Similarly, we can show that

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) \leq s^2\epsilon. \tag{2.17}$$

Finally, we obtain that

$$\begin{aligned} d(x_{m(k)+1}, x_{n(k)+1}) &\leq s[d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)+1})] \\ &\leq sd(x_{m(k)+1}, x_{m(k)}) + s^2[d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})]. \end{aligned} \tag{2.18}$$

Taking limit supremum as $k \to \infty$ in (2.18), we have

$$\limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq s^3\epsilon. \tag{2.19}$$

Using (B_3) again, we have

$$\begin{aligned} d(x_{m(k)}, x_{n(k)}) &\leq s[d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)})] \\ &\leq sd(x_{m(k)}, x_{m(k)+1}) + s^2[d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})]. \end{aligned} \tag{2.20}$$

Taking limit supremum as $k \to \infty$ and using (2.7) and (2.11), we have,

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}). \tag{2.21}$$

From (2.19) and (2.21), we get,

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq s^3\epsilon. \tag{2.22}$$

By using transitivity property of α, we get

$$\alpha(x_{m(k)}, x_{n(k)}) \geq 1.$$

Since $f \in \Xi_s(X, \alpha, \psi, \varphi)$, we have

$$\begin{aligned} \psi(s^3 d(x_{m(k)+1}, x_{n(k)+1})) = \psi(s^3 d(fx_{m(k)}, fx_{n(k)})) &\leq \psi(M_s(x_{m(k)}, x_{n(k)})) \\ &\quad -\varphi(M_s(x_{m(k)}, x_{n(k)})) + L\psi(N(x_{m(k)}, x_{n(k)})), \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} &M_s(x_{m(k)}, x_{n(k)}) \\ &= \max\left\{d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, fx_{m(k)}), d(x_{n(k)}, fx_{n(k)}),\right. \\ &\quad \left.\times\frac{d(x_{m(k)}, fx_{n(k)}) + d(x_{n(k)}, fx_{m(k)})}{2s}\right\} \\ &= \max\left\{d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}),\right. \\ &\quad \left.\times\frac{d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1})}{2s}\right\} \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} N(x_{m(k)}, x_{n(k)}) &= \min\{d(x_{m(k)}, fx_{m(k)}), d(x_{n(k)}, fx_{m(k)})\} \\ &= \min\{d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{m(k)+1})\}. \end{aligned} \tag{2.25}$$

Taking limit supremum as $k \to \infty$ in Eqs. (2.24) and (2.25) and using (2.7), (2.11), (2.16) and (2.17), we have

$$\epsilon = \max\left\{\epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{2s}\right\} \leq \limsup_{k\to\infty} M_s(x_{m(k)}, x_{n(k)}) \leq \max\left\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2s}\right\} = s\epsilon$$

and

$$\limsup_{k\to\infty} N(x_{m(k)}, x_{n(k)}) = 0.$$

Similarly, we can show that

$$\epsilon = \max\left\{\epsilon, \frac{\frac{\epsilon}{s}+\frac{\epsilon}{s}}{2s}\right\} \leq \liminf_{k\to\infty} M_s(x_{m(k)}, x_{n(k)}) \leq \max\left\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2s}\right\} = s\epsilon.$$

Taking limit as $k \to \infty$ in (2.23), we have

$$\begin{aligned}\psi(s\epsilon) &= \psi\left(s^3\left(\frac{\epsilon}{s^2}\right)\right)\\ &\leq \psi\left(s^3 \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1})\right)\\ &\leq \psi\left(\limsup_{k\to\infty} M_s(x_{m(k)}, x_{n(k)})\right) - \varphi\left(\liminf_{k\to\infty} M_s(x_{m(k)}, x_{n(k)})\right)\\ &\quad + L\psi\left(\limsup_{k\to\infty} N(x_{m(k)}, x_{n(k)})\right)\\ &\leq \psi(s\epsilon) - \varphi(\epsilon) + L\psi(0). \end{aligned} \tag{2.26}$$

This implies that $\varphi(\epsilon) = 0$ and hence $\epsilon = 0$, which is a contradiction. Therefore, $\{x_n\}$ is a b-Cauchy sequence. By the b-completeness of b-metric space X, there exists $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

By b-continuity of f, we get

$$\lim_{n\to\infty} d(fx_n, fx) = 0.$$

From the triangle inequality, we have

$$d(x, fx) \leq s[d(x, fx_n) + d(fx_n, fx)] \tag{2.27}$$

for all $n \in \mathbb{N} \cup \{0\}$. Taking limit as $n \to \infty$ in above inequality, we obtain that

$$d(x, fx) = 0$$

and then $fx = x$. This shows that $Fix(f) \neq \emptyset$. □

Example 2.3 Let $X = \mathbb{R}$ and $d : X \times X \to [0, \infty)$ be defined by

$$d(x, y) = |x - y|^2$$

for all $x, y \in X$. Then (X, d) is a b-complete b-metric space with coefficient $s = 2$. Define mappings $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$fx = \begin{cases} \sinh^{-1}\frac{x}{6}, & x \in [0, 8];\\ \ln(2x - 13), & x \in (8, \infty)\end{cases}$$

and

$$\alpha(x, y) = \begin{cases} x + \cosh(2x + y), & x, y \in [0, 8];\\ \tanh(x - y), & \text{otherwise}.\end{cases}$$

Also, define two altering distance functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ by $\psi(t) = rt$ and $\varphi(t) = (r - 1)t$ for all $t \in [0, \infty)$, where $r \in (1, 4)$.

Next, we show that $f \in \Xi_s(X, \alpha, \psi, \varphi)$. Assume that $\alpha(x, y) \geq 1$ and hence $x, y \in [0, 8]$. By using the mean value theorem simultaneously for the inverse hyperbolic sine function we get,

$$\begin{aligned}
\psi(2^3 d(fx, fy)) &= 8r|fx - fy|^2 \\
&= 8r\left|\sinh^{-1}\frac{x}{6} - \sinh^{-1}\frac{y}{6}\right|^2 \\
&\leq 8r\left|\frac{x}{6} - \frac{y}{6}\right|^2 \\
&\leq |x - y|^2 \\
&\leq M_s(x, y) \\
&\leq \psi(M_s(x, y)) - \varphi(M_s(x, y)) + L\psi(N(x, y))
\end{aligned}$$

for each $L \geq 0$. This implies that (2.1) holds and thus $f \in \Xi_s(X, \alpha, \psi, \varphi)$.

It is easy to see that $f \in \mathcal{WA}(X, \alpha)$. Indeed, if $x \in X$ such that $\alpha(x, fx) \geq 1$, then $x, fx \in [0, 8]$. This implies that $ffx \in [0, 8]$ and hence $\alpha(fx, ffx) \geq 1$. Also, we can see that f is continuous and there is $x_0 = 1$ such that

$$\alpha(x_0, fx_0) = \alpha(1, f1) = \alpha(1, \sin^{-1} 1/6) = 1 + \cosh(2 + \sin^{-1} 1/6) \geq 1.$$

Therefore, all the conditions of Theorem 2.2 are satisfied. Then we can conclude that $Fix(f) \neq \emptyset$. In this example, it is easy to see that $0 \in Fix(f)$.

Theorem 2.4 *Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$, $\psi, \varphi : [0, \infty) \to [0, \infty)$ be altering distance functions and $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be two given mappings. Suppose that the following conditions hold:*

- (AS_1) *$f \in \Xi_s(X, \alpha, \psi, \varphi) \cap \mathcal{WA}_s(X, \alpha)$;*
- (AS_2) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
- (AS_3) *α has transitive property;*
- $(\widetilde{AS_4})$ *X is α-regular, that is, if $\{x_n\}$ is sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then $Fix(f) \neq \emptyset$.

Proof Following the proof of Theorem 2.2, we know that $\{x_n\}$ is a b-Cauchy sequence in the b-complete b-metric space (X, d). Then, there exists $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0, \tag{2.28}$$

that is, $x_n \to x$ as $n \to \infty$. From α-regularity of X, we get

$$\alpha(x_n, x) \geq 1$$

for all $n \in \mathbb{N}$. Since $f \in \Xi_s(X, \alpha, \psi, \varphi)$, we have

$$\psi(s^3 d(fx_n, fx)) \leq \psi(M_s(x_n, x)) - \varphi(M_s(x_n, x)) + L\psi(N(x_n, x)), \tag{2.29}$$

where

$$M_s(x_n, x) = \max\left\{d(x_n, x), d(x_n, fx_n), d(x, fx), \frac{d(x_n, fx) + d(x, fx_n)}{2s}\right\}$$

and

$$N(x_n, x) = \min\{d(x_n, fx_n), d(x, fx_n)\}.$$

Taking limit as $n \to \infty$ in (2.29) and using Lemma 1.10, we obtain that

$$\begin{aligned}
\psi(d(x, fx)) &\leq \psi(s^2 d(x, fx)) \\
&= \psi\left(s^3 \frac{1}{s} d(x, fx)\right) \\
&\leq \psi\left(\limsup_{n\to\infty} s^3 d(x_{n+1}, fx)\right) \\
&= \psi\left(\limsup_{n\to\infty} s^3 d(fx_n, fx)\right) \\
&\leq \psi\left(\limsup_{n\to\infty} M_s(x_n, x)\right) - \varphi\left(\liminf_{n\to\infty} M_s(x_n, x)\right) \\
&\quad + L\psi\left(\limsup_{n\to\infty} N(x_n, x)\right) \\
&\leq \psi(d(x, fx)) - \varphi(d(x, fx)) + L\psi(0).
\end{aligned}$$

This implies that $\varphi(d(x, fx)) = 0$, equivalently, $d(x, fx) = 0$ and so $x = fx$. Therefore, $Fix(f) \neq \emptyset$. This completes the proof. □

From Remark 1.15, we get the following results for class $\mathcal{A}(X, \alpha)$.

Corollary 2.5 *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$, $\psi, \varphi : [0, \infty) \to [0, \infty)$ be altering distance functions and $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be two given mappings. Suppose that the following conditions hold:*

$(\widetilde{AS_1})$ *$f \in \Xi_s(X, \alpha, \psi, \varphi) \cap \mathcal{A}(X, \alpha)$;*
(AS_2) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(AS_3) *α has transitive property;*
(AS_4) *f is b-continuous.*

Then $Fix(f) \neq \emptyset$.

Corollary 2.6 *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$, $\psi, \varphi : [0, \infty) \to [0, \infty)$ be altering distance functions and $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be two given mappings. Suppose that the following conditions hold:*

$(\widetilde{AS_1})$ *$f \in \Xi_s(X, \alpha, \psi, \varphi) \cap \mathcal{A}(X, \alpha)$;*
(AS_2) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(AS_3) *α has transitive property;*
$(\widetilde{AS_4})$ *X is α-regular.*

Then $Fix(f) \neq \emptyset$.

Theorems 2.2, 2.4, Corollaries 2.5 and 2.6 unify, extend and improve several fixed point results in b-metric spaces. Also, since a b-metric is a metric when $s = 1$, so our results can be viewed as a generalization and extension of the following results:

- The classical Banach contraction principle [3], Kannan's fixed point theorem [14], Chaterjia's fixed point theorem [6] in the framework of metric space;

- Alber et al.'s fixed point theorem [2] in the setup of Hilbert spaces;
- Rhoades's fixed point theorem [22];
- Dutta and Choudhury's fixed point theorem [12].

Note that, it has been pointed out in some studies that the following fixed point results can be concluded from the fixed point results related with α-admissible mappings:

- fixed point results in b-metric spaces endowed with partially ordered;
- fixed point results in b-metric spaces endowed with an arbitrary binary relation;
- fixed point results in b-metric spaces endowed with graph;
- fixed point results for cyclic mappings.

Next, we show that the fixed point results for nonlinear mappings satisfying some Lebesgue integral conditions can be obtained by our results.

Let Θ denote the set of all functions $\theta : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(θ_1) θ is a Lebesgue integrable function on each compact subset of $[0, \infty)$;
(θ_2) for each $\epsilon > 0$, we have $\int_0^\epsilon \theta(s)ds > 0$.

Remark 2.7 It is an easy matter to check that the mapping $\psi : [0, \infty) \to [0, \infty)$ defined by

$$\psi(t) = \int_0^t \theta(s)ds$$

is an altering distance function.

From above remark, fixed point results for nonlinear mappings satisfying some Lebesgue integral conditions can be obtained by our main results.

3 Applications to the existence of a solution for a nonlinear integral equation

In this section, we prove the existence theorem for a solution of the following integral equation by using our main result in Sect. 2:

$$x(c) = \Omega(\phi(c), c) + K(c, c, \phi(c)) + \int_a^b K(c, r, x(r))dr, \tag{3.1}$$

where $a, b \in \mathbb{R}$ with $a < b$, $x \in C[a, b]$ (the set of all continuous real functions defined on $[a, b]$), $\phi : [a, b] \to \mathbb{R}$, $\Omega : \mathbb{R} \times [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are given mappings.

Theorem 3.1 *Consider the integral equation* (3.1). *Suppose that the following conditions hold:*

($\spadesuit_1$) $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ *is continuous and non-decreasing in the third ordered;*
($\spadesuit_2$) *there exists* $p > 1$ *satisfies the following condition:*
for each $r, c \in [a, b]$ *and* $x, y \in X$ *with* $x(w) \leq y(w)$ *for all* $w \in [a, b]$, *we have*

$$|K(c, r, x(r)) - K(c, r, y(r))| \leq \xi(c, r)(\Gamma(|x(r) - y(r)|^p)),$$

where $\xi : [a,b] \times [a,b] \to [0,\infty)$ *is a continuous function satisfying*

$$\sup_{c\in[a,b]} \left(\int_a^b \xi(c,r)^p dr \right) < \frac{1}{2^{3p^2-3p}(b-a)^{p-1}}$$

and $\Gamma : [0,\infty) \to [0,\infty)$ *is continuous non-decreasing and satisfying the following condition:*

(Γ_1) $\Gamma(t) = 0$ *if and only if* $t = 0$*;*
(Γ_2) $\Gamma(t) < t$ *and* $\frac{d}{dt}(\Gamma(t)) < 1$ *for all* $t > 0$.

($\spadesuit_3$) *there exists* $x_0 \in X$ *such that* $x_0(c) \leq \Omega(\phi(c),c) + K(c,c,\phi(c)) + \int_a^b K(c,r,x_0(r))dr$ *for all* $c \in [a,b]$.

Then the integral equation (3.1) *has a solution.*

Proof Let $X = C[a,b]$ and define a mapping $f : X \to X$ by

$$(fx)(c) = \Omega(\phi(c),c) + K(c,c,\phi(c)) + \int_a^b K(c,r,x(r))dr$$

for all $x \in X$ and $c \in [a,b]$. Define a mapping $d : X \times X \to \mathbb{R}_+$ by

$$d(x,y) = \sup_{c\in[a,b]} |x(c) - y(c)|^p$$

for all $x, y \in X$. Clearly, (X,d) is a b-complete b-metric space with coefficient $s = 2^{p-1} > 1$. Next, we define a mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & x(c) \leq y(c) \text{ for all } c \in [a,b]; \\ \lambda, & \text{otherwise,} \end{cases}$$

where $\lambda \in (0,1)$. It is easy to see that α has a transitive property. It follows from K is non-decreasing in the third ordered that $f \in \mathcal{A}(X,\alpha)$. From ($\spadesuit_3$), we get $\alpha(x_0, fx_0) \geq 1$. In [18], we get condition $(\widetilde{AS_4})$ in Theorem 2.4 holds.

Now define functions $\psi, \varphi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t^p$ and $\varphi(t) = t^p - (\Gamma(t))^p$ for all $t \in [0,\infty)$. It is easy to see that ψ is altering distance function. By (Γ_1) and (Γ_2), we can prove that φ is also altering distance.

Next, we show that $f \in \Xi_s(X,\alpha,\psi,\varphi)$. Choosing $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $x, y \in X$ be such that $\alpha(x,y) \geq 1$, that is, $x(c) \leq y(c)$ for all $c \in [a,b]$. From ($\spadesuit_1$), ($\spadesuit_2$) and and Hölder inequality, for each $c \in [a,b]$ we get

$$\begin{aligned}
&(2^{3p-3}|(fx)(c) - (fy)(c)|)^p \\
&\leq 2^{3p^2-3p} \left(\int_a^b |K(c,r,x(r)) - K(c,r,y(r))| dr \right)^p \\
&\leq 2^{3p^2-3p} \left[\left(\int_a^b 1^q dr \right)^{\frac{1}{q}} \left(\int_a^b |K(c,r,x(r)) - K(c,r,y(r))|^p dr \right)^{\frac{1}{p}} \right]^p
\end{aligned}$$

$$
\begin{aligned}
&\leq 2^{3p^2-3p}(b-a)^{\frac{p}{q}}\left(\int_a^b \xi(c,r)^p(\Gamma(|x(r)-y(r)|^p))^p dr\right) \\
&\leq 2^{3p^2-3p}(b-a)^{\frac{p}{q}}\left(\int_a^b \xi(c,r)^p(\Gamma(d(x,y)))^p dr\right) \\
&\leq 2^{3p^2-3p}(b-a)^{\frac{p}{q}}\left(\int_a^b \xi(c,r)^p(\Gamma(M_s(x,y)))^p dr\right) \\
&= 2^{3p^2-3p}(b-a)^{p-1}\left(\int_a^b \xi(c,r)^p dr\right)(\Gamma(M_s(x,y)))^p \\
&< (\Gamma(M_s(x,y)))^p \\
&\leq M_s(x,y)^p - [M_s(x,y)^p - (\Gamma(M_s(x,y)))^p] + LN(x,y)^p.
\end{aligned}
$$

for each $L \geq 0$. This implies that

$$
\begin{aligned}
\psi(s^3 d(fx,fy)) &= (s^3 d(fx,fy))^p \\
&= \left(2^{3p-3}\sup_{c\in[a,b]}|(fx)(c)-(fy)(c)|\right)^p \\
&\leq M_s(x,y)^p - [M_s(x,y)^p - (\Gamma(M_s(x,y)))^p] + LN(x,y)^p \\
&= \psi(M_s(x,y)) - \varphi(M_s(x,y)) + L\psi(N(x,y))
\end{aligned}
$$

for all $x, y \in X$. This claims that $f \in \Xi_s(X, \alpha, \psi, \varphi)$.

Therefore, by using Theorem 2.4, we can conclude that $Fix(f) \neq \emptyset$, that is, there exists $x \in X$ such that x is a fixed point of f. This implies that x is a solution for (3.1) because the existence of a solution of (3.1) is equivalent to the existence of a fixed point of f. This completes the proof. □

Under some setting function Γ, we get the following result:

Corollary 3.2 *Consider the integral equation* (3.1). *Suppose that the following conditions hold:*

($\spadesuit_1$) $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ *is continuous and nondecreasing at the third ordered;*
($\spadesuit_2'$) *there exists* $p > 1$ *satisfies the following condition:*
for each $r, c \in [a,b]$ *and* $x, y \in X$ *with* $x(w) \leq y(w)$ *for all* $w \in [a,b]$, *we have*

$$|K(c,r,x(r)) - K(c,r,y(r))| \leq \xi(c,r)(\ln(1 + |x(r)-y(r)|^p)),$$

where $a > 1$ *and* $\xi : [a,b] \times [a,b] \to [0,\infty)$ *is a continuous function satisfying*

$$\sup_{c\in[a,b]}\left(\int_a^b \xi(c,r)^p dr\right) < \frac{1}{2^{3p^2-3p}(b-a)^{p-1}}.$$

($\spadesuit_3$) *there exists* $x_0 \in X$ *such that* $x_0(c) \leq \Omega(\phi(c),c) + K(c,c,\phi(c)) + \int_a^b K(c,r,x_0(r))dr$ *for each* $c \in [a,b]$.

Then the integral equation (3.1) *has a solution.*

Proof Follows from Theorem 3.1 by taking $\Gamma(t) = \ln(1+t)$, we get this result. □

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