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ORIGINAL PAPER

Homoclinic orbits for second order p-Laplacian difference equations containing both advance and retardation

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Abstract Using the critical point theory, we obtain the existence of a nontrivial homoclinic orbit for second order p-Laplacian difference equations containing both advance and retardation. The proof is based on the Mountain Pass Lemma in combination with periodic approximations. One of our results generalizes and improves the results in the literature.

Keywords Homoclinic orbits · Second order · p-Laplacian difference equations · Discrete variational methods · Advance and retardation

Mathematics Subject Classification 34C37 · 37J45 · 39A12

1 Introduction

Below **N**, **Z** and **R** denote the sets of all natural numbers, integers and real numbers respectively. For any $a, b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \ldots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \ldots, b\}$ when $a \leq b$. l^p denotes the space of all real functions whose pth powers are summable on **Z**.

In this paper, we consider the following difference equation

$$\Delta \left(\varphi_p \left(\Delta u_{n-1}\right)\right) - q_n \varphi_p \left(u_n\right) + f \left(n, u_{n+M}, u_n, u_{n-M}\right) = 0, \; n \in \mathbf{Z}, \tag{1.1}$$

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containing both advance and retardation, where Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, $\varphi_p(s)$ is the p-Laplacian operator $\varphi_p(s) = |s|^{p-2}s (1 < p < \infty)$, $\{q_n\}$ is a real sequence, M is a given nonnegative integer, $f \in C(\mathbf{Z} \times \mathbf{R}^3, \mathbf{R})$, q_n and $f(n, v_1, v_2, v_3)$ are T-periodic in n for a given positive integer T. We mention that (1.1) is a kind of difference equation containing both advance and retardation. This kind of difference equation has many applications both in theory and practice [1–4,27].

Equation (1.1) can be considered as a discrete analogue of the following second-order functional differential equation

$$\left(\varphi_p(u')\right)' + q(t)\varphi_p(u(t)) + f(t, u(t+M), u(t), u(t-M)) = 0, \ t \in \mathbf{R}. \tag{1.2}$$

Equation (1.2) includes the following equation

$$\left(p(t)\psi(u')\right)' + f(t, u(t)) = 0, \ t \in \mathbf{R},$$

which has arose in the study of fluid dynamics, combustion theory, gas diffusion through porous media, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, and adiabatic reactor [9,18]. Equations similar in structure to (1.2) arise in the study of homoclinic orbits [12,14–16] of functional differential equations.

In the theory of differential equations, the trajectories which are asymptotic to a constant state as the time variable $|t| \to \infty$ are called homoclinic orbits (or homoclinic solutions). Such orbits have been found in various models of continuous dynamical systems and frequently have tremendous effects on the dynamics of such nonlinear systems. So homoclinic orbits have been extensively studied since the time of Poincaré, see [11–16,24] and the references therein. Recently, Ma and Guo [21,22] have found that the trajectories which are asymptotic to a constant state as the time variable $|n| \to \infty$ also exist in discrete dynamical systems [3–8,10,19–22]. These trajectories are also called homoclinic orbits (or homoclinic solutions).

If $q_n \equiv 0$ and $M = 1$, Chen and Fang [2] have obtained a sufficient condition for the existence of periodic solutions of the second-order p-Laplacian difference equation (1.1).

In 2011, Chen and Tang [3] established some existence criteria to guarantee the following fourth-order difference equation

$$\Delta^4 u_{n-2} - q_n u_n = f(n, u_{n+1}, u_n, u_{n-1}), \ n \in \mathbf{Z}, \tag{1.3}$$

containing both advance and retardation has infinitely many homoclinic orbits.

In some recent papers [2,5–8,19,21,22], the authors studied the existence of periodic solutions and homoclinic orbits of some special forms of (1.1) by using the critical point theory. These papers show that the the critical point theory is an effective approach to study of periodic solutions and homoclinic orbits for difference equations. Ma and Guo [21] (without periodicity assumption) and [22] (with periodicity assumption) applied variational methods to prove the existence of homoclinic orbits for the special form of (1.1) (with $p = 2$ and $M = 0$)

$$\Delta(\Delta u_{n-1}) - q_n u_n + f(n, u_n) = 0, \ n \in \mathbf{Z}. \tag{1.4}$$

A crucial role that the Ambrosetti-Rabinowitz condition plays is to ensure the boundedness of Palais-Smale sequences. This is very crucial in applying the critical point theory.

The boundary value problems, periodic solutions and homoclinic orbits of difference equations has been a very active area of research in the last decade, and for surveys of recent results, we refer the reader to the monographs and papers by Agarwal et al. [1–8,10,17,19–22,26,27]. However, to the best of our knowledge, the results on homoclinic orbits of p-Laplacian difference equations are scarce in the literature. Furthermore, since (1.1) contains both advance and retardation, there are very few manuscripts dealing with this subject. The main purpose of this paper is to develop a new approach to above problem without the classical Ambrosetti-Rabinowitz condition. Particularly, one of our results generalizes and improves the results in the literature. In fact, one can see the following Remarks 1.2 and 1.3 for details. The motivation for the present work stems from the recent papers [2,6,11].

For the basic knowledge of variational methods, the reader is referred to [23,25].

Let

$$\underline{q} = \min_{n\in\mathbf{Z}(1,T)} \{q_n\},\ \bar{q} = \max_{n\in\mathbf{Z}(1,T)} \{q_n\}.$$

Our main results are as follows.

Theorem 1.1 *Assume that the following hypotheses are satisfied:*

(F_1) *there exists a functional* $F(n, v_1, v_2) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ *with* $F(n + T, v_1, v_2) = F(n, v_1, v_2)$ *and it satisfies*

$$\frac{\partial F(n-M, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3);$$

(F_2) *there exist positive constants* ϱ *and* $a < \frac{\underline{q}}{2p}\left(\frac{\kappa_1}{\kappa_2}\right)^p$ *such that*

$$|F(n, v_1, v_2)| \le a\left(|v_1|^p + |v_2|^p\right) \textit{ for all } n \in \boldsymbol{Z} \textit{ and } \sqrt{v_1^2 + v_2^2} \le \varrho;$$

(F_3) *there exist constants* ρ, $c > \frac{1}{2p}\left(\frac{\kappa_2}{\kappa_1}\right)^p (2^p + \bar{q})$ *and* b *such that*

$$F(n, v_1, v_2) \ge c\left(|v_1|^p + |v_2|^p\right) + b \textit{ for all } n \in \boldsymbol{Z} \textit{ and } \sqrt{v_1^2 + v_2^2} \ge \rho;$$

(F_4) $\frac{\partial F(n,v_1,v_2)}{\partial v_1}v_1 + \frac{\partial F(n,v_1,v_2)}{\partial v_2}v_2 - pF(n, v_1, v_2) > 0,$ *for all* $(n, v_1, v_2) \in \boldsymbol{Z}\times\boldsymbol{R}^2\setminus\{(0,0)\}$;

(F_5) $\frac{\partial F(n,v_1,v_2)}{\partial v_1}v_1 + \frac{\partial F(n,v_1,v_2)}{\partial v_2}v_2 - pF(n, v_1, v_2) \to +\infty$ *as* $\sqrt{v_1^2 + v_2^2} \to +\infty$.

Then (1.1) *has a nontrivial homoclinic orbit.*

Remark 1.1 By (F_3), it is easy to see that there exists a constant $\zeta > 0$ such that

$$(F_3')\ F(n, v_1, v_2) \ge c\left(|v_1|^p + |v_2|^p\right) + b - \zeta,\ \forall (n, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2.$$

As a matter of fact, let $\zeta = \max\left\{ \left|F(n, v_1, v_2) - c\left(|v_1|^p + |v_2|^p\right) - b\right| : n \in \mathbf{Z}, \sqrt{v_1^2 + v_2^2} \le \rho \right\}$, we can easily get the desired result.

Remark 1.2 Theorem 1.1 extends Theorem 1.1 in [22] which is the special case of our Theorem 1.1 by letting $p = 2$ and $M = 0$.

Remark 1.3 In many studies (see e.g. [2,17,21,22]) of second order difference equations, the following classical Ambrosetti-Rabinowitz condition is assumed.

(**AR**) there exists a constant $\beta > 2$ such that

$$0 < \beta F(n, u) \le uf(n, u) \text{ for all } n \in \mathbf{Z} \text{ and } u \in \mathbf{R}\backslash 0.$$

Note that $(F_3) - (F_5)$ are much weaker than (**AR**). Thus our result improves that the existing ones.

Theorem 1.2 *Assume that* $(F_1) - (F_5)$ *and the following hypothesis are satisfied:*

(F_6) $q_{-n} = q_n, \ F(-n, v_1, v_2) = F(n, v_1, v_2)$.

Then (1.1) *has a nontrivial even homoclinic orbit.*

2 Preliminaries

In order to apply the critical point theory, we shall establish the corresponding variational framework for (1.1) and give some lemmas which will be of fundamental importance in proving our results. We start by some basic notations.

Let S be the set of sequences $u = (\ldots, u_{-n}, \ldots, u_{-1}, u_0, u_1, \ldots, u_n, \ldots) = \{u_n\}_{n=-\infty}^{+\infty}$, that is

$$S = \{\{u_n\} | u_n \in \mathbf{R}, \ n \in \mathbf{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_n + bv_n\}_{n=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers m and T, E_m is defined as a subspace of S by

$$E_m = \{u \in S | u_{n+2mT} = u_n, \ \forall n \in \mathbf{Z}\}.$$

Clearly, E_m is isomorphic to $\mathbf{R}^{2mT}$. E_m can be equipped with the inner product

$$\langle u, v \rangle = \sum_{j=-mT}^{mT-1} u_j v_j, \ \forall u, v \in E_m, \tag{2.1}$$

by which the norm $\| \cdot \|$ can be induced by

$$\|u\| = \left(\sum_{j=-mT}^{mT-1} u_j^2 \right)^{\frac{1}{2}}, \ \forall u \in E_m. \tag{2.2}$$

It is obvious that E_m with the inner product (2.1) is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbf{R}^{2mT}$.

On the other hand, we define the norm $\| \cdot \|_s$ on E_m as follows:

$$\|u\|_s = \left(\sum_{j=-mT}^{mT-1} |u_j|^s \right)^{\frac{1}{s}}, \tag{2.3}$$

for all $u \in E_m$ and $s > 1$.

Since $\|u\|_s$ and $\|u\|_2$ are equivalent, there exist constants $\kappa_1, \ \kappa_2$ such that $\kappa_2 \geq \kappa_1 > 0$, and

$$\kappa_1 \|u\|_2 \leq \|u\|_s \leq \kappa_2 \|u\|_2, \ \forall u \in E_m. \tag{2.4}$$

Clearly, $\|u\| = \|u\|_2$. For all $u \in E_m$, define the functional J on E_m as follows:

$$J(u) = \frac{1}{p} \sum_{n=-mT}^{mT-1} |\Delta u_{n-1}|^p + \frac{1}{p} \sum_{n=-mT}^{mT-1} q_n |u_n|^p - \sum_{n=-mT}^{mT-1} F(n, u_{n+M}, u_n). \tag{2.5}$$

Clearly, $J \in C^1(E_m, \mathbf{R})$ and for any $u = \{u_n\}_{n\in\mathbf{Z}} \in E_m$, by the periodicity of $\{u_n\}_{n\in\mathbf{Z}}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_n} = -\Delta\left(\varphi_p\left(\Delta u_{n-1}\right)\right) + q_n\varphi_p(u_n) - f\left(n, u_{n+M}, u_n, u_{n-M}\right),\ \forall n \in \mathbf{Z}(-mT, mT-1). \tag{2.6}$$

Thus, u is a critical point of J on E_m if and only if

$$\Delta\left(\varphi_p(\Delta u_{n-1})\right) - q_n\varphi_p(u_n) + f\left(n, u_{n+M}, u_n, u_{n-M}\right) = 0,\ \forall n \in \mathbf{Z}(-mT, mT-1).$$

Due to the periodicity of $u = \{u_n\}_{n\in\mathbf{Z}} \in E_m$ and $f(n, v_1, v_2, v_3)$ in the first variable n, we reduce the existence of periodic solutions of (1.1) to the existence of critical points of J on E_m. That is, the functional J is just the variational framework of (1.1).

In what follows, we define a norm $\|\cdot\|_\infty$ in E_m by

$$\|u\|_\infty = \max_{j\in\mathbf{Z}(-mT,mT-1)} \left|u_j\right|,\ \forall u \in E_m.$$

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\{u_n\} \subset E$ for which $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ $(n \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Mountain Pass Lemma [25]). *Let E be a real Banach space and $J \in C^1(E, \boldsymbol{R})$ satisfy the P.S. condition. If $J(0) = 0$ and*

- (J_1) *there exist constants ρ, $\alpha > 0$ such that $J|_{\partial B_\rho} \geq \alpha$, and*
- (J_2) *there exists $e \in E\backslash B_\rho$ such that $J(e) \leq 0$.*

Then J possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} J(g(s)), \tag{2.7}$$

where

$$\Gamma = \{g \in C([0,1], E)\,|\,g(0) = 0,\ g(1) = e\}. \tag{2.8}$$

Lemma 2.2 *The following inequality is true:*

$$\frac{1}{p}\sum_{n=-mT}^{mT-1} |\Delta u_{n-1}|^p \leq \frac{\kappa_2^p 2^p}{p}\|u\|^p. \tag{2.9}$$

Proof

$$\begin{aligned}\frac{1}{p}\sum_{n=-mT}^{mT-1} |\Delta u_{n-1}|^p &= \frac{1}{p}\left[\left(\sum_{n=-mT}^{mT-1} |\Delta u_n|^p\right)^{\frac{1}{p}}\right]^p \\ &\leq \frac{1}{p}\left[\kappa_2\left(\sum_{n=-mT}^{mT-1} |\Delta u_n|^2\right)^{\frac{1}{2}}\right]^p\end{aligned}$$

$$\le \frac{1}{p}\kappa_2^p \left[\sum_{n=-mT}^{mT-1} 2\left(u_{n+1}^2 + u_n^2\right)\right]^{\frac{p}{2}}$$
$$= \frac{\kappa_2^p 2^p}{p}\|u\|^p.$$

□

3 Proof of theorems

In this section, we shall prove the main results stated in Sect. 1 by using the critical point method.

Lemma 3.1 *Assume that* $(F_1)-(F_5)$ *are satisfied. Then* J *satisfies the P.S. condition.*

Proof Assume that $\{u^{(i)}\}_{i\in\mathbf{N}}$ in E_m is a sequence such that $\{J(u^{(i)})\}_{i\in\mathbf{N}}$ is bounded. Then there exists a constant $K>0$ such that $-K \le J(u^{(i)})$. By (2.9) and (F_3'), we have

$$-K \le J\left(u^{(i)}\right) \le \frac{\kappa_2^p 2^p}{p}\left\|u^{(i)}\right\|^p + \frac{\bar{q}}{p}\left[\left(\sum_{n=-mT}^{mT-1}\left|u_n^{(i)}\right|^p\right)^{\frac{1}{p}}\right]^p$$
$$-\sum_{n=-mT}^{mT-1}\left[c\left(\left|u_{n+M}^{(i)}\right|^p + \left|u_n^{(i)}\right|^p\right) + b - \zeta\right]$$
$$\le \left(\frac{\kappa_2^p 2^p}{p} + \frac{\bar{q}\kappa_2^p}{p} - 2c\kappa_1^p\right)\left\|u^{(i)}\right\|^p + 2mT(\zeta - b).$$

Therefore,

$$\left(2c\kappa_1^p - \frac{\kappa_2^p 2^p}{p} - \frac{\bar{q}\kappa_2^p}{p}\right)\left\|u^{(i)}\right\|^p \le 2mT(\zeta - b) + K. \tag{3.1}$$

Since $c > \frac{1}{2p}\left(\frac{\kappa_2}{\kappa_1}\right)^p(2^p+\bar{q})$, (3.1) implies that $\{u^{(i)}\}_{i\in\mathbf{N}}$ is bounded in E_m. Thus, $\{u^{(i)}\}_{i\in\mathbf{N}}$ possesses a convergence subsequence in E_m. The desired result follows. □

Lemma 3.2 *Assume that* $(F_1)-(F_5)$ *are satisfied. Then for any given positive integer* m, (1.1) *possesses a* $2mT$*-periodic solution* $u^{(m)} \in E_m$.

Proof In our case, it is clear that $J(0)=0$. By Lemma 3.1, J satisfies the P.S. condition. By (F_2), we have

$$J(u) \ge \frac{1}{p}\sum_{n=-mT}^{mT-1}|\Delta u_n|^p + \frac{\bar{q}}{p}\sum_{n=-mT}^{mT-1}|u_n|^p - a\sum_{n=-mT}^{mT-1}\left(|u_{n+M}|^p + |u_n|^p\right)$$
$$\ge \frac{\bar{q}\kappa_1^p}{p}\|u\|^p - 2a\kappa_2^p\|u\|^p$$
$$= \left(\frac{\bar{q}\kappa_1^p}{p} - 2a\kappa_2^p\right)\|u\|^p.$$

Taking $\alpha = \left(\frac{\bar{q}\kappa_1^p}{p} - 2a\kappa_2^p\right)\varrho^p > 0$, we obtain

$$J(u)|_{\partial B_\varrho} \geq \alpha > 0,$$

which implies that J satisfies the condition (J_1) of the Mountain Pass Lemma.

Next, we shall verify the condition (J_2).

There exists a sufficiently large number $\varepsilon > \max\{\varrho, \rho\}$ such that

$$\left(2c\kappa_1^p - \frac{\kappa_2^p 2^p}{p} - \frac{\bar{q}\kappa_2^p}{p}\right)\varepsilon^p \geq |b|. \tag{3.2}$$

Let $e \in E_m$ and

$$e_n = \begin{cases} \varepsilon, \text{ if } n = 0, \\ 0, \text{ if } n \in \{j \in \mathbf{Z} : -mT \leq j \leq mT - 1 \text{ and } j \neq 0\}, \end{cases}$$

$$e_{n+M} = \begin{cases} \varepsilon, \text{ if } n = 0, \\ 0, \text{ if } n \in \{j \in \mathbf{Z} : -mT \leq j \leq mT - 1 \text{ and } j \neq 0\}. \end{cases}$$

Then

$$F(n, e_{n+M}, e_n) = \begin{cases} F(0, \varepsilon, \varepsilon), & \text{if } n = 0 \\ 0, & \text{if } n \in \{j \in \mathbf{Z} : -mT \leq j \leq mT - 1 \text{ and } j \neq 0\}. \end{cases}$$

With (3.2) and (F_3), we have

$$\begin{aligned} J(e) &= \frac{1}{p}\sum_{n=-mT}^{mT-1}(\Delta e_{n-1})^p + \frac{1}{p}\sum_{n=-mT}^{mT-1} q_n e_n^p - \sum_{n=-mT}^{mT-1} F(n, e_{n+M}, e_n) \\ &\leq \frac{\kappa_2^p 2^p}{p}\|e\|^p + \frac{\bar{q}\kappa_2^p}{p}\|e\|^p - 2c\kappa_1^p\|e\|^p - b \\ &= -\left(2c\kappa_1^p - \frac{\kappa_2^p 2^p}{p} - \frac{\bar{q}\kappa_2^p}{p}\right)\varepsilon^{\delta+1} - b \leq 0. \end{aligned} \tag{3.3}$$

All the assumptions of the Mountain Pass Lemma have been verified. Consequently, J possesses a critical value c_m given by (2.7) and (2.8) with $E = E_m$ and $\Gamma = \Gamma_m$, where $\Gamma_m = \{g_m \in C([0,1], E_m) | g_m(0) = 0,\ g_m(1) = e,\ e \in E_m \backslash B_\varepsilon\}$. Let $u^{(m)}$ denote the corresponding critical point of J on E_m. Note that $\left\|u^{(m)}\right\| \neq 0$ since $c_m > 0$. □

Lemma 3.3 *Assume that $(F_1) - (F_5)$ are satisfied. Then there exist positive constants ϱ and η independent of m such that*

$$\varrho \leq \left\|u^{(m)}\right\|_\infty \leq \eta. \tag{3.4}$$

Proof The continuity of $F(0, v_1, v_2)$ with respect to the second and third variables implies that there exists a constant $\tau > 0$ such that $|F(0, v_1, v_2)| \leq \tau$ for $\sqrt{v_1^2 + v_2^2} \leq \varrho$. It is clear that

$$\begin{aligned} J\left(u^{(m)}\right) &\leq \max_{0 \leq s \leq 1}\left\{\frac{1}{p}\sum_{n=-mT}^{mT-1}\left[|\Delta(se)_{n-1}|^p + q_n |(se)_n|^p\right] - \sum_{n=-mT}^{mT-1} F(n, (se)_{n+M}, (se)_n)\right\} \\ &\leq \frac{\kappa_2^p 2^p + \bar{q}\kappa_2^p}{p}\|e\|^p + \tau \\ &= \frac{\kappa_2^p 2^p + \bar{q}\kappa_2^p}{p}\varepsilon^p + \tau. \end{aligned}$$

Let $\xi = \frac{\kappa_2^p 2^p + \bar{q}\kappa_2^p}{p}\varepsilon^p + \tau$, we have that $J\left(u^{(m)}\right) \le \xi$, which is independent of m. From (2.5) and (2.6), we have

$$\begin{aligned}
J\left(u^{(m)}\right) &= \frac{1}{p}\sum_{n=-mT}^{mT-1}\left[\frac{\partial F\left(n-M, u_n^{(m)}, u_{n-M}^{(m)}\right)}{\partial v_2}u_n^{(m)} + \frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_2}u_n^{(m)}\right] \\
&\quad - \sum_{n=-mT}^{mT-1} F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right) \\
&= \frac{1}{p}\sum_{n=-mT}^{mT-1}\left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_1}u_{n+M}^{(m)} + \frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_2}u_n^{(m)}\right] \\
&\quad - \sum_{n=-mT}^{mT-1} F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right) \\
&\le \xi.
\end{aligned}$$

By (F_4) and (F_5), there exists a constant $\eta > 0$ such that
$\frac{1}{p}\left(\frac{\partial F(n,v_1,v_2)}{\partial v_1}v_1 + \frac{\partial F(n,v_1,v_2)}{\partial v_2}v_2\right) - F(n, v_1, v_2) > \xi$, for all $n \in \mathbf{Z}$ and $\sqrt{v_1^2 + v_2^2} \ge \eta$, which implies that $\left|u_n^{(m)}\right| \le \eta$ for all $n \in \mathbf{Z}$, that is $\left\|u^{(m)}\right\|_\infty \le \eta$.

From the definition of J, we have

$$\begin{aligned}
0 &= \left\langle J'(u^{(m)}), u^{(m)}\right\rangle \\
&\ge \underline{q}\sum_{n=-mT}^{mT-1}\left|u_n^{(m)}\right|^p - \sum_{n=-mT}^{mT-1}\left[\frac{\partial F\left(n-M, u_n^{(m)}, u_{n-M}^{(m)}\right)}{\partial v_2}u_n^{(m)} + \frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_2}u_n^{(m)}\right] \\
&\ge \underline{q}\kappa_1^p\|u^{(m)}\|^p - \sum_{n=-mT}^{mT-1}\left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_1}u_{n+M}^{(m)} + \frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_2}u_n^{(m)}\right].
\end{aligned}$$

Therefore, combined with (F_2), we get

$$\begin{aligned}
\underline{q}\kappa_1^p\|u^{(m)}\|^p &\le \sum_{n=-mT}^{mT-1}\left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_1}u_{n+M}^{(m)} + \frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_2}u_n^{(m)}\right] \\
&\le \left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_1}\right]^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\|u^{(m)}\|_p \\
&\quad + \left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_2}\right]^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\|u^{(m)}\|_p
\end{aligned}$$

$$\leq \kappa_2 \|u^{(m)}\| \left\{ \left\{ \sum_{n=-mT}^{mT-1} \left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_1} \right]^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} + \left\{ \sum_{n=-mT}^{mT-1} \left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_2} \right]^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} \right\}.$$

That is,

$$\frac{q\kappa_1^p}{\kappa_2} \|u^{(m)}\|^{p-1} \leq \left\{ \sum_{n=-mT}^{mT-1} \left[\frac{\partial F(n, u_{n+M}^{(m)}, u_n^{(m)})}{\partial v_1} \right]^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} + \left\{ \sum_{n=-mT}^{mT-1} \left[\frac{\partial F(n, u_{n+M}^{(m)}, u_n^{(m)})}{\partial v_2} \right]^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}}.$$

Thus,

$$\frac{q^{\frac{p}{p-1}} \kappa_1^{\frac{p^2}{p-1}}}{\kappa_2^{\frac{p}{p-1}}} \|u^{(m)}\|^p \leq \left\{ \left\{ \sum_{n=-mT}^{mT-1} \left[\frac{\partial F(n, u_{n+M}^{(m)}, u_n^{(m)})}{\partial v_1} \right]^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} + \left\{ \sum_{n=-mT}^{mT-1} \left[\frac{\partial F(n, u_{n+M}^{(m)}, u_n^{(m)})}{\partial v_2} \right]^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} \right\}^{\frac{p}{p-1}}. \quad (3.5)$$

Combined with (F_2), we get

$$\frac{q^p \kappa_1^{\frac{p^2}{p-1}}}{\kappa_2^{\frac{p}{p-1}}} \|u^{(m)}\|^p \leq \left\{ \left\{ \sum_{n=-mT}^{mT-1} \left[pa \left| u_{n+M}^{(m)} \right|^{p-1} \right]^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} + \left\{ \sum_{n=-mT}^{mT-1} \left[pa \left| u_n^{(m)} \right|^{p-1} \right]^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} \right\}^{\frac{p}{p-1}}$$
$$\leq 2^{\frac{p}{p-1}} (ap)^{\frac{p}{p-1}} \kappa_2^p \|u^{(m)}\|^p.$$

Thus, we have $u^{(m)} = 0$. But this contradicts $\|u^{(m)}\| \neq 0$, which shows that

$$\|u^{(m)}\|_\infty \geq \varrho,$$

and the proof of Lemma 3.3 is finished. □

Proof of Theorem 1.1 In the following, we shall give the existence of a nontrivial homoclinic orbit.

Consider the sequence $\left\{u_n^{(m)}\right\}_{n\in\mathbf{Z}}$ of $2mT$-periodic solutions found in Lemma 3.2. First, by (3.4), for any $m \in \mathbf{N}$, there exists a constant $n_m \in \mathbf{Z}$ independent of m such that

$$\left|u_{n_m}^{(m)}\right| \geq \varrho. \tag{3.6}$$

Since q_n and $f(n, v_1, v_2, v_3)$ are all T-periodic in n, $\left\{u_{n+jT}^{(m)}\right\}$ $(\forall j \in \mathbf{N})$ is also $2mT$-periodic solution of (1.1). Hence, making such shifts, we can assume that $n_m \in \mathbf{Z}(0, T-1)$ in (3.6). Moreover, passing to a subsequence of ms, we can even assume that $n_m = n_0$ is independent of m.

Next, we extract a subsequence, still denote by $u^{(m)}$, such that

$$u_n^{(m)} \to u_n,\ m \to \infty,\ \forall n \in \mathbf{Z}.$$

Inequality (3.6) implies that $\left|u_{n_0}\right| \geq \xi$ and, hence, $u = \{u_n\}$ is a nonzero sequence. Moreover,

$$\begin{aligned}&\Delta\left(\varphi_p(\Delta u_{n-1})\right) - q_n\varphi_p(u_n) + f(n, u_{n+M}, u_n, u_{n-M})\\&= \lim_{n\to\infty}\left[\Delta\left(\varphi_p\left(\Delta u_{n-1}^{(m)}\right)\right) - q_n\varphi_p\left(u_n^{(m)}\right) + f\left(n, u_{n+M}^{(m)}, u_n^{(m)}, u_{n-M}^{(m)}\right)\right] = 0.\end{aligned}$$

So $u = \{u_n\}$ is a solution of (1.1).

Finally, we show that $u \in l^p$. For $u_m \in E_m$, let

$$P_m = \left\{n \in \mathbf{Z} : \left|u_n^{(m)}\right| < \frac{\sqrt{2}}{2}\varrho, -mT \leq n \leq mT-1\right\},$$

$$Q_m = \left\{n \in \mathbf{Z} : \left|u_n^{(m)}\right| \geq \frac{\sqrt{2}}{2}\varrho, -mT \leq n \leq mT-1\right\}.$$

Since $F(n, v_1, v_2) \in C^1(\mathbf{Z}\times\mathbf{R}^2, \mathbf{R})$, there exist constants $\bar{\xi} > 0$, $\underline{\xi} > 0$ such that

$$\begin{aligned}\max\Bigg\{&\Bigg\{\left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n, v_1, v_2)}{\partial v_1}\right]^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\\&+\left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n, v_1, v_2)}{\partial v_2}\right]^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\Bigg\}^{\frac{p}{p-1}} : \varrho \leq \sqrt{v_1^2+v_2^2} \leq \eta, n \in \mathbf{Z}\Bigg\} \leq \bar{\xi},\end{aligned}$$

$$\begin{aligned}\min\Bigg\{&\frac{1}{p}\left[\frac{\partial F(n, v_1, v_2)}{\partial v_1}v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2}v_2\right]\\&-F(n, v_1, v_2) : \varrho \leq \sqrt{v_1^2+v_2^2} \leq \eta, n \in \mathbf{Z}\Bigg\} \geq \underline{\xi}.\end{aligned}$$

For $n \in Q_m$,

$$\left\{\left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_1}\right]^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}} + \left\{\left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_2}\right]^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}$$

$$\leq \frac{\bar{\xi}}{\underline{\xi}} \left\{ \frac{1}{p} \left[\frac{\partial F(n, u_{n+M}^{(m)}, u_n^{(m)})}{\partial v_1} u_{n+M}^{(m)} + \frac{\partial F(n, u_{n+M}^{(m)}, u_n^{(m)})}{\partial v_2} u_n^{(m)}) \right] \right.$$

$$\left. -F(n, u_{n+M}^{(m)}, u_n^{(m)}) \right\}.$$

By (3.5), we have

$$\frac{\underline{q}^{\frac{p}{p-1}} \kappa_1^{\frac{p^2}{p-1}}}{\kappa_2^{\frac{p}{p-1}}} \|u^{(m)}\|^p$$

$$\leq \left\{ \left\{ \sum_{n \in P_m} \left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_1} \right]^{\frac{p}{p-1}} \right\}^{\frac{p}{p-1}} \right.$$

$$\left. + \left\{ \sum_{n \in P_m} \left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_2} \right]^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} \right\}^{\frac{p}{p-1}}$$

$$+ \left\{ \left\{ \sum_{n \in Q_m} \left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_1} \right]^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} \right.$$

$$\left. + \left\{ \sum_{n \in Q_m} \left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_2} \right]^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} \right\}^{\frac{p}{p-1}}$$

$$\leq \left\{ \left\{ \sum_{n \in P_m} \left[pa \left| u_{n+M}^{(m)} \right|^{p-1} \right]^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} + \left\{ \sum_{n \in P_m} \left[pa \left| u_n^{(m)} \right|^{p-1} \right]^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}} \right\}^{\frac{p}{p-1}}$$

$$+ \frac{\bar{\xi}}{\underline{\xi}} \left\{ \frac{1}{p} \sum_{n \in Q_m} \left[\frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_1} u_{n+M}^{(m)} + \frac{\partial F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right)}{\partial v_2} u_n^{(m)}) \right] \right.$$

$$\left. -F\left(n, u_{n+M}^{(m)}, u_n^{(m)}\right) \right\}$$

$$\leq 2^{\frac{p}{p-1}} (ap)^{\frac{p}{p-1}} \kappa_2^p \|u^{(m)}\|^p + \frac{\bar{\xi}\xi}{\underline{\xi}}.$$

Thus,

$$\left\| u^{(m)} \right\|^p \leq \frac{\bar{\xi}\xi\kappa_2^{\frac{p}{p-1}}}{\underline{\xi}\left\{ \underline{q}^{\frac{p}{p-1}} \kappa_1^{\frac{p^2}{p-1}} - (2ap)^{\frac{p}{p-1}} \kappa_2^{\frac{p^2}{p-1}} \right\}}.$$

For any fixed $D \in \mathbf{Z}$ and m large enough, we have that

$$\sum_{n=-D}^{D} \left|u_n^{(m)}\right|^p \leq \|u^{(m)}\|^p \leq \frac{\bar{\xi}\xi\kappa_2^{\frac{p}{p-1}}}{\underline{\xi}\left\{\underline{q}^{\frac{p}{p-1}}\kappa_1^{\frac{p^2}{p-1}} - (2ap)^{\frac{p}{p-1}}\kappa_2^{\frac{p^2}{p-1}}\right\}}.$$

Since $\bar{\xi}, \ \underline{\xi}, \ \xi, \ \underline{q}, \ p, \ a, \ \kappa_1$ and κ_2 are constants independent of m, passing to the limit, we have that

$$\sum_{n=-D}^{D} |u_n|^p \leq \frac{\bar{\xi}\xi\kappa_2^{\frac{p}{p-1}}}{\underline{\xi}\left\{\underline{q}^{\frac{p}{p-1}}\kappa_1^{\frac{p^2}{p-1}} - (2ap)^{\frac{p}{p-1}}\kappa_2^{\frac{p^2}{p-1}}\right\}}.$$

Due to the arbitrariness of D, $u \in l^p$. Therefore, u satisfies $u_n \to 0$ as $|n| \to \infty$. The existence of a nontrivial homoclinic orbit is obtained. □

Proof of Theorem 1.2 Consider the following boundary problem:

$$\begin{cases} \Delta\left(\varphi_p(\Delta u_{n-1})\right) - q_n\varphi_p(u_n) + f(n, u_{n+M}, u_n, u_{n-M}) = 0, \ n \in \mathbf{Z}(-mT, mT), \\ q_{-mT} = q_{mT} = 0, \\ q_{-n} = q_n, \qquad n \in \mathbf{Z}(-mT, mT). \end{cases}$$

Let S be the set of sequences $u = (\ldots, u_{-n}, \ldots, u_{-1}, u_0, u_1, \ldots, u_n, \ldots) = \{u_n\}_{n=-\infty}^{+\infty}$, that is

$$S = \{\{u_n\} | u_n \in \mathbf{R}, \ n \in \mathbf{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_n + bv_n\}_{n=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers m and T, $\tilde{E}_m$ is defined as a subspace of S by

$$\tilde{E}_m = \{u \in S | u_{-n} = u_n, \ \forall n \in \mathbf{Z}\}.$$

Clearly, $\tilde{E}_m$ is isomorphic to $\mathbf{R}^{2mT+1}$. $\tilde{E}_m$ can be equipped with the inner product

$$\langle u, v\rangle = \sum_{j=-mT}^{mT} u_j v_j, \ \forall u, v \in \tilde{E}_m,$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=-mT}^{mT} u_j^2\right)^{\frac{1}{2}}, \ \forall u \in \tilde{E}_m.$$

It is obvious that $\tilde{E}_m$ is Hilbert space with $2mT + 1$-periodicity and linearly homeomorphic to $\mathbf{R}^{2mT+1}$.

Similarly to the proof of Theorem 1.1, we can also prove Theorem 1.2. For simplicity, we omit its proof. □

4 Example

In this section, we give an example to illustrate our results.

Example 4.1 Let

$$f(n, v_1, v_2, v_3) = \gamma v_2 \left(\frac{v_1^2 + v_2^2}{v_1^2 + v_2^2 + 1} + \frac{v_2^2 + v_3^2}{v_2^2 + v_3^2 + 1} \right)$$

and

$$F(n, v_1, v_2) = \frac{\gamma}{2} \left[v_1^2 + v_2^2 - \ln \left(v_1^2 + v_2^2 + 1 \right) \right],$$

where $\gamma > \bar{q}$. It is easy to verify all the assumptions of Theorem 1.1 are satisfied. Consequently, a nontrivial homoclinic orbit is obtained.

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