RACSAM (2016) 110:433–456
DOI 10.1007/s13398-015-0242-6

ORIGINAL PAPER

Last remarks on *G*-metric spaces and related fixed point theorems

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Received: 12 January 2015 / Accepted: 22 July 2015 / Published online: 6 August 2015

Abstract In this report, we present some new fixed points theorems in the context of quasi-metric spaces that can be particularized in a wide range of different frameworks (metric spaces, partially ordered metric spaces, G-metric spaces, etc.). Our contractivity conditions involve different classes of functions and we study the case in which they only depend on a unique variable. Furthermore, we do not only introduce new contractivity conditions, but also expansivity conditions. As a consequence of our results, we announce that many fixed point results in G-metric spaces can be derived from the existing results if all arguments are not distinct.

Keywords Fixed point · Quasi-metric space · G-metric spaces · Contractivity condition · Picard sequence

Mathematics Subject Classification 47H10 · 47H09 · 54H25 · 46T99

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1 Introduction

In Mathematics, it is very natural to search for a new algebraic structure in order to improve the obtained results in the literature. One of the most suitable examples for this motivation is the investigation on the extension of the notion of *metric*, which was initiated by Fréchet. The notion of metric has been examined and extended in various ways such as quasi-metrics, b-metrics, fuzzy metrics, probabilistic metrics, partial metrics, two-metrics, D-metrics, G-metrics, cone metrics (Banach-valued metrics), etc. In this paper, we focus on the notion of G-metric, introduced by Mustafa and Sims [48] in 2007, as a generalization of the concept D-metric defined by Dhage [26], and, in particular, as a generalization of the concept of metric. Mustafa and Sims published a series of papers in which they investigated the topology of G-metric spaces and also discussed some results about existence and uniqueness of fixed points in the context of G-metric spaces. Following these results, several authors have continued this study in the setting of G-metric spaces using more and more general operators (see, e.g., [3,5–51,53–55,57,58] and references therein).

Recently, Samet et al. [56], and Jleli and Samet [31], announced that G-metric spaces and quasi-metric spaces coincide under certain conditions. In these papers, the authors also proved that several published fixed point results in the context of G-metric spaces can be derived from the existing theorems in the context of associative metric spaces. On the other hand, in the recent papers [2,6,55], some authors suggested another approach for which the techniques in [31,56] are not applicable. At the first glance, the authors seem right due to their explanation that all arguments should be distinct. However, it is possible to express the contractive conditions in [2,6,55] in terms of quasi-metric spaces after a suitable substitution.

One of the main goals of this note is to investigate the existence and uniqueness of a fixed point of certain mapping in the context of quasi-metric spaces. Additionally, we prove that fixed point theorems in the context of G-metric spaces that were suggested in [2,6,55] can also be considered as a consequence of their corresponding existing results via the techniques showed in [31,56].

2 Preliminaries

For the sake of the completeness, we collect in this section some basic definitions and well known results in this field. Firstly, let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of all positive integers and all real numbers, respectively. If $A \subseteq \mathbb{R}$ is a nonempty subset of $\mathbb{R}$, the *Euclidean metric on A* is $d(x, y) = |x - y|$ for all $x, y \in A$. In the sequel, let X be a nonempty set. Given a natural number n, we use X^n to denote the nth *Cartesian power of* X, that is, $X \times X \times \cdots \times X$ (n times).

From now on, let $T : X \to X$ be a self-mapping (also called *operator*). For simplicity, we denote $T(x)$ by Tx and $T \circ T$ by T^2. In general, the *iterates* of a self-mapping T are the mappings $\{T^n : X \to X\}_{n\geq 0}$ defined by

$$T^0 = \text{identity on } X, \quad T^1 = T, \quad T^2 = T \circ T, \quad T^{n+1} = T \circ T^n \quad \text{for all} \quad n \geq 2.$$

Given a point $x \in X$, the *Picard sequence of the operator T (based on x)* is the sequence $\{T^n x\}_{n\geq 0}$, which we will denote by $\{x_n\}$.

The main aim of this manuscript if to show some sufficient conditions to ensure existence and uniqueness of the following kind of points. A *fixed point of* T is a point $x \in X$ such that $Tx = x$. Henceforth, $\mathrm{Fix}(T)$ will mean the family of all fixed points of T.

A function $\phi : [0, \infty) \to [0, \infty)$ is *lower semi-continuous* if $\phi(t) \leq \liminf_{s \to t} \phi(s)$ for all $t \geq 0$. A lower semi-continuous function ϕ must verify the following condition:

$$\text{if } \{t_n\} \subset [0, \infty) \text{ verifies } \{t_n\} \to t \geq 0, \quad \text{then} \quad \phi(t) \leq \liminf_{n \to \infty} \phi(t_n).$$

A *metric* (or a *distance function*) on a set X is a function $d : X \times X \to [0, \infty)$ verifying the following conditions: for all $x, y, z \in X$,

- $d(x, x) = 0.$ (1)
- $d(x, y) > 0 \quad \text{if } x \neq y.$ (2)
- $d(x, y) = d(y, x).$ (3)
- $d(x, y) \leq d(x, z) + d(z, y).$ (4)

In such a case, the pair (X, d) is called a *metric space*.

2.1 G-metric spaces

The notion of G-metric space is defined as follows.

Definition 2.1 (Mustafa and Sims [48]) Let X be a set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G_1) $G(x, y, z) = 0$ if $x = y = z$;
(G_2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ (rectangle inequality) for all $x, y, z, a \in X$.

Then the function G is called a *generalized metric*, or, more specifically, a *G-metric* on X, and the pair (X, G) is called a *G-metric space*.

Note that every G-metric on X induces a metric d_G on X defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \quad \text{for all} \quad x, y \in X.$$

For a better understanding of the subject we give the following examples of G-metrics:

Example 2.1 Let (X, d) be a metric space. The function $G : X \times X \times X \to [0, +\infty)$, defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$$

for all $x, y, z \in X$, is a G-metric on X.

Example 2.2 (See e.g. [48]) Let $X = [0, \infty)$. The function $G : X \times X \times X \to [0, \infty)$, defined by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|$$

for all $x, y, z \in X$, is a G-metric on X.

In their initial paper, Mustafa and Sims [48] also defined the basic topological concepts in G-metric spaces as follows:

Definition 2.2 (Mustafa and Sims [48]) Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is *G-convergent to $x \in X$* if

$$\lim_{n,m\to\infty} G(x, x_n, x_m) = 0,$$

that is, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all $n, m \geq N$. We call x the *limit* of the sequence and we write $\{x_n\} \to x$ or $\lim_{n\to\infty} x_n = x$.

It is clear that the limit of a convergent sequence is unique.

Proposition 2.1 (Mustafa and Sims [48]) *In a G-metric space (X, G), the following conditions are equivalent.*

1. *$\{x_n\}$ is G-convergent to x.*
2. *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*
3. *$G(x_n, x, x) \to 0$ as $n \to \infty$.*

Definition 2.3 (Mustafa and Sims [48]) Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called a *G-Cauchy sequence* if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2.2 (Mustafa and Sims [48]) *In a G-metric space (X, G), the following conditions are equivalent.*

1. *The sequence $\{x_n\}$ is G-Cauchy.*
2. *For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geq N$.*

Definition 2.4 (Mustafa and Sims [48]) A G-metric space (X, G) is called *G-complete* if every G-Cauchy sequence is G-convergent in (X, G).

Definition 2.5 Let (X, G) be a G-metric space. A mapping $T : X \to X$ is said to be *G-continuous* if $\{Tx_n\}$ G-converges to Tx for any G-convergent sequence $\{x_n\}$ to $x \in X$. In general, given $m \in \mathbb{N}$, a mapping $F : X^m \to X$ is said to be *G-continuous* if $\{F(x_n^1, x_n^2, \ldots, x_n^m)\}$ G-converges to $F(x^1, x^2, \ldots, x^m)$ for any G-convergent sequences $\{x_n^1\}, \{x_n^2\}, \ldots, \{x_n^m\} \subseteq X$ such that $\{x_n^i\} \to x^i \in X$ for all $i \in \{1, 2, \ldots, m\}$.

Mustafa [37] extended the well-known Banach Contraction Mapping Principle to the framework of G-metric spaces as follows.

Theorem 2.1 (Mustafa [37]) *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition:*

$$G(Tx, Ty, Tz) \leq k\, G(x, y, z) \quad \textit{for all} \quad x, y, z \in X, \tag{5}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Theorem 2.2 (Mustafa [37]) *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition*:

$$G(Tx, Ty, Ty) \leq k\, G(x, y, y) \quad \textit{for all} \quad x, y, z \in X, \tag{6}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Remark 2.1 We notice that condition (5) implies condition (6). The converse is only true if $k \in [0, \frac{1}{2})$. For details, see [37].

By the rectangle inequality (G_5) together with the symmetry (G_4), we have, for all $x, y \in X$,

$$G(x, y, y) = G(y, y, x) \leq G(y, x, x) + G(x, y, x) = 2G(y, x, x).$$

Lemma 2.1 [37] *If (X, G) is a G-metric space and $x, y \in X$, then*

$$G(x, y, y) \leq 2G(y, x, x). \tag{7}$$

Theorem 2.3 (See [47]) *Let (X, G) be a G-metric space. Let $T : X \to X$ be a mapping such that*

$$G(Tx, Ty, Tz) \leq aG(x, y, z) + bG(x, Tx, Tx) + cG(y, Ty, Ty) + dG(z, Tz, Tz) \tag{8}$$

for all $x, y, z \in X$, where a, b, c, d are nonnegative constants such that $k = a+b+c+d < 1$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem 2.4 (See [45]) *Let (X, G) be a G-metric space. Let $T : X \to X$ be a mapping such that*

$$G(Tx, Ty, Tz) \leq k[G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)] \tag{9}$$

for all $x, y, z \in X$, where $k \in [0, \frac{1}{3})$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem 2.5 (See [47]) *Let (X, G) be a G-metric space. Let $T : X \to X$ be a mapping such that*

$$G(Tx, Ty, Tz) \leq aG(x, y, z) + b[G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)] \tag{10}$$

for all $x, y, z \in X$, where a, b are nonnegative constants such that $a + b < 1$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem 2.6 (See [47]) *Let (X, G) be a G-metric space. Let $T : X \to X$ be a mapping such that*

$$G(Tx, Ty, Tz) \leq aG(x, y, z) + b \max\{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\} \tag{11}$$

for all $x, y, z \in X$, where a, b are nonnegative constants such that $a + b < 1$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem 2.7 (See [49]) *Let (X, G) be a G-metric space. Let $T : X \to X$ be a mapping such that*

$$G(Tx, Ty, Tz) \leq k \max\Big\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), G(z, Tx, Tx), G(x, Ty, Ty), G(y, Tz, Tz)\Big\} \tag{12}$$

for all $x, y, z \in X$, where $k \in [0, \frac{1}{2})$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem 2.8 (See e.g. [31]) *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a given mapping satisfying*

$$G(Tx, Ty, Tz) \leq G(x, y, z) - \varphi(G(x, y, z)) \tag{13}$$

if for all $x, y \in X$, where $\varphi : [0, \infty) \to [0, \infty)$ is continuous with $\varphi^{-1}(\{0\}) = 0$. Then there is a unique $x \in X$ such that $Tx = x$.

Theorem 2.9 [23] *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping with the property:*

$$d(Tx, Ty) \leq q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{14}$$

for all $x \in X$, where q is a constant such that $q \in [0, 1)$. Then T has a unique fixed point.

2.2 Quasi-metric spaces

The notion of *quasi-metric space* was introduced by Wilson [60], and it constitutes a field with an extensive research from the paper [34] by Kelly. Furthermore, the notions of a (left, right) Cauchy sequence of a quasi-metric space are clearly inspired in Reilly et al.'s work [52]. The classical monograph by Fletcher and Lindgren [29] and the updated book [25] by Cobzaş provide two basic references on quasi-metric spaces.

Definition 2.6 A function $q : X \times X \to [0, \infty)$ is a *quasi-metric on X* if it satisfies (1), (2) and (4), that is, if it verifies, for all $x, y, z \in X$:

(q_1) $q(x, y) = 0$ if, and only if, $x = y$,
(q_2) $q(x, y) \leq q(x, z) + q(z, y)$.

In such a case, the pair (X, q) is called a *quasi-metric space*.

Remark 2.2 Any metric space is a quasi-metric space, but the converse is not true in general.

Now, we recollect some basic topological notions and related results about quasi-metric spaces.

Definition 2.7 Let (X, q) be a quasi-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. We will say that:

- $\{x_n\}$ *converges to x* (and we will denote it by $\{x_n\} \xrightarrow{q} x$) if $\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x_n) = 0$;
- $\{x_n\}$ is a *Cauchy sequence* if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $q(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$.

The quasi-metric space (X, q) is said to be *complete* if every Cauchy sequence is convergent on (X, q).

As q is not necessarily symmetric, some authors distinguished between left/right Cauchy/convergent sequences and completeness.

Definition 2.8 (Jleli and Samet [31]) Let (X, q) be a quasi-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. We say that:

- $\{x_n\}$ *right-converges to x* if $\lim_{n\to\infty} q(x_n, x) = 0$;
- $\{x_n\}$ *left-converges to x* if $\lim_{n\to\infty} q(x, x_n) = 0$;
- $\{x_n\}$ is a *right-Cauchy sequence* if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $q(x_n, x_m) < \varepsilon$ for all $m > n \geq n_0$;
- $\{x_n\}$ is a *left-Cauchy sequence* if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $q(x_m, x_n) < \varepsilon$ for all $m > n \geq n_0$;
- (X, q) is *right-complete* if every right-Cauchy sequence is right-convergent;
- (X, q) is *left-complete* if every left-Cauchy sequence is left-convergent.

Remark 2.3 (See e.g. [31]) A sequence $\{x_n\}$ in a quasi-metric space is Cauchy if, and only if, it is left-Cauchy and right-Cauchy.

Remark 2.4 1. The limit of a sequence in a quasi-metric space, if there exists, is unique. However, this is false if we consider right-limits or left-limits.
2. If $\{x_n\} \to x$ and $\{y_n\} \to y$ in a quasi-metric space, then $\{q(x_n, y_n)\} \to q(x, y)$, that is, q is continuous on both arguments. It follows from:

$$q(x, y) - q(x, x_n) - q(y_n, y) \leq q(x_n, y_n) \leq q(x_n, x) + q(x, y) + q(y, y_n)$$

for all n. In particular, $\{q(x_n, z)\} \to q(x, z)$ and $\{q(z, x_n)\} \to q(z, x)$ for all $z \in X$.
3. If a sequence $\{x_n\}$ has a right-limit x and a left-limit y, then $x = y$, $\{x_n\}$ converges and it has an only limit (from the right and from the left). However, it is possible that a sequence has two different right-limits when it has no left-limit.

Example 2.3 Let X be a subset of $\mathbb{R}$ containing [0,1] and define, for all $x, y \in X$,

$$q(x, y) = \begin{cases} x - y, & \textit{if} \quad x \geq y, \\ 1, & \textit{otherwise}. \end{cases}$$

Then q is the Sorgenfrey quasi-metric on X. Notice that $\{q(1/n, 0)\} \to 0$ but $\{q(0, 1/n)\} \to 1$. Therefore, $\{1/n\}$ right-converges to 0 but it does not converge from the left.

Lemma 2.2 (See e.g. [31]) *Let (X, G) be a G-metric space and let define $q_G, q'_G : X^2 \to [0, \infty)$ by*

$$q_G(x, y) = G(x, y, y) \quad \textit{and} \quad q'_G(x, y) = G(x, x, y) \quad \textit{for all} \quad x, y \in X.$$

Then the following properties hold.

1. *q_G and q'_G are quasi-metrics on X. Moreover*

$$q'_G(x, y) \leq 2q_G(x, y) \leq 4q'_G(x, y) \quad \textit{for all} \quad x, y \in X. \tag{15}$$

2. *In (X, q_G) and in (X, q'_G), a sequence is right-convergent (respectively, left-convergent) if, and only if, it is convergent. In such a case, its right-limit, its left-limit and its limit coincide.*
3. *In (X, q_G) and in (X, q'_G), a sequence is right-Cauchy (respectively, left-Cauchy) if, and only if, it is Cauchy.*
4. *In (X, q_G) and in (X, q'_G), every right-convergent (respectively, left-convergent) sequence has a unique right-limit (respectively, left-limit).*
5. *If $\{x_n\} \subseteq X$ and $x \in X$, then $\{x_n\} \xrightarrow{G} x \iff \{x_n\} \xrightarrow{q_G} x \iff \{x_n\} \xrightarrow{q'_G} x$.*
6. *If $\{x_n\} \subseteq X$, then $\{x_n\}$ is G-Cauchy $\iff \{x_n\}$ is q_G-Cauchy $\iff \{x_n\}$ is q'_G-Cauchy.*
7. *(X, G) is complete $\iff (X, q_G)$ is complete $\iff (X, q'_G)$ is complete.*

Every quasi-metric induces a metric, that is, if (X, q) is a quasi-metric space, then the function $d_q : X \times X \to [0, \infty)$, defined by

$$d_q(x, y) = \max\{q(x, y), q(y, x)\} \quad \text{for all} \quad x, y \in X,$$

is a metric on X. As an immediate consequence of the definition above and Theorem 2.2, the following theorem is obtained.

Theorem 2.10 (See e.g. [31]) *Let (X, G) be a G-metric space.*

1. (X, d_{q_G}) *is a metric space.*
2. $\{x_n\} \subset X$ *is* G*-convergent to* $x \in X$ *if, and only if,* $\{x_n\}$ *is convergent to* x *in* (X, d_{q_G}).
3. $\{x_n\} \subset X$ *is* G*-Cauchy if, and only if,* $\{x_n\}$ *is Cauchy in* (X, d_{q_G}).
4. (X, G) *is* G*-complete if, and only if,* (X, d_{q_G}) *is complete.*

Samet et al. [56] proved that Theorems 2.4–2.7 are consequence of Theorem 2.9 by using Theorem 2.10. It is natural to ask whether all fixed point results in the context of G-metric spaces can be derived as a consequence of the related existing results in the context of quasi-metric spaces and/or metric spaces. In [2,6,55], the authors suggested new approaches and they have seemed successful at the first glance since they proposed to consider distinct points on each argument (for instance, $G(x, Tx, y)$). In fact, if x, y, z are distinct, then $G(x, y, z)$ can be formulated in terms of metrics and/or quasi-metrics (see Theorem 2.2). On the other hand, in the expression $G(x, Tx, y)$, nobody guarantee that all argument are distinct since one can take $y = Tx$. In the following section, based on the discussion above, we prove that existing fixed point theorems, such as Theorem 2.9, imply several fixed point results in the context of G-metric spaces, including the results in [2,6,55].

3 Fixed point theorems on quasi-metric spaces

We start this section introducing the following notion.

Definition 3.1 An operator $T : X \to X$ from a quasi-metric space (X, q) into itself is said to be *Picard-continuous* if for all convergent Picard sequence $\{x_n\}$ we have that

$$T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n.$$

Remark 3.1 1. If T is continuous on (X, q), then T is Picard-continuous.
2. Obviously, there exist Picard sequences that are not convergent. The previous definition only involves Picard sequences that also are convergent.

Lemma 3.1 *An operator* T *is Picard-continuous if, and only if, the limit of any convergent Picard sequence is a fixed point of* T.

Proof It follows from $u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\left(\lim_{n\to\infty} x_n\right) = Tu$. □

Example 3.1 If we consider the Euclidean metric on $X = [0, 2]$, then the mapping $T : X \to X$, defined by

$$Tx = \begin{cases} 0, & if \quad x \in [0, 1], \\ 1, & if \quad x \in (1, 2]\,, \end{cases}$$

is Picard-continuous but it is not continuous.

Next, we present the following classes of auxiliary functions.

$$\Psi = \left\{\psi : [0, \infty) \to [0, \infty) : \psi \text{ is non-decreasing, continuous and } \psi^{-1}(\{0\}) = \{0\}\right\},$$
$$\Phi = \left\{\varphi : [0, \infty) \to [0, \infty)\text{: } \varphi \text{ is lower semi-continuous and } \varphi^{-1}(\{0\}) = \{0\}\right\}.$$

Functions in Ψ are known as *altering distance functions* (see Khan et al. [35]).

Definition 3.2 Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. We say that T is:

- *α-admissible* if for all $x, y \in X$ we have

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1. \tag{16}$$

- *triangular α-admissible* [33] if it is α-admissible and it satisfies, for all $x, y, z \in T(X)$:

$$\alpha(x, z) \geq 1 \quad \text{and} \quad \alpha(z, y) \geq 1 \Rightarrow \alpha(x, y) \geq 1. \tag{17}$$

- *orbitally α-admissible at $x_0 \in X$* if

$$\alpha(T^n x_0, T^m x_0) \geq 1 \quad \text{for all} \quad n, m \geq 1 \text{ such that } \quad n \neq m.$$

Lemma 3.2 ([33]) *Let $T : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then the Picard sequence $\{x_n\}$ of T based on x_0 satisfies $\alpha(x_n, x_m) \geq 1$ for all $m, n \geq 1$ with $n < m$.*

Furthermore, if T is triangular α-admissible and $\min(\alpha(x_0, Tx_0), \alpha(Tx_0, x_0)) \geq 1$, *then T is orbitally α-admissible at $x_0 \in X$.*

Definition 3.3 Let $\alpha : X \times X \to [0, \infty)$ be a function and let (X, q) be a quasi-metric space. A convergent sequence $\{x_n\}$ in X verifying $\min(\alpha(x_{n+1}, x_n), \alpha(x_n, x_{n+1})) \geq 1$ for all n is said to be *α-regular* if there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\left[\alpha(x, x_{n(k)}) \geq 1 \quad \text{for all} \quad k\right] \quad \text{or} \quad \left[\alpha(x_{n(k)}, x) \geq 1 \quad \text{for all} \quad k\right], \tag{18}$$

where x is the limit of $\{x_n\}$.

Next, we introduce the following kind of contractive mappings.

Definition 3.4 Let (X, q) be a quasi-metric space and let $T : X \to X$ be a mapping. We say that T is an *(α, ψ, ϕ)-contractive mapping* if there exist three functions $\alpha : X \times X \to [0, \infty)$, $\psi \in \Psi$ and $\phi \in \Phi$ such that, for all $x, y \in X$,

$$\alpha(x, y)\psi(q(Tx, Ty)) \leq \psi(q(x, y)) - \phi(q(x, y)). \tag{19}$$

The main result of this section guarantees that the Picard sequence of any (α, ψ, ϕ)-contractive mapping T from a complete quasi-metric space into itself based on a point in which T is orbitally α-admissible is always convergent and its limit, under continuity or regularity, is a fixed point of T.

Theorem 3.1 *Let (X, q) be a quasi-metric space and let $T : X \to X$ be an (α, ψ, ϕ)-contractive mapping. If T is orbitally α-admissible at $x_0 \in X$, then the Picard sequence $\{x_n\}$ of T based on x_0 is a Cauchy sequence in (X, q).*

Furthermore, assume that (X, q) is complete and, at least, one of the following conditions holds.

(a) T is Picard-continuous.
(b) T is continuous.
(c) $\{x_n\}$ is α-regular.

Then $\{x_n\}$ converges to a fixed point of T. In particular, T has a fixed point.

Notice that the arguments of the following proof will show that the completeness of (X, q) can be replaced by the completeness of $(T(X), q)$.

Proof Let $\{x_n\}$ be the Picard sequence of T based on x_0, that is, $x_{n+1} = Tx_n$ for all $n \geq 0$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T. On the contrary case, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. In particular, $q(x_n, x_{n+1}) > 0$ and $q(x_{n+1}, x_n) > 0$ for all $n \geq 0$. Since T is orbitally α-admissible at x_0, then

$$\alpha(x_n, x_m) \geq 1 \quad \text{for all} \quad n, m \geq 1 \quad \text{such that} \quad n \neq m. \tag{20}$$

Taking into account (19) and (20) and using $x = x_n$ and $y = x_{n-1}$, we find that

$$\begin{aligned} \psi(q(x_{n+1}, x_n)) &= \psi(q(Tx_n, Tx_{n-1})) \leq \alpha(x_n, x_{n-1})\psi(q(Tx_n, Tx_{n-1})) \\ &\leq \psi(q(x_n, x_{n-1})) - \phi(q(x_n, x_{n-1})) < \psi(q(x_n, x_{n-1})) \end{aligned} \tag{21}$$

for all $n \geq 1$. Since ψ is nondecreasing, we derive that

$$q(x_{n+1}, x_n) \leq q(x_n, x_{n-1}) \quad \text{for all} \quad n \geq 1. \tag{22}$$

Therefore $\{q(x_{n+1}, x_n)\}_{n\geq 1}$ is a decreasing sequence in $\mathbb{R}^+$ and, thus, it is convergent. Let $L \in \mathbb{R}^+$ be its limit. We claim that $L = 0$. Suppose, on the contrary, that $L > 0$. Since ψ is continuous and ϕ is lower continuous, taking limit as $n \to \infty$ in (21) we get $\psi(L) \leq \psi(L) - \phi(L)$, which implies that $\phi(L) = 0$. Therefore, $L = 0$, which is a contradiction. Hence, we have that $\lim_{n\to\infty} q(x_{n+1}, x_n) = 0$. Similarly, it can be proved that $\lim_{n\to\infty} q(x_n, x_{n+1}) = 0$, so we conclude that

$$\lim_{n\to\infty} q(x_{n+1}, x_n) = \lim_{n\to\infty} q(x_n, x_{n+1}) = 0. \tag{23}$$

Next, we shall show that the sequence $\{x_n\}$ is left-Cauchy in (X, q) reasoning by contradiction. Suppose, on the contrary, that $\{x_n\}$ is not left-Cauchy. Thus there exists $\varepsilon > 0$ for which one can find two partial subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that

$$q(x_{n(k)}, x_{m(k)}) > \varepsilon \quad \text{for all} \quad k \geq 0 \tag{24}$$

with $n(k) > m(k) \geq k$. Further, corresponding to each $m(k)$, we can choose $n(k)$ as the smallest integer with $n(k) > m(k)$ and satisfying (24). Hence, we have

$$q(x_{n(k)-1}, x_{m(k)}) \leq \varepsilon \quad \text{for all} \quad k \geq 0.$$

Moreover, taking into account (23), we can suppose that $n(k) - 1 > m(k)$ for all k. Due to the triangle inequality, we deduce that

$$\varepsilon \leq q(x_{n(k)}, x_{m(k)}) \leq q(x_{n(k)}, x_{n(k)-1}) + q(x_{n(k)-1}, x_{m(k)}) \leq q(x_{n(k)}, x_{n(k)-1}) + \varepsilon. \tag{25}$$

Letting $k \to \infty$ in (25) we derive that

$$\lim_{k\to\infty} q(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{26}$$

On the other hand, we have

$$q(x_{n(k)-1}, x_{m(k)-1}) \leq q(x_{n(k)-1}, x_{n(k)}) + q(x_{n(k)}, x_{m(k)}) + q(x_{m(k)}, x_{m(k)-1}) \tag{27}$$

and

$$q(x_{n(k)}, x_{m(k)}) \leq q(x_{n(k)}, x_{n(k)-1}) + q(x_{n(k)-1}, x_{m(k)-1}) + q(x_{m(k)-1}, x_{m(k)}) \tag{28}$$

By taking (26) into account, and letting $k \to \infty$ in (27) and (28), we find that

$$\lim_{k\to\infty} q(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon. \tag{29}$$

Regarding (20) and the contractivity condition (19), we have

$$\begin{aligned}
&\psi(q(x_{n(k)}, x_{m(k)})) \\
&\quad = \psi(q(Tx_{n(k)-1}, Tx_{m(k)-1})) \leq \alpha(x_{n(k)-1}, x_{m(k)-1})\psi(q(Tx_{n(k)-1}, Tx_{m(k)-1})) \\
&\quad \leq \psi(q(x_{n(k)-1}, x_{m(k)-1})) - \phi(q(x_{n(k)-1}, x_{m(k)-1}))
\end{aligned} \tag{30}$$

Since ψ is continuous and ϕ is lower continuous, letting $k \to \infty$ in (30) we derive that $\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon)$, which yields the contradiction $\varepsilon = 0$. Hence, we conclude that the sequence $\{x_n\}$ is left-Cauchy. Analogously, we may derive that the sequence $\{x_n\}$ is right-Cauchy. Therefore, the sequence $\{x_n\}$ is a Cauchy sequence in the quasi-metric space (X, q).

Now suppose that (X, q) is complete. Thus, there exists $u \in X$ such that $\{x_n\} \xrightarrow{q} u$, that is,

$$\lim_{n\to\infty} q(x_n, u) = \lim_{n\to\infty} q(u, x_n) = 0.$$

Case (a) *Assume that* T *is Picard-continuous.* By using Lemma 3.1, u is a fixed point of T.

Case (b) *Assume that* T *is continuous.* It follows from the fact that, in this case, T is also Picard-continuous.

Case (c) *Assume that* $\{x_n\}$ *is* α*-regular.* In this case, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ verifying one of the two possibilities of (18). Suppose, without loss of generality, that $\alpha(u, x_{n(k)}) \geq 1$ for all k. Applying (19), for all k, we get that

$$\begin{aligned}
\psi(q(Tu, x_{n(k)+1})) = \psi(q(Tu, Tx_{n(k)})) &\leq \alpha(u, x_{n(k)})\psi(q(Tu, Tx_{n(k)})) \\
&\leq \psi(q(u, x_{n(k)})) - \phi(q(u, x_{n(k)})).
\end{aligned}$$

Letting $k \to \infty$ in the above equality, we obtain, by Remark 2.4 and the continuity of ψ, that $q(Tu, u) \leq 0$. Thus, we have $q(Tu, u) = 0$, that is, $Tu = u$. □

The following results are easy particularizations of Theorem 3.1 using Lemma 3.2.

Corollary 3.1 *Let* (X, q) *be a complete quasi-metric space. Suppose that* $T : X \to X$ *is an* (α, ψ, ϕ)*-contractive mapping which satisfies*:

(i) T *is triangular* α*-admissible*;
(ii) *there exists* $x_0 \in X$ *such that* $\alpha(Tx_0, x_0) \geq 1$ *and* $\alpha(x_0, Tx_0) \geq 1$;
(iii) T *is continuous.*

Then T *has a fixed point.*

Corollary 3.2 *Let* (X, q) *be a complete quasi-metric space. Suppose that* $T : X \to X$ *is a* (α, ψ, ϕ)*-contractive mapping which satisfies*:

(i) T *is triangular* α*-admissible*;
(ii) *there exists* $x_0 \in X$ *such that* $\alpha(Tx_0, x_0) \geq 1$;
(iii) $\{x_n\}$ *is* α*-regular.*

Then T *has a fixed point.*

For the uniqueness of a fixed point of a generalized (α, ψ, ϕ)-contractive mapping, we propose the following hypotheses.

Theorem 3.2 *Under the hypotheses of Theorem* 3.1 (*respectively*, *Corollary* 3.1, *Corollary* 3.2), *assume that*, *at least*, *one of the following conditions holds.*

- (U) $\alpha(u,v) \geq 1$ *for all* $u, v \in \operatorname{Fix}(T)$.
- (U') *T is* α*-admissible and for all* $u, v \in \operatorname{Fix}(T)$, *there exists* $z \in X$ *such that* $\alpha(u,z) \geq 1$, $\alpha(z,u) \geq 1$, $\alpha(v,z) \geq 1$ *and* $\alpha(z,v) \geq 1$.

Then T has a unique fixed point.

Proof *Case* (U). Suppose that u and v are distinct fixed points of T. Thus, from the contractive condition, we have

$$\psi(q(u,v)) = \psi(q(Tu,Tv)) \leq \alpha(u,v)\psi(q(Tu,Tv)) \leq \psi(q(u,v)) - \phi(q(u,v)), \quad (31)$$

which is a contradiction. Hence u is the unique fixed point of T.

Case (U'). Given two fixed points $u, v \in \operatorname{Fix}(T)$, let z be given as in (U'). Let $z_0 = z$ and let $\{z_n\}$ be the Picard sequence of T based in z_0. Using the condition $\alpha(u,z) \geq 1$, we are going to show that $\{q(u,z_n)\} \to 0$. Indeed, since T is α-admissible, then $\alpha(u,z_1) = \alpha(Tu,Tz_0) = \alpha(Tu,Tz) \geq 1$. By induction, $\alpha(u,z_n) \geq 1$ for all $n \geq 0$. Using that T is an (α,ψ,ϕ)-contractive mapping,

$$\begin{aligned} \psi(q(u,z_{n+1})) &= \psi(q(Tu,Tz_n)) \leq \alpha(u,z_n)\psi(q(Tu,Tz_n)) \\ &\leq \psi(q(u,z_n)) - \phi(q(u,z_n)). \end{aligned}$$

Reasoning as in the proof of Theorem 3.1, we obtain that $q(u,z_{n+1}) \leq q(u,z_n)$ for all n. Then $\{q(u,z_n)\} \to L \geq 0$, and similarly we can deduce $L = 0$. The same arguments, using $\alpha(z,u) \geq 1$, show that $\{q(z_n,u)\} \to 0$, so $\{z_n\}$ converges to u. The uniqueness of the limit in a quasi-metric space guarantees that $u = v$. □

The following result is a consequence of Theorem 3.2 taking $\psi(t) = t$ for all $t \geq 0$.

Corollary 3.3 *Let* (X,q) *be a complete quasi-metric space and let* $T : X \to X$ *and* $\alpha : X \times X \to [0,\infty)$ *be mappings. Suppose that there exists a function* $\phi \in \Phi$ *such that*

$$\alpha(x,y)q(Tx,Ty) \leq q(x,y) - \phi(q(x,y)). \quad (32)$$

for all $x, y \in X$. *Also assume that*:

(i) *there exists* $x_0 \in X$ *such that T is orbitally* α*-admissible at* x_0;
(ii) *T is continuous, or Picard-continuous or the Picard sequence of T based on* x_0 *is* α*-regular.*

Then T has a fixed point $u \in X$, *that is,* $Tu = u$. *Moreover, if at least one of the hypotheses* (U) *and* (U') *is satisfied, then u is the unique fixed point of T.*

If $k \in (0,1)$ and $\phi(t) = (1-k)t$ for all $t \geq 0$, we obtain the following statement.

Corollary 3.4 *Let* (X,q) *be a complete quasi-metric space and let* $T : X \to X$ *and* $\alpha : X \times X \to [0,\infty)$ *be mappings. Suppose that there exists* $k \in (0,1)$ *such that*

$$\alpha(x,y)q(Tx,Ty) \leq kq(x,y). \quad (33)$$

for all $x, y \in X$. *Also assume that*:

(i) *there exists* $x_0 \in X$ *such that T is orbitally* α*-admissible at* x_0;
(ii) *T is continuous, or Picard-continuous or the Picard sequence of T based on* x_0 *is* α*-regular.*

Then T has a fixed point $u \in X$, *that is,* $Tu = u$. *Moreover, if at least one of the hypotheses* (U) *and* (U') *is satisfied, then u is the unique fixed point of T.*

4 Fixed point theorems on quasi-metric spaces using contractivity conditions depending on a unique variable

Some authors used $y = Tx$ in the previous results to obtain different fixed point theorems, that is, they considered contractivity conditions depending on only one variable $x \in X$ such as the following one:

$$\psi(q(Tx, T^2x)) \leq \psi(q(x, Tx)) - \phi(q(x, Tx)) \quad \text{for all} \quad x \in X. \tag{34}$$

However, these kind of contractivity conditions have four drawbacks.

- In general, unless additional hypotheses on the auxiliary functions ψ and ϕ, these contractivity conditions are not sufficient to ensure that a Picard sequence of T is Cauchy even from one side (because $q(x_m, x_n)$ does not appear in (34) when $m + 1 < n$).
- Although we can prove that a Picard sequence of T is Cauchy from one side, this is not sufficient to use the completeness of the quasi-metric space, because there are Cauchy sequences from one side that are not Cauchy from the other side.
- Even if we know that a Picard sequence $\{x_n\}$ is convergent to u, the terms $q(x_n, u)$ and $q(x_n, Tu)$ do not appear in (34) in order to prove that u is a fixed point of T.
- The uniqueness of the fixed point cannot be deduced from the contractivity condition (34) because it does not permit to compare, in general, u and v when $u, v \in \text{Fix}(T)$. The uniqueness of the fixed point must be deduced from another arguments.

The last two objections appear even if q is a metric in X. In order to overcome the first three disadvantages, in this section we present some additional conditions to ensure existence of fixed points of nonlinear operators.

Let $\mathcal{F}_{\text{com}}$ be the family of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(ψ_1) φ is nondecreasing;
(ψ_2) $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$, where φ^n is the nth iterate of φ.

These functions are known in the literature as *(c)-comparison functions*. One can easily deduce that if ψ is a (c)-comparison function, then

$$\varphi(t) < t \quad \text{for all} \quad t > 0. \tag{35}$$

Examples of (c)-comparison functions are $\varphi_k(t) = kt$ for all $t \geq 0$, where $k \in [0, 1)$.

Definition 4.1 Let (X, q) be a quasi-metric space and let $T : X \to X$ be a mapping. We say that q is *T-Picard-bilateral-Cauchy* if any Picard sequence $\{x_n\}$ of T is left-Cauchy in (X, q) if, and only if, it is right-Cauchy in (X, q).

The following property follows from item 3 of Lemma 2.2 and permit consider a wide range of Picard-bilateral-Cauchy quasi-metrics.

Lemma 4.1 *Every quasi-metric $q = q_G$ (or $q = q'_G$) associated to a G-metric G in X (as in Lemma 2.2) is T-Picard-bilateral-Cauchy, whatever the operator $T : X \to X$.*

The main result of this section is the following one.

Theorem 4.1 *Let (X, q) be a quasi-metric space and let $T : X \to X$, $\alpha : X \times X \to [0, \infty)$ and $\varphi \in \mathcal{F}_{\text{com}}$ be three mappings such that*

$$\alpha(x, Tx)q(Tx, T^2x) \leq \varphi(q(x, Tx)) \quad \textit{for all} \quad x \in X. \tag{36}$$

If there exists $x_0 \in X$ *such that* $\alpha(T^n x_0, T^{n+1}x_0) \geq 1$ *for all* $n \geq 1$, *then the Picard sequence* $\{x_n\}$ *of* T *based on* x_0 *is right-Cauchy in* (X, q).

Furthermore, assume that (X, q) *is complete*, q *is* T*-Picard-bilateral-Cauchy and* T *is Picard-continuous. Then* $\{x_n\}$ *converges to a fixed point of* T. *In particular*, T *has a fixed point.*

Remark 4.1 As we shall show in the proof, the previous theorem also holds if we do one or more of the following replacements:

1. the condition "(X, q) is complete" may be replaced by "$(T(X), q)$ is complete";
2. the condition "T is Picard-continuous" may be replaced by the stronger property "T is continuous";
3. the condition "there exists $x_0 \in X$ such that $\alpha(T^n x_0, T^{n+1}x_0) \geq 1$ for all $n \geq 1$" may be replaced by the stronger property: "T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$".

Proof Let $\{x_n\}$ be the Picard sequence of T based on x_0, that is, $x_{n+1} = Tx_n$ for all $n \geq 0$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T. On the contrary case, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. In particular, $q(x_n, x_{n+1}) > 0$ for all $n \geq 0$. By hypothesis,

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all} \quad n \geq 1. \tag{37}$$

Taking into account (35) and (37), and using $x = x_n$ in (36), we find that, for all $n \geq 1$,

$$\begin{aligned} q(x_{n+1}, x_{n+2}) &= q(Tx_n, T^2x_n) \leq \alpha(x_n, x_{n+1})q(Tx_n, T^2x_n) = \alpha(x_n, Tx_n)q(Tx_n, T^2x_n) \\ &\leq \varphi(q(x_n, Tx_n)) = \varphi(q(x_n, x_{n+1})). \end{aligned} \tag{38}$$

Therefore, for all $n \geq 2$, as φ is nondecreasing,

$$q(x_n, x_{n+1}) \leq \varphi(q(x_{n-1}, x_n)) \leq \varphi^2(q(x_{n-2}, x_{n-1})) \leq \cdots \leq \varphi^{n-1}(q(x_1, x_2)).$$

Let $n, m \in \mathbb{N}$ be such that $n < m$. In such a case,

$$q(x_n, x_m) \leq \sum_{k=n}^{m-1} q(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \varphi^{k-1}(q(x_1, x_2)).$$

Since $q(x_1, x_2) > 0$ and $\varphi \in \mathcal{F}_{\text{com}}$, the series $\sum_{k\geq 1}\varphi^{k-1}(q(x_1, x_2))$ is convergent. Therefore, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\sum_{k=n}^{\infty} \varphi^{k-1}(q(x_1, x_2)) < \varepsilon.$$

In particular, if $m > n \geq n_0$, we have that

$$q(x_n, x_m) \leq \sum_{k=n}^{m-1} \varphi^{k-1}(q(x_1, x_2)) \leq \sum_{k=n}^{\infty} \varphi^{k-1}(q(x_1, x_2)) < \varepsilon,$$

which means that the sequence $\{x_n\}$ is right-Cauchy in (X, q).

Now assume that (X, q) is complete, q is T-Picard-bilateral-Cauchy and T is Picard-continuous. The second property guarantees that $\{x_n\}$ is also left-Cauchy, and Remark 2.3 shows that $\{x_n\}$ is Cauchy. As (X, q) is complete, there exists $u \in X$ such that $\{x_n\} \xrightarrow{q} u$. Finally, by Lemma 3.1, as T is Picard-continuous, u is a fixed point of T. □

Using $\alpha(x, y) = 1$ for all $x, y \in X$, we have the following result.

Corollary 4.1 *Let (X, q) be a complete quasi-metric space and let $T : X \to X$ and $\varphi \in \mathcal{F}_{\text{com}}$ be two mappings such that*

$$q(Tx, T^2x) \le \varphi(q(x, Tx)) \quad \textit{for all} \quad x \in X.$$

If q is T-Picard-bilateral-Cauchy and T is Picard-continuous, then any Picard sequence $\{x_n\}$ of T converges to a fixed point of T. In particular, T has a fixed point.

If we employ $\varphi_k(t) = kt$ for all $t \ge 0$, we deduce the following statement.

Corollary 4.2 *Let (X, q) be a complete quasi-metric space, let $T : X \to X$ be a mapping and let $k \in [0, 1)$ be such that*

$$q(Tx, T^2x) \le kq(x, Tx) \quad \textit{for all} \quad x \in X.$$

If q is T-Picard-bilateral-Cauchy and T is Picard-continuous, then any Picard sequence $\{x_n\}$ of T converges to a fixed point of T. In particular, T has a fixed point.

An interesting case in which we can ensure that q is T-Picard-bilateral-Cauchy occurs when $q = q_G$ (or $q = q'_G$) is the quasi-metric associated to a G-metric G in X as in Lemma 2.2.

Corollary 4.3 *Let (X, G) be a complete G-metric space and let $T : X \to X$ be an operator satisfying, at least, one of the following conditions.*

(C_1) *There exist $\alpha : X \times X \to [0, \infty)$, a point $x_0 \in X$ such that $\alpha(T^n x_0, T^{n+1} x_0) \ge 1$ for all $n \ge 1$, and a function $\varphi \in \mathcal{F}_{\text{com}}$ verifying*

$$\alpha(x, Tx)G(Tx, T^2x, T^2x) \le \varphi(G(x, Tx, Tx)) \quad \textit{for all} \quad x \in X. \tag{39}$$

(C_2) *There exists a function $\varphi \in \mathcal{F}_{\text{com}}$ verifying*

$$G(Tx, T^2x, T^2x) \le \varphi(G(x, Tx, Tx)) \quad \textit{for all} \quad x \in X. \tag{40}$$

(C_3) *There exists a constant $k \in [0, 1)$ verifying*

$$G(Tx, T^2x, T^2x) \le kG(x, Tx, Tx) \quad \textit{for all} \quad x \in X. \tag{41}$$

If T is a Picard-continuous mapping, then T has a fixed point.

We finish this section including a result in which the contractivity condition involves a unique variable, but it is symmetric in the arguments of q.

Theorem 4.2 *Let (X, q) be a quasi-metric space and let $T : X \to X$, $\alpha : X \times X \to [0, \infty)$ and $\varphi \in \mathcal{F}_{\text{com}}$ be three mappings such that*

$$\alpha(x, Tx)q(Tx, T^2x) \le \varphi(q(x, Tx)) \quad \textit{and} \tag{42}$$

$$\alpha(Tx, x)q(T^2x, Tx) \le \varphi(q(Tx, x)) \quad \textit{for all} \quad x \in X. \tag{43}$$

If there exists $x_0 \in X$ such that T is orbitally α-admissible at x_0, then the Picard sequence $\{x_n\}$ of T based on x_0 is Cauchy in (X, q).

Furthermore, assume that (X, q) is complete and T is Picard-continuous. Then $\{x_n\}$ converges to a fixed point of T. In particular, T has a fixed point.

The same commentaries of Remark 4.1 can be pointed out with respect to the previous statement.

Proof Exactly as in the proof of Theorem 4.1, condition (42) implies that $\{x_n\}$ is right-Cauchy, and condition (43) guarantees that $\{x_n\}$ is left-Cauchy. By Remark 2.3, $\{x_n\}$ is Cauchy. If (X, q) is complete, $\{x_n\}$ is convergent. As T is Picard continuous, its limit is a fixed point of T. □

5 Consequences

In this section, in order to show the powerful and usability of our main results, we list some consequences that can be easily deduced.

5.1 Standard fixed point results in the context of quasi-metric spaces

By taking $\alpha(x, y) = 1$ for all $x, y \in X$ in Corollary 3.2 and Theorem 3.2, we derive immediately the following fixed point theorem.

Corollary 5.1 *Let (X, q) be a complete quasi-metric space and let $T : X \to X$ be a given mapping. Suppose that there exist functions $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\psi(q(Tx, Ty)) \leq \psi(q(x, y)) - \phi(q(x, y))$$

for all $x, y \in X$. Then T has a unique fixed point.

Notice that the following result (which is obtained by taking $\psi(t) = t$ for all $t \geq 0$ in the previous corollary) is slightly more general than the main theorem in [31].

Corollary 5.2 *Let (X, q) be a complete quasi-metric space and let $T : X \to X$ be a given mapping. Suppose that there exists a function $\phi \in \Phi$ such that*

$$q(Tx, Ty) \leq q(x, y) - \phi(q(x, y))$$

for all $x, y \in X$. Then T has a unique fixed point.

If $k \in [0, 1)$ and $\phi(t) = (1 - k)t$ for all $t \geq 0$, we obtain the following version of the Banach Contraction Mapping Principle for quasi-metric spaces.

Corollary 5.3 *Let (X, q) be a complete quasi-metric space and let $T : X \to X$ be a given mapping. Suppose that there exists a $k \in [0, 1)$ such that*

$$q(Tx, Ty) \leq kq(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point.

In other words, a contractive mapping from a complete quasi-metric space into itself has a unique fixed point.

5.2 Fixed point theorems on metric spaces endowed with a partial order

One of the interesting topics in the field of Fixed Point Theory is to investigate existence of fixed points of operators on metric spaces endowed with partial orders. The initial results in this direction were given by Turinici [59], Ran and Reurings [51], and Nieto and Rodríguez-López [50].

We first need to recall the following notions.

Definition 5.1 Let $(X, \preceq)$ be a partially ordered set and let $T : X \to X$ be a given mapping. We say that T is nondecreasing with respect to $\preceq$ if

$$x, y \in X, \quad x \preceq y \Longrightarrow Tx \preceq Ty.$$

Definition 5.2 Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all n.

Definition 5.3 Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. We say that $(X, \preceq, d)$ is *regular* if for every nondecreasing sequence $\{x_n\} \subset X$ such that $\{x_n\} \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.

Obviously, under the last condition, all sequence $\{x_n\}$ verifies $x_n \preceq x$ for all n. In this framework, we have the following result.

Corollary 5.4 *Let $(X, \preceq)$ be a partially ordered set and let d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist functions $\psi \in \Psi$ and $\phi \in \Phi$ such that $\phi \leq \psi$ and*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) \tag{44}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(ii) *T is continuous or $(X, \preceq, d)$ is regular.*

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

Notice that the condition $\phi \leq \psi$ is only necessary in the range of d.

Proof Notice that the symmetry of d means that the contractivity condition (44) holds for all $x, y \in X$ such that $x \preceq y$ or $x \succeq y$. Define the symmetric function $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if} \quad x \preceq y \quad \text{or} \quad x \succeq y, \\ 0, & \text{otherwise}. \end{cases} \tag{45}$$

Clearly, T is an (α, ψ, ϕ)-contractive mapping, that is,

$$\alpha(x, y)\psi(q(Tx, Ty)) \leq \psi(q(x, y)) - \phi(q(x, y)) \quad \text{for all} \quad x, y \in X.$$

Notice that T is α-admissible because

$$\alpha(x, y) \geq 1 \Longrightarrow [x \succeq y \text{ or } x \preceq y] \Longrightarrow [Tx \succeq Ty \text{ or } Tx \preceq Ty] \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Let x_0 be the point given by condition (i) and let $\{x_n\}_{n \geq 0}$ be the Picard sequence of T based on x_0. Since $x_0 \preceq Tx_0$ and T is nondecreasing, then $x_1 = Tx_0 \preceq T^2x_0 = x_2$. By induction, $\{x_n\}$ is a nondecreasing sequence, that is, $x_n \preceq x_{n+1}$ for all $n \geq 0$. As $\preceq$ is transitive, then $x_n \preceq x_m$ for all $m \geq n \geq 0$. Therefore $\alpha(x_n, x_m) \geq 1$ for all $m \geq n \geq 0$. But as α is symmetric, then $\alpha(x_n, x_m) \geq 1$ for all $m, n \geq 0$ and T is orbitally α-admissible at x_0. Taking into account that the regularity of $(X, \preceq, d)$ implies that $\{x_n\}$ is α-regular, Theorem 3.1 guarantees that T has a fixed point, and Theorem 3.2 proves the uniqueness.

□

Remark 5.1 1. If we do not suppose $\phi \preceq \psi$ in the previous result, then it is possible that x and y are not $\preceq$-comparable and, in this case, as $\alpha(x, y) = 0$, it is not guaranteed that $0 \leq \psi(d(x, y)) - \phi(d(x, y))$. Then, in this case, the condition $\alpha(x, y)\psi(q(Tx, Ty)) \leq \psi(q(x, y)) - \phi(q(x, y))$ would not have to be valid.
2. Notice that, if α is defined as in (45), then T is not necessarily triangular α-admissible. For instance, $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ can be true when $x \preceq z$ and $y \preceq z$, and in this case we have no any kind of relationship between x and y. However, it is possible to prove that T is α-admissible and T is orbitally α-admissible at x_0.

5.3 More theorems using (c)-comparison functions

Definition 5.4 ([19]) Let (X, q) be a quasi-metric space and let $T : X \to X$ be a given mapping. We say that T is an (α, φ)-*contractive mapping* if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\varphi \in \mathcal{F}_{\text{com}}$ such that

$$\alpha(x, y)q(Tx, Ty) \leq \varphi(q(x, y)) \quad \text{for all} \quad x, y \in X. \tag{46}$$

Following the same arguments of the proof of Theorem 3.1, the following statement can be proved.

Theorem 5.1 *Let (X, q) be a complete quasi-metric space and let $T : X \to X$ be an (α, φ)-contractive mapping. If T is orbitally α-admissible at $x_0 \in X$, then the Picard sequence $\{x_n\}$ of T based on x_0 converges in (X, q).*

Furthermore, assume that, at least, one of the following conditions holds.

(a) *T is Picard-continuous.*
(b) *T is continuous.*
(c) *$\{x_n\}$ is α-regular.*

Then $\{x_n\}$ converges to a fixed point of T. In particular, T has a fixed point.

Additionally, assume that, at least, one of the conditions (U) or (U') (given in Theorem 3.2) holds. Then T has a unique fixed point.

Proof First at all, notice that (46) implies (36) taking $y = Tx$. Therefore, the same arguments of the proof of Theorem 4.1 ensure that $\{x_n\}$ is right-Cauchy. As the contractivity condition (46) is symmetric on x and y, the same reasoning proves that $\{x_n\}$ is left-Cauchy (we do not need to assume that q is T -Picard-bilateral-Cauchy). Hence $\{x_n\}$ is Cauchy and repeating the same proof, we deduce that the limit u of $\{x_n\}$ is a fixed point of T when T is Picard-continuous (or continuous).

Next assume that $\{x_n\}$ is α-regular. By Definition 3.3, we can suppose, without loss of generality, that there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(u, x_{n(k)}) \geq 1$ for all k. Therefore, using (46), for all k,

$$q(Tu, x_{n(k)+1}) = q(Tu, Tx_{n(k)}) \leq \alpha(u, x_{n(k)})q(Tu, Tx_{n(k)}) \leq \varphi(q(u, x_{n(k)})) \leq q(u, x_{n(k)})$$

As $\{x_n\} \to u$, we deduce that $q(Tu, u) = 0$, so $Tu = u$. The uniqueness follows as in Theorem 3.2. □

The following theorems can be found in [19].

Theorem 5.2 ([19]) *Let (X, q) be a complete quasi-metric space and let $T : X \to X$ be an (α, ψ)-contractive mapping which satisfies:*

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *T is continuous.*

Then T has a fixed point.

Theorem 5.3 ([19]) *Let (X, q) be a complete quasi-metric space. Suppose that $T : X \to X$ is a (α, ψ)-contractive mapping which satisfies*:

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$*;
(iii) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_{n+1}, x_n) \geq 1$ for all n and $\{x_n\} \to x \in X$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x, x_{n(k)}) \geq 1$ for all k.*

Then T has a fixed point.

Thus, we are able to deduce the following statement.

Corollary 5.5 *Theorems 5.2 and 5.3 follows from Theorem 5.1.*

In [19], the authors did not discussed the uniqueness of the fixed point guaranteed by Theorems 5.2 and 5.3. However, it is clear that conditions (U) or (U') are sufficient to ensure the uniqueness in both theorems.

5.4 Fixed point theorems in the context of G-metric spaces

The main aim of this subsection is to show many different fixed point theorems in the context of G-metric spaces that can be seen as simple consequences of Corollary 4.3. Notice that the uniqueness of the fixed point must be proved on each case.

Theorem 5.4 ([2]) *Let (X, G) be a complete G-metric space and let $T : X \to X$ be mapping such that*

$$G(Tx, Ty, Tz) \leq k M(x, y, z) \tag{47}$$

for all $x, y, z \in X$, where $k \in \left[0, \frac{1}{2}\right)$ and

$$M(x, y, z) = \max \left\{ \begin{array}{lll} G(x, Tx, y), & G(y, T^2x, Ty), & G(Tx, T^2x, Ty), \\ G(y, Tx, Ty), & G(x, Tx, z), & G(z, T^2x, Tz), \\ G(Tx, T^2x, Tz), & G(z, Tx, Ty), & G(x, y, z), \\ G(x, Tx, Tx), & G(y, Ty, Ty), & G(z, Tz, Tz), \\ G(z, Tx, Tx), & G(x, Ty, Ty), & G(y, Tz, Tz) \end{array} \right\} \tag{48}$$

Then there is a unique $x \in X$ such that $Tx = x$.

Theorem 5.5 *Theorem 5.4 is a particular case of Corollary 4.3.*

Proof If we take $z = y = Tx$ in (47), then all terms in $M(x, y, z)$ are $G(x, Tx, Tx)$ or $G(Tx, T^2x, T^2x)$, unless $G(x, Ty, Ty)$, which is $G(x, T^2x, T^2x)$. Therefore, if q_G is the quasi-metric in X associated to G as in Lemma 2.2, then

$$\begin{aligned} q_G(Tx, T^2x) &= G(Tx, T^2x, T^2x) \\ &\leq k \max\left\{G(x, Tx, Tx), G(Tx, T^2x, T^2x), G(x, T^2x, T^2x)\right\}. \end{aligned} \tag{49}$$

If the maximum in (49) is $G(Tx, T^2x, T^2x)$ for some $x \in X$, then $G(Tx, T^2x, T^2x) \leq kG(Tx, T^2x, T^2x)$ yields $Tx = T^2x$, so $y = Tx$ is a fixed point of T. In other case,

$$\begin{aligned} q_G(Tx, T^2x) &\leq k \max\left\{G(x, Tx, Tx), G(x, T^2x, T^2x)\right\} \\ &\leq k\left(G(x, Tx, Tx) + G(Tx, T^2x, T^2x)\right) = k\left(q_G(x, Tx) + q_G(Tx, T^2x)\right), \end{aligned}$$

which is equivalent to

$$q_G(Tx, T^2x) \leq \frac{k}{1-k}\, q_G(x, Tx).$$

Since $k \in [0, 1/2)$, then $\lambda = k/(1-k) \in [0, 1)$. Therefore, the contractivity condition (41) holds.

Let show that T is a Picard continuous operator. Let $\{x_n\}$ be a Picard sequence of T converging to $u \in X$. Then taking $x = y = x_n$ and $z = u$ in (47), we have that, for all n,

$$\begin{aligned} &G(x_{n+1}, x_{n+1}, Tu) = G(Tx_n, Tx_n, Tu) \\ &\leq k \max \left\{ \begin{array}{lll} G(x_n, x_{n+1}, x_n), & G(x_n, x_{n+2}, x_{n+1}), & G(x_{n+1}, x_{n+2}, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}), & G(x_n, x_{n+1}, u), & G(u, x_{n+2}, Tu), \\ G(x_{n+1}, x_{n+2}, Tu), & G(u, x_{n+1}, x_{n+1}), & G(x_n, x_n, u), \\ G(x_n, x_{n+1}, x_{n+1}), & G(x_n, x_{n+1}, x_{n+1}), & G(u, Tu, Tu), \\ G(u, x_{n+1}, x_{n+1}), & G(x_n, x_{n+1}, x_{n+1}), & G(x_n, Tu, Tu) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, we deduce that

$$G(u, u, Tu) \leq k \max\{G(u, u, Tu), G(u, Tu, Tu)\}.$$

By Lemma 2.1, since $G(u, Tu, Tu) \leq 2G(u, u, Tu)$, it follows that $G(u, u, Tu) \leq 2k \max G(u, u, Tu)$, being $2k < 1$. Then $G(u, u, Tu) = 0$ and $Tu = u$.

The uniqueness of the fixed point follows similarly from (47) using $x = y = u$ and $z = v$, where $u, v \in \mathrm{Fix}(T)$. □

Notice that we can improve Theorem 5.4 adding, for instance, $G(x, Tz, Tz)$ to M, which was not considered in (47), because it does not change the previous proof.

Corollary 5.6 *Theorem 5.4 also holds if we add to M in (48) the term $G(x, Tz, Tz)$.*

Also notice that the proof of Theorem 5.5 shows that only the term $G(x, Ty, Ty)$ of M in (48) yields $G(x, T^2x, T^2x)$. Then, it is necessary $k \in [0, 1/2)$. However, if we remove this term from M, we could take $k \in [0, 1)$.

Corollary 5.7 *Theorem 5.4 also holds for all $k \in [0, 1)$ if we remove from M in (48) the term $G(x, Ty, Ty)$.*

Theorem 5.6 (See, e.g., [6]) *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$:*

$$G(Tx, Ty, Ty) \leq kG(x, Tx, y), \tag{50}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Corollary 5.8 *Theorem 5.6 is a particular case of Corollary 4.3.*

Proof If we take $y = Tx$, then the contractive condition (50) turns into

$$q_G(Tx, T^2x) = G(Tx, T^2x, T^2x) \leq kG(x, Tx, Tx) = kq_G(x, Tx)$$

for all $x \in X$, which is a particular case of Corollary 4.3. The same arguments of the proof of Theorem 5.5 show that T is Picard-continuous and the uniqueness of the fixed point. □

Corollary 5.9 (See e.g.[6]) *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $x, y, z \in X$:*

$$G(Tx, Ty, Tz) \leq aG(x, Tx, z) + bG(x, Tx, y),$$

where $0 \leq a + b < 1$. Then T has a unique fixed point.

Corollary 5.10 *Corollary 5.9 is a particular case of Corollary 4.2.*

Proof If we take $z = y = Tx$, then the involved contractive condition turns into

$$\begin{aligned} q_G(Tx, T^2x) &= G(Tx, T^2x, T^2x) \leq aG(x, Tx, Tx) + bG(x, Tx, Tx) \\ &= (a + b)q_G(x, Tx), \end{aligned}$$

where $k = a + b \in [0, 1)$. The same arguments of the proof of Theorem 5.5 show that T is Picard-continuous and the uniqueness of the fixed point. □

Theorem 5.7 ([6]) *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$:*

$$G(Tx, T^2x, Ty) \leq kG(x, Tx, y),$$

where $0 \leq k < 1$. Then T has a unique fixed point.

Corollary 5.11 *Theorem 5.7 is a particular case of Corollary 4.2.*

Proof If we take $y = Tx$, then the involved contractive condition turns into

$$q_G(Tx, T^2x) = G(Tx, T^2x, T^2x) \leq kG(x, Tx, Tx) = kq_G(x, Tx).$$

The same arguments of the proof of Theorem 5.5 show that T is Picard-continuous and the uniqueness of the fixed point. □

Theorem 5.8 ([6]) *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $x, y, z \in X$:*

$$G(Tx, T^2x, Ty) + G(Tx, T^2x, Tz) \leq a\, G(x, Tx, y) + b\, G(x, Tx, z),$$

where $0 \leq a + b < 2$. Then T has a unique fixed point.

Corollary 5.12 *Theorem 5.8 is a particular case of Corollary 5.11.*

Proof Letting $y = z$, we get

$$G(Tx, T^2x, Ty) \leq \frac{a + b}{2}\, G(x, Tx, y),$$

where $k = (a + b)/2 \in [0, 1)$. □

Acknowledgments We are very thankful to the editor and to the anonymous reviewers for their careful reading of our manuscript and for their constructive reports, which have been very useful to improve the paper. The third author is grateful to the Department of Quantitative Methods for Economics and Business of the University of Granada (Spain). This manuscript has been partially supported by Junta de Andalucía by project FQM-268 of the Andalusian CICYE.

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