RACSAM (2016) 110:1–6
DOI 10.1007/s13398-014-0214-2

ORIGINAL PAPER

Algebras of holomorphic functions and the Michael problem

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Received: 22 August 2014 / Accepted: 27 December 2014 / Published online: 9 January 2015

Abstract Clayton, Schottenloher and Mujica have reduced the study of the Michael problem to certain specific algebras of holomorphic functions on infinite dimensional spaces. In this note we establish a general theorem that yields as special cases the aforementioned results.

Keywords Locally m-convex algebra · Fréchet algebra · Michael problem · Locally convex space · Holomorphic function · Schauder basis

Mathematics Subject Classification Primary 46G20; Secondary 46H40 · 46J99 · 46A35

1 Introduction

In 1952 Michael [6] posed the following two problems:

(a) If A is a commutative Fréchet algebra, is every complex homomorphism on A necessarily continuous?
(b) If A is a complete commutative locally m-convex algebra, is every complex homomorphism on A necessarily bounded?

Clearly a positive solution to the second problem implies a positive solution to the first problem, and in 1972 Dixon and Fremlin [3] proved that the reverse implication is also true.

Clayton [1], Schottenloher [9] and Mujica [8] have reduced the study of the Michael problem to certain specific algebras of holomorphic functions on infinite dimensional spaces. In this note we establish a general theorem that yields as special cases the aforementioned results of Clayton [1], Schottenloher [9] and Mujica [8].

Dedicated to the memory of Manuel Valdivia (1928–2014).

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2 The main results

Let E and F denote locally convex spaces, always assumed complex and Hausdorff, and let $cs(E)$ denote the set of all continuous seminorms on E. Let E'_b (resp. E'_c) denote the dual E' of E, with the topology of uniform convergence on the bounded (resp. compact) subsets of E. Let $\mathcal{L}(E; F)$ denote the space of all continuous linear mappings from E into F, and let τ_c denote the topology of uniform convergence on the compact subsets of E.

We recall that a sequence $(e_n)_{n=1}^{\infty} \subset E$ is said to be a *basis* if every $x \in E$ admits a unique representation as a series $x = \sum_{j=1}^{\infty} \xi_j e_j = \lim_{n\to\infty} \sum_{j=1}^{n} \xi_j e_j$, with $(\xi_j)_{j=1}^{\infty} \subset \mathbb{C}$. The linear functionals $\phi_j : x \in E \to \xi_j \in \mathbb{C}$ are called *coordinate functionals*, and the linear mappings $T_n : x \in E \to \sum_{j=1}^{n} \xi_j e_j$ are called *canonical projections*. A basis $(e_n)_{n=1}^{\infty}$ is said to be a *Schauder basis* if the coordinate functionals are continuous. A Schauder basis $(e_n)_{n=1}^{\infty}$ is said to be an *equicontinuous Schauder basis* if the sequence of canonical projections is equicontinuous. A Schauder basis $(e_n)_{n=1}^{\infty}$ is said to be a *compactly convergent Schauder basis* if the sequence of canonical projections converges to the identity uniformly on the compact subsets of E. Every basis in a Fréchet space is a Schauder basis (see [5, p. 249]). Clearly every Schauder basis in a barrelled space is an equicontinuous Schauder basis. And clearly every equicontinuous Schauder basis is a compactly convergent Schauder basis.

Let $\mathcal{H}(E)$ denote the algebra of all complex-valued holomorphic functions on E, and let τ_c denote the topology of uniform convergence on the compact subsets of E. Let $\mathcal{H}_b(E)$ denote the subalgebra of all $f \in \mathcal{H}(E)$ which are bounded on the bounded subsets of E, and let τ_b denote the topology of uniform convergence on the bounded subsets of E.

We recall that A is said to be a *topological algebra* if A is a complex algebra and a topological vector space such that ring multiplication is continuous. We require that complex algebras have a unit element, and if A and B are complex algebras, we require that a homomorphism $T : A \to B$ map the unit element of A onto the unit element of B. A topological algebra A is said to be *locally m-convex* if its topology is defined by a family of continuous seminorms q such that $q(xy) \le q(x)q(y)$ for all $x, y \in A$. A complete metrizable locally m-convex algebra is called a *Fréchet algebra*.

Theorem 2.1 *Let E be a sequentially complete infinite dimensional locally convex space with a compactly convergent Schauder basis $(e_n)_{n=1}^{\infty}$. Let $(\phi_n)_{n=1}^{\infty}$ denote the sequence of coordinate functionals, and assume that $(e_n)_{n=1}^{\infty}$ is bounded in E. Let A be a sequentially complete commutative locally m-convex algebra. If $(a_n)_{n=1}^{\infty}$ is a sequence in A such that*

$$\sum_{n=1}^{\infty} \sqrt{q(a_n)} < \infty \quad \textit{for every} \quad q \in cs(A),$$

then there exists a continuous homomorphism $T : (\mathcal{H}(E), \tau_c) \to A$ such that $T\phi_n = a_n$ for every $n \in \mathbb{N}$.

Proof The proof is a straightforward adaptation of the proof of [8, Theorem 33.3], which is reproduced here for the convenience of the reader, with the corresponding modifications in our more general situation. Let $f \in \mathcal{H}(E)$, and let $(T_n)_{n=1}^{\infty}$ denote the sequence of canonical projections. Since the sequence $(T_n)_{n=1}^{\infty}$ converges to the identity in $(\mathcal{L}(E; E), \tau_c)$, it follows that the sequence $(f \circ T_n)_{n=1}^{\infty}$ converges to f in $(\mathcal{H}(E), \tau_c)$. For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ let

$$c_\alpha f = (2\pi i)^{-n} \int_{|\zeta_1|=R_1,\ldots,|\zeta_n|=R_n} \frac{f(\zeta_1 e_1 + \cdots + \zeta_n e_n)}{\zeta_1^{\alpha_1+1} \ldots \zeta_n^{\alpha_n+1}} d\zeta_1 \ldots d\zeta_n, \tag{1}$$

with $R_1 > 0, \ldots, R_n > 0$. Then each $c_\alpha f$ is independent from the choice of $R_1, \ldots, R_n$, and the multiple series $\sum_{\alpha\in\mathbb{N}_0^n} c_\alpha f \phi_1^{\alpha_1} \ldots \phi_n^{\alpha_n}$ converges to $f \circ T_n$ in $(\mathcal{H}(E), \tau_c)$. It follows that

$$f = \lim_{n\to\infty} f \circ T_n = \lim_{n\to\infty} \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha f \phi_1^{\alpha_1} \ldots \phi_n^{\alpha_n},$$

with uniform convergence on the compact subsets of E (see [8, Corollary 7.8] or [2, p. 237]).

The topology of A is given by a family Q of continuous seminorms q satisfying the condition $q(xy) \le q(x)q(y)$ for all $x, y \in A$. Given $q \in Q$ we have by hypothesis that $\sum_{n=1}^{\infty} \sqrt{q(a_n)} < \infty$. Choose $0 < \varepsilon < 1$ such that $\varepsilon \sum_{n=1}^{\infty} \sqrt{q(a_n)} < 1$, and set $r_n = \varepsilon\sqrt{q(a_n)}$, $R_n = \varepsilon^{-1}\sqrt{q(a_n)}$ for every n.

We assert that the set

$$L_q = \left\{ \sum_{n=1}^{\infty} \zeta_n e_n : \zeta_n \in \mathbb{C}, |\zeta_n| \le R_n \quad \text{for every} \quad n \right\}$$

is a compact subset of E. Indeed consider the set

$$K_q = \left\{ (\zeta_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}} : |\zeta_n| \le R_n \ \text{ for every } \quad n \right\}.$$

By the Tychonoff theorem K_q is a compact subset of $\mathbb{C}^{\mathbb{N}}$. Consider the mappings $S : K_q \to E$ and $S_N : K_q \to E$ defined by

$$S\left((\zeta_n)_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} \zeta_n e_n, \quad S_N\left((\zeta_n)_{n=1}^{\infty}\right) = \sum_{n=1}^{N} \zeta_n e_n.$$

Clearly each S_N is continuous. To show that S is continuous we show that the sequence $(S_N)_{N=1}^{\infty}$ converges to S absolutely and uniformly on K_q. Indeed for each $p \in cs(E)$, let $c_p = \sup_n p(e_n)$. Then

$$\sup_{((\zeta_n)_{n=1}^{\infty})\in K_q} \sum_{n=1}^{\infty} p\left(\zeta_n e_n\right) \le c_p \sum_{n=1}^{\infty} R_n < \infty$$

and

$$\sup_{((\zeta_n)_{n=1}^{\infty})\in K_q} p\left((S - S_N)((\zeta_n)_{n=1}^{\infty})\right) \le c_p \sum_{n=N+1}^{\infty} R_n.$$

Thus S is continuous and $L_q = S(K_q)$ is compact, as asserted.

It follows from (1) that

$$|c_\alpha f| \le \left(R_1^{\alpha_1} \ldots R_n^{\alpha_n}\right)^{-1} \sup_{L_q} |f|$$

for every $\alpha \in \mathbb{N}_0^n$. Since $q(a_n) = R_n r_n$ for every n, it follows that

$$\sum_{\alpha\in\mathbb{N}_0^n} q\left(c_\alpha f a_1^{\alpha_1} \ldots a_n^{\alpha_n}\right) \le \sum_{\alpha\in\mathbb{N}_0^n} |c_\alpha f| q(a_1)^{\alpha_1} \ldots q(a_n)^{\alpha_n}$$
$$= \sum_{\alpha\in\mathbb{N}_0^n} |c_\alpha f| R_1^{\alpha_1} \ldots R_n^{\alpha_n} r_1^{\alpha_1} \ldots r_n^{\alpha_n} \le \sup_{L_q} |f| \,(1 - r_1)^{-1} \ldots (1 - r_n)^{-1}.$$

Since $\sum_{n=1}^{\infty} r_n = \theta < 1$, it follows that

$$\sum_{n=1}^{\infty} \frac{r_n}{1-r_n} \leq \sum_{n=1}^{\infty} \frac{r_n}{1-\theta} = \frac{\theta}{1-\theta} < \infty.$$

Hence it follows that the infinite product

$$\prod_{n=1}^{\infty} (1-r_n)^{-1} = \prod_{n=1}^{\infty} \left(1 + \frac{r_n}{1-r_n}\right)$$

converges. Hence there exists a constant $d_q > 0$ such that

$$\sum_{\alpha \in \mathbb{N}_0^n} q\left(c_\alpha f a_1^{\alpha_1} \dots a_n^{\alpha_n}\right) \leq d_q \sup_{L_q} |f|$$

for every $f \in \mathcal{H}(E)$ and $n \in \mathbb{N}$. Since A is sequentially complete, it follows that the multiple series

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha f a^\alpha = \sum_{n=1}^{\infty} \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha f a_1^{\alpha_1} \dots a_n^{\alpha_n}$$

converges absolutely in A for every $f \in \mathcal{H}(E)$. Let $T : (\mathcal{H}(E), \tau_c) \to A$ be defined by

$$Tf = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha f a^\alpha.$$

Then $T\phi_j = a_j$ for every j, and we can readily verify that T is a homomorphism. Since

$$q(Tf) \leq d_q \sup_{L_q} |f|$$

for every $f \in \mathcal{H}(E)$, it follows that T is continuous.

Theorem 2.2 *let E be a sequentially complete infinite dimensional locally convex space with a compactly convergent Schauder basis $(e_n)_{n=1}^{\infty}$. Let $(\phi_n)_{n=1}^{\infty}$ denote the sequence of coordinate functionals, and assume that:*

(i) $(e_n)_{n=1}^{\infty}$ is bounded in E;
(ii) there exists a sequence $(\lambda_n)_{n=1}^{\infty}$ of strictly positive numbers such that $(\lambda_n \phi_n)_{n=1}^{\infty}$ is bounded in E_b'.

Let A be a sequentially complete commutative locally m-convex algebra. If there exists an unbounded complex homomorphism on A, then there exists a complex homomorphism on $\mathcal{H}(E)$ whose restriction to E_b' is unbounded. In particular there exists an unbounded complex homomorphism on $(\mathcal{H}(E), \tau_c)$ whose restriction to $(\mathcal{H}_b(E), \tau_b)$ is unbounded as well.

Proof Let $\psi : A \to \mathbb{C}$ be an unbounded homomorphism. Then there is a bounded sequence $(b_n)_{n=1}^{\infty}$ in A such that $|\psi(b_n)| > 8^n/\lambda_n$ for every $n \in \mathbb{N}$. Let $a_n = 4^{-n} b_n$ for every $n \in \mathbb{N}$. Then for each $q \in cs(A)$ there is a constant $c > 0$ such that $q(b_n) \leq c$ for every n. Hence it follows that $q(a_n) \leq 4^{-n} c$ for every n, and therefore $\sum_{n=1}^{\infty} \sqrt{q(a_n)} < \infty$. By Theorem 2.1 there exists a continuous homomorphism $T : (\mathcal{H}(E), \tau_c) \to A$ such that $T\phi_n = a_n$ for every n. Since

$$|\psi \circ T(\lambda_n \phi_n)| = |\psi(\lambda_n a_n)| > 2^n$$

for every n, it follows that the homomorphism $\psi \circ T : \mathcal{H}(E) \to \mathbb{C}$ is unbounded on E_b', as asserted.

Example 2.3 In [8, Theorem 33.5] Mujica reduces the study of the Michael problem to the Fréchet algebra $(\mathcal{H}_b(E), \tau_b)$, where E is any infinite dimensional Banach space with a normalized Schauder basis $(e_n)_{n=1}^{\infty}$. Every Schauder basis in a Banach space is an equicontinuous Schauder basis. Since $(e_n)_{n=1}^{\infty}$ is bounded in E, and $(\phi_n)_{n=1}^{\infty}$ is bounded in E_b', Theorem 2.2 applies to E, and therefore yields [8, Theorem 33.5] as a special case.

We recall that a (DFN)-space is the strong dual of a Fréchet-nuclear space. Then we have the following example.

Example 2.4 In [9, Theorem 6] Schottenloher reduces the study of the Michael problem to the Fréchet algebra $(\mathcal{H}(E), \tau_c)$, where E is any infinite dimensional (DFN)-space with a Schauder basis $(e_n)_{n=1}^{\infty}$ wich satisfies a certain condition (B). The space $E = s'$ of slowly increasing sequences, and the space $E = \mathcal{H}(0_{\mathbb{C}^n})$ of germs of holomorphic functions at $0 \in \mathbb{C}^n$, are examples of (DFN)-spaces which satisfy condition (B). Since E is a Montel space, it is in particular quasi-complete. Since E is barrelled, the Schauder basis $(e_n)_{n=1}^{\infty}$ is an equicontinuous Schauder basis. Condition (B) implies that $(e_n)_{n=1}^{\infty}$ is bounded in E. And since E_b' is metrizable, the Mackey countability condition implies the existence of a sequence $(\lambda_n)_{n=1}^{\infty}$ of strictly positive numbers such that $(\lambda_n \phi_n)_{n=1}^{\infty}$ is bounded in E_b' (see [4, p. 116, Proposition 3]). Thus Theorem 2.2 applies to E and therefore yields [9, Theorem 6] as a special case.

Our next example rests on the following auxiliary result.

Proposition 2.5 *Let F be a barrelled locally convex space, and let $\left((f_n)_{n=1}^{\infty}, (f_n')_{n=1}^{\infty}\right)$ be a biorthogonal sequence in $F \times F'$, that is $f_n'(f_m) = \delta_{nm}$ for all n, m. Then $(f_n)_{n=1}^{\infty}$ is a compactly convergent Schauder basis in F if and only if $(f_n')_{n=1}^{\infty}$ is a compactly convergent Schauder basis in F_c'.*

Proof On the one hand the polars L° of the compact subsets L of F form a 0-neighborhood base in F_c'. On the other hand, since F is barrelled, the polars V° of the 0-neighborhoods V in F form a fundamental family of compact subsets of F_c'. Consider the mapping $T_n \in \mathcal{L}(F; F)$ and the dual mapping $T_n' \in \mathcal{L}(F_c'; F_c')$ given by

$$T_n y = \sum_{j=1}^{n} f_j'(y) f_j, \quad T_n' y' = \sum_{j=1}^{n} y'\left(f_j\right) f_j'.$$

Then we can prove that the sequence $(T_n)_{n=1}^{\infty}$ converges to I_F in $(\mathcal{L}(F; F), \tau_c)$ if and only if the sequence $(T_n')_{n=1}^{\infty}$ converges to $I_{F'}$ in $(\mathcal{L}(F_c'; F_c'), \tau_c)$. Indeed if L is a convex balanced compact set in F, and V is a closed convex balanced 0-neighborhood in F, then we can readily verify that

$$(T_n - I_F)(L) \subset V \quad \text{if and only if} \quad \left(T_n' - I_{F'}\right)\left(V^\circ\right) \subset L^\circ.$$

We will say that E is a (DBC)-space if $E = F_c'$ for some Banach space F. Then we have the following example.

Example 2.6 Let F be an infinite dimensional Banach space with a normalized Schauder basis $(f_n)_{n=1}^{\infty}$, and let $(f_n')_{n=1}^{\infty}$ denote the sequence of coordinate functionals. Then $(f_n)_{n=1}^{\infty}$ is an equicontinuous Schauder basis of F. By the preceding proposition the sequence $(f_n')_{n=1}^{\infty}$

is a compactly convergent Schauder basis of the (DBC)-space $E = F'_c$. Moreover $(f'_n)_{n=1}^{\infty}$ is bounded in F'_b, and therefore bounded in $E = F'_c$, whereas $(f_n)_{n=1}^{\infty}$ is bounded in $F = E'_b$. Moreover E is a semi-Montel space, in particular quasi-complete (see [7, Proposition 7.2]). Thus Theorem 2.2 applies to E, and therefore reduces the study of the Michael problem to the Fréchet algebra $(\mathcal{H}(E), \tau_c)$. That $(\mathcal{H}(E), \tau_c)$ is a Fréchet algebra follows from the fact that E is a hemicompact k-space (see [7, Proposition 7.2] and [7, p. 513]).

Example 2.7 It is well known that

$$\mathcal{H}(\mathbb{C}^{\mathbb{N}}) = \bigcup_{n=1}^{\infty} \{f_n \circ \pi_n : f_n \in \mathcal{H}(\mathbb{C}^n)\},$$

where $\pi_n : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^n$ denotes the canonical projection (see [2, p. 66, Example 2.25]). In [1, Theorem 9] Clayton reduces the study of the Michael problem to the Fréchet algebra A which is defined as the completion of the algebra $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ with respect to uniform convergence on the bounded subsets of ℓ_∞. In [9, Remark 7c] Schottenloher observes that A is isomorphic to the Fréchet algebra $(\mathcal{H}(E), \tau_c)$, where $E = (\ell_1)'_c$. Thus Clayton's example is a special case of Example 2.6.

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