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Copies of c_0 in the space of Pettis integrable functions revisited

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Abstract If (Ω, Σ, μ) is a finite measure space and X a Banach space whose dual has a countable norming set we provide a proof of the fact that the space of all weakly μ-measurable (classes of scalarly equivalent) Pettis integrable functions $f\colon \Omega \to X$ of finite variation, equipped with the variation norm, contains a copy of c_0 if and only if X does.

Keywords Pettis integrable · Countably additive vector measure · Copy of c_0

Mathematics Subject Classification 28B05 · 46B03

1 Preliminaries

Throughout this paper X will stand for a Banach space over the real or complex field $\mathbb{K}$, X^* for its dual space and (Ω, Σ, μ) for a finite measure space. As usual $ca(\Sigma, X)$ will denote the Banach space over $\mathbb{K}$ of all X-valued countably additive measures F on Σ equipped with the semivariation norm $\|F\|$ whereas $bvca(\Sigma, X)$ will stand for the Banach space of all X-valued countably additive measures F of bounded variation on Σ equipped with the variation norm $|F|$.

Let us recall that a weakly μ-measurable function $f\colon \Omega \to X$ is said to be Dunford integrable if $x^* f \in \mathcal{L}_1(\mu)$ for every $x^* \in X^*$. If f is Dunford integrable and $E \in \Sigma$, the map $x^* \mapsto \int_E x^* f \, d\mu$, usually denoted by $(D)\int_E f \, d\mu$, is a continuous linear form on X^*. If $(D)\int_E f \, d\mu \in X$ for each $E \in \Sigma$ then f is said to be Pettis integrable and we write $(P)\int_E f \, d\mu$ instead of $(D)\int_E f \, d\mu$. The Pettis space of all weakly measurable

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(classes of scalarly equivalent) Pettis integrable functions $f\colon \Omega \to X$ is denoted by $\mathcal{P}_1(\mu, X)$ whereas the subspace of all those strongly measurable (classes of) functions is represented by $P_1(\mu, X)$. We will consider these two spaces $\mathcal{P}_1(\mu, X)$ and $P_1(\mu, X)$ endowed with the semivariation norm

$$\|f\|_{\mathcal{P}_1(\mu,X)} = \sup\left\{\int_\Omega \left|x^* f(\omega)\right| d\mu(\omega) : x^* \in X^*,\ \left\|x^*\right\| \le 1\right\}.$$

By a result of Pettis, if $f\colon \Omega \to X$ is weakly measurable and Pettis integrable, the map $F\colon \Sigma \to X$ defined by $F(E) = (P)\int_E f\, d\mu$ is a μ-continuous countably additive X-valued measure and the linear operator $S\colon \mathcal{P}_1(\mu, X) \to ca(\Sigma, X)$ defined by $Sf = F$ is a linear isometry from $\mathcal{P}_1(\mu, X)$ into $ca(\Sigma, X)$, that is, it holds that $\|Sf\| = \|f\|_{\mathcal{P}_1(\mu,X)}$. If in addition f is strongly measurable, then $Sf(\Sigma)$ is a relatively compact subset of X [5, Chapter VIII].

We also recall that a subspace M of X^* is said to be norming over a subspace Y of X if for some $c > 0$ it holds that for all $y \in Y$, $\sup_{f\in S(M)} |f(y)| \ge c\|y\|$ where $S(M)$ denotes the unit sphere of M. For short M is said to be norming when M happens to be norming on X.

The existence of copies of c_0 in the Pettis space $P_1(\mu, X)$ has been studied by a number of authors, e.g. Freniche [10,11] showed that $P_1(\mu, X)$ contains a copy of c_0 if and only if X does (see also [7]). In [9, Theorem 1.2] we find that if X^* contains a norming sequence and X has the weak Radon-Nikodým property (WRNP) with respect to each positive and finite measure defined on Σ, then the space $bvca(\Sigma, X)$ contains a copy of c_0 if and only if X does. Recently, Ferrando [8, Theorem 2.2] has shown that the Musiał space $\mathcal{M}(\mu, X)$ formed by all those functions $f\colon \Omega \to X$ of $\mathcal{P}_1(\mu, X)$ whose associated measure Sf has bounded variation, endowed with the variation norm of the integral, enjoys the property that if each $f \in \mathcal{M}(\mu, X)$ has a Pettis integral with separable range, then $\mathcal{M}(\mu, X)$ contains a copy of c_0 if and only if X does. This happens in a number of situations, for instance if: (A) (Ω, Σ, μ) is a perfect space; (B) X has the weak** Radon-Nikodým property (W**RNP); or (C) X is weakly compactly generated (WCG).

In this note, by making a timing use of [13, Proposition 3.2], under which the existence of a norming sequence in X^* provides that each $f \in \mathcal{M}(\mu, X)$ satisfies that $\|f(\cdot)\| \in L_1(\mu)$ although it may happen that $f \notin L_1(\mu)$ as f is not strongly measurable in general, we provide a proof of the fact that in the particular case that X^* contains a norming sequence, then the space $\mathcal{M}(\mu, X)$ contains a copy of c_0 if and only if X does. This result is not included in those covered by [8, Theorem 2.2] as we will provide examples of some Banach spaces with a norming sequence that fail to have the W**RNP and are not WCG.

2 Copies of c_0 in $\mathcal{M}(\mu, X)$

Our main theorem reads as follows,

Theorem 2.1 *Assuming that X^* has a norming sequence, then the normed space $\mathcal{M}(\mu, X)$ contains a copy of c_0 if and only if X does.*

Proof Let S be the isometric embedding map of $\mathcal{M}(\mu, X)$ in $bvca(\Sigma, X)$ defined by

$$Sf(E) = (P)\int_E f(\omega)\, d\mu(\omega),\ E \in \Sigma.$$

Since $Sf \in bvca(\Sigma, X)$ for each $f \in \mathcal{M}(\mu, X)$, by [13, Proposition 3.2] the fact that X^* has a norming sequence implies that the function $\omega \mapsto \|f(\omega)\|$ belongs to $L_1(\mu)$ and

$$|Sf| = \int_\Omega \|f(\omega)\| \, d\mu(\omega). \tag{2.1}$$

Let $J : c_0 \to \mathcal{M}(\mu, X)$ be an isomorphism from c_0 into $\mathcal{M}(\mu, X)$. Set $f_n := Je_n$ and

$$F_n(E) := (P) \int_E Je_n \, d\mu$$

for each $n \in \mathbb{N}$, so that $F_n = (S \circ J) e_n$ for each $n \in \mathbb{N}$.

Since the series $\sum_{n=1}^{\infty} F_n$ in $bvca(\Sigma, X)$ is weakly unconditionally Cauchy (wuC), there is $C > 0$ such that $\left|\sum_{i=1}^{n} \varepsilon_i F_i\right| < C$ for all finite set of signs ε_i. Using the fact that S is a linear map onto its range, then

$$\sum_{i=1}^{n} \varepsilon_i F_i = \sum_{i=1}^{n} \varepsilon_i Sf_i = S\left(\sum_{i=1}^{n} \varepsilon_i f_i\right) \tag{2.2}$$

for each $n \in \mathbb{N}$. Then Eqs. (2.1) and (2.2) provide

$$\int_\Omega \left\| \sum_{i=1}^{n} \varepsilon_i f_i(\omega) \right\| d\mu(\omega) = \left| \sum_{i=1}^{n} \varepsilon_i F_i \right| < C \tag{2.3}$$

for each $\varepsilon_i \in \{-1, 1\}$, $1 \le i \le n$ and $n \in \mathbb{N}$.

The rest of the proof follows a similar argument to that of [8, Theorem 2.2], but working with norms instead of seminorms by properly using Rosenthal's disjointification lemma [5] and Bourgain averaging theorem [1] in order to provide a basic sequence in X equivalent to the unit vector basis of c_0. □

With the aim of providing some example concerning the Theorem 2.1 hypothesis, let us recall that for a dual Banach space X^*, the complementarity of X^* in X^{***} guarantees that X^* has the W**RNP if and only if X^* has the WRNP [14].

*Example 2.1 A Banach space with a norming sequence that fails to have the W** RNP and is not WCG*. The dual unit vector sequence $\left(e_n^*\right)_n$ of ℓ_∞^* is norming for ℓ_∞, the latter failing to have the WRNP, see [16, Example 5.13]. The space ℓ_∞ is not WCG either as every weakly compact set in ℓ_∞ is norm separable.

This given example contains a copy of c_0 and so does $\mathcal{M}(\mu, \ell_\infty)$ under this note's setting. Next example does not contain a copy of c_0 and keeps all the other characteristics of the aforementioned example. It is worth while recalling that according to [2, Corollary 6.8] a normed space X has got a norming sequence if and only if X is isometric to a subspace of ℓ_∞, and, according to [16], every dual Banach space X^* failing the WRNP, consequently the W**RNP, must enjoy ℓ_∞ as a quotient or, equivalently, X must contain a copy of ℓ_1.

*Example 2.2 A Banach space, not containing a copy of c_0, with a norming sequence that fails to have the W**RNP and is not WCG*. Let X be the Banach space $\mathcal{C}[0, 1]$ of real-valued continuous functions defined on the closed unit interval $[0, 1]$ of $\mathbb{R}$ equipped with the supremum norm. Since $\mathcal{C}[0, 1]$ is separable, X^* is topologically isomorphic to a (weakly closed) linear subspace of ℓ_∞ , see [12, 22.4.(4)]. Hence, the isomorphic copy E of X^* in ℓ_∞ has a norming sequence. On the other hand, since the compact set $[0, 1]$ is not scattered, a classic result of Pełczyńsky and Semadeni (see [3, Theorem 3.1.1]) guarantees that X has a copy of ℓ_1. According to the previous considerations, this fact ensures that the dual Banach space E does not have the W ** RNP. Furthermore, a result of Saab and Saab [15], (see [3, Theorem 3.1.4]) prevents any copy of ℓ_1 in X to be complemented, which implies thatX^*, hence E, does not contain a copy of c_0, see [4, Chapter V, Theorem 10]. Finally, since E has a quotient isomorphic to the non WCG Banach space ℓ_∞ and the class of WCG Banach spaces is stable under quotients, it follows that E is not WCG. Consequently the space E fulfills the stated properties. As a consequence of Theorem 2.1, $\mathcal{M}(\mu, E)$ does not contain a copy of c_0.

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