RACSAM (2015) 109:603–621
DOI 10.1007/s13398-014-0204-4

ORIGINAL PAPER

Existence and data dependence of the fixed points of generalized contraction mappings with applications

Mujahid Abbas · Basit Ali · Asma Rashid Butt

Received: 12 November 2013 / Accepted: 6 November 2014 / Published online: 14 December 2014

Abstract The aim of this paper is to introduce a new type of generalized multivalued contraction mappings and to present some results regarding fixed points of new class of multivalued contractions. As applications we obtain some basic results in fixed point theory like characterization of metric completeness, data dependence of fixed points and homotopy result. We prove the existence and uniqueness of bounded solution of functional equation arising in dynamic programming. Our results generalize, extend and unify various comparable results in the existing literature.

Keywords Metric space · Multivalued mapping · Fixed point · Data dependence

Mathematics Subject Classification 47H10 · 47H04 · 47H07

1 Introduction and preliminaries

The Hausdorff metric H induced by the metric d of X is given by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}$$

M. Abbas · B. Ali (✉)
Department of Mathematics and Applied Mathematics,
University of Pretoria, Lynnwood road, Pretoria 0002, South Africa
e-mail: basit.aa@gmail.com

M. Abbas
e-mail: mujahid.abbas@up.ac.za

A. Rashid Butt
Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan

A. Rashid Butt
Department of Mathematics, Brock University, St. Catharines, ON, Canada
e-mail: asma.rbutt@gmail.com

for every $A, B \in CB(X)$, where $CB(X)$ denotes the collection of closed and bounded subsets of X. It is well known that if (X, d) is a complete metric space, then the pair $(CB(X), H)$ is a complete metric space. In 1969, Nadler [16] obtained the following multivalued version of Banach contraction principle.

Theorem 1.1 *Let (X, d) be a complete metric space and $T : X \longrightarrow CB(X)$ a multivalued mapping such that*

$$H(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$ and for some $k \in (0, 1)$. Then there exists a fixed point $x \in X$ of T, i.e., $x \in Tx$.

A number of fixed point theorems (see [5,6,8,9,12,14,19,21]) have been proved in the context of generalization of Theorem 1.1. Kikkawa and Suzuki [13] refined Nadler's result by proving the following result.

Theorem 1.2 *Let (X, d) be a complete metric space and $T : X \to CB(X)$ a multivalued mapping. Define the mapping $\beta : [0, 1) \to (\frac{1}{2}, 1]$ by $\beta(b) = \dfrac{1}{1+b}$. If there exists a $b \in [0, 1)$ such that*

$$\beta(b)d(x, Tx) \leq d(x, y) \text{ implies } H(Tx, Ty) \leq bd(x, y)$$

for all $x, y \in X$. Then T has a fixed point. In this case, we call T as b-KS multivalued operator.

Theorem 1.2 has further been generalized in [7,10,11,15,23].

Definition 1.3 [20] Let (X, d) be a metric space. A mapping $T : X \to CB(X)$ is called a *multivalued weakly Picard* operator (MWP operator), if for all $x \in X$ and $y \in Tx$, there exists a sequence $\{x_n\}_{n\geq 0}$ satisfying (a) $x_0 = x$, $x_1 = y$ (b) $x_{n+1} \in Tx_n$ for all $n \geq 0$ (c) the sequence $\{x_n\}_{n\geq 0}$ converges to a fixed point of T.

The sequence $\{x_n\}$ satisfying (a) and (b) is called a sequence of successive approximations (briefly s.s.a.) of T starting from x_0.

Let (X, d) be a metric space and $T : X \longrightarrow CB(X)$ a multivalued mapping. We define

$$M_T(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \tag{1}$$

for all $x, y \in X$.

Recently Popescu [18] introduced the following class of multivalued operators.

Definition 1.4 [18] Let (X, d) be a complete metric space. A mapping $T : X \longrightarrow CB(X)$ is called an (s, r)-contractive multivalued operator if $r \in [0, 1)$, $s \geq r$ and $x, y \in X$ with $d(y, Tx) \leq sd(y, x)$ implies $H(Tx, Ty) \leq rM_T(x, y)$.

Theorem 1.5 [18] *Let (X, d) be a complete metric space and $T : X \longrightarrow CB(X)$ an (s, r)-contractive multivalued operator with $s > r$. Then T is a MWP operator.*

In this paper, we introduce a new type of generalized multivalued contraction in metric spaces. As a result we generalize results given in [5,13,15,16,18].

2 Main results

Let $\psi : [0, 1) \to (0, \frac{1}{2}]$ be a strictly decreasing mapping defined by

$$\psi(s) = \begin{cases} \frac{1}{2(1+s)} & \text{if } 0 \le s < \frac{1}{2} \\ \frac{1-s}{2} & \text{if } \frac{1}{2} \le s < 1 \end{cases}. \tag{2}$$

We define (ψ, r)-contractive multivalued operators as follows:

Definition 2.1 Let (X, d) be a metric space. A mapping $T : X \to CB(X)$ is said to be a (ψ, r)-contractive multivalued operator if $r \in [0, 1)$, $s \ge r$ and $x, y \in X$ with

$$\psi(s)(d(x, Tx) + d(y, Tx)) \le d(x, y) \tag{3}$$

implies

$$H(Tx, Ty) \le rM_T(x, y). \tag{4}$$

Theorem 2.2 *Let (X, d) be a complete metric space and $T : X \longrightarrow CB(X)$ a (ψ, r)-contractive multivalued operator. Then T is a MWP operator and has a fixed point.*

Proof Let r_1 be a real number such that $0 \le r < r_1 < 1$ and $r_1 \le s$. Let u_1 be a given point in X. We can arbitrary choose $u_2 \in Tu_1$. If $h = \frac{1}{\sqrt{r}}$, then there exists $u_3 \in Tu_2$ such that $d(u_2, u_3) \le \frac{1}{\sqrt{r}} H(Tu_1, Tu_2)$. As $\psi(s) \le 1$, so we have

$$\begin{aligned} \psi(s)(d(u_1, Tu_1) + d(u_2, Tu_1)) &\le d(u_1, Tu_1) + d(u_2, Tu_1) \\ &\le d(u_1, Tu_1) \le d(u_1, u_2), \end{aligned}$$

which implies that $\psi(s)(d(u_1, Tu_1) + d(u_2, Tu_1)) \le d(u_1, u_2)$. Now by (4), we have

$$\begin{aligned} d(u_2, u_3) &\le \frac{1}{\sqrt{r}} H(Tu_1, Tu_2) \le r\frac{1}{\sqrt{r}} M_T(u_1, u_2) \\ &= \sqrt{r} \max\left\{ d(u_1, u_2), d(u_1, Tu_1), d(u_2, Tu_2), \frac{d(u_1, Tu_2) + d(u_2, Tu_1)}{2} \right\} \\ &\le \sqrt{r} \max\left\{ d(u_1, u_2), d(u_2, u_3), \frac{d(u_1, u_2) + d(u_2, u_3)}{2} \right\}. \end{aligned}$$

Thus

$$d(u_2, u_3) \le \sqrt{r} \max\{d(u_1, u_2), d(u_2, u_3)\}.$$

If $\max\{d(u_1, u_2), d(u_2, u_3)\} = d(u_1, u_2)$, then we have $d(u_2, u_3) \le \sqrt{r} d(u_1, u_2)$. If $\max\{d(u_1, u_2), d(u_2, u_3)\} = d(u_2, u_3)$, then we get $d(u_2, u_3) \le \sqrt{r} d(u_2, u_3)$ which implies that $d(u_2, u_3) = 0$, that is, $u_2 = u_3 \in Tu_2$. Hence the result follows. So we assume that $\max\{d(u_1, u_2), d(u_2, u_3)\} = d(u_1, u_2)$. Thus

$$d(u_2, u_3) \le \sqrt{r} d(u_1, u_2) \le \sqrt{r_1} d(u_1, u_2).$$

By continuing this way, we can obtain a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$, we have

$$d(u_n, u_{n+1}) \le (\sqrt{r_1})^{n-1} d(u_1, u_2), \tag{5}$$

which implies that $\lim_{n\to\infty} d(u_n, u_{n+1}) = 0$. Now we show that $\{u_n\}$ is a Cauchy sequence. For a positive integer p, we have

$$\begin{aligned} d(u_n, u_{n+p}) &\leq (d(u_n, u_{n+1}) + \ldots + d(u_{n+p-1}, u_{n+p})) \\ &\leq ((\sqrt{r_1})^{n-1} d(u_1, u_2) + \ldots + (\sqrt{r_1})^{n+p-2} d(u_1, u_2)) \\ &\leq (\sqrt{r_1})^{n-1} \frac{1}{1-\sqrt{r_1}} d(u_1, u_2), \end{aligned}$$

which on taking limit as n tends to infinity implies that

$$\lim_{n\to\infty} d(u_n, u_{n+p}) = 0. \tag{6}$$

Therefore $\{u_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists an element $z \in X$ such that $\lim_{n\to\infty} u_n = z$, that is, $\lim_{n\to\infty} d(u_n, z) = 0$. Next we show that

$$d(z, Tx) \leq r \max\{d(z, x), d(x, Tx)\} \tag{7}$$

for all $x \neq z$. As $\lim_{n\to\infty} d(u_n, z) = 0$, so there exists a positive integer n_0 such that $d(z, u_n) < \frac{1}{9} d(z, x)$ for all $n \geq n_0$. Using $u_{n+1} \in Tu_n$, we obtain

$$\begin{aligned} 2\psi(s)(d(u_n, Tu_n) + d(x, Tu_n)) &\leq d(u_n, Tu_n) + d(x, Tu_n) \leq d(u_n, u_{n+1}) + d(x, u_{n+1}) \\ &\leq d(u_n, z) + d(z, u_{n+1}) + d(x, z) + d(z, u_{n+1}) \leq \frac{4}{3} d(z, x) \\ &= 2[d(z, x) - \frac{1}{3} d(z, x)] \leq 2[d(z, x) - \frac{1}{9} d(z, x)] \\ &\leq 2[d(z, x) - d(u_n, z)] \leq 2d(u_n, x). \end{aligned}$$

So for any $n \geq n_0$,

$$\psi(s)(d(u_n, Tu_n) + d(x, Tu_n)) \leq d(u_n, x).$$

Also from (4), we have

$$\begin{aligned} d(u_{n+1}, Tx) &\leq H(Tu_n, Tx) \\ &\leq r \max\left\{d(u_n, x), d(u_n, Tu_n), d(x, Tx), \frac{d(u_n, Tx) + d(x, Tu_n)}{2}\right\} \\ &\leq r \max\left\{d(u_n, x), d(u_n, u_{n+1}), d(x, Tx), \frac{d(u_n, Tx) + d(x, u_{n+1})}{2}\right\}. \end{aligned}$$

On taking limit as $n \to \infty$ on both sides of above inequality, we have

$$d(z, Tx) \leq r \max\left\{d(z, x), d(x, Tx), \frac{d(z, Tx) + d(x, z)}{2}\right\}.$$

Now we claim that

$$d(z, Tx) \leq r \max\{d(z, x), d(x, Tx)\}$$

holds for all $x \neq z$. Indeed, if we suppose that

$$\max\left\{d(z, x), d(x, Tx), \frac{d(z, Tx) + d(x, z)}{2}\right\} = \frac{d(z, Tx) + d(x, z)}{2},$$

then we have $d(z, Tx) \le r\frac{d(z,Tx)+d(x,z)}{2}$. As $r < 1$, so we have $d(z, Tx) \le \frac{2r}{2-r}d(x, z) < rd(x, z) \le r \max\{d(z, x), d(x, Tx)\}$. Thus

$$d(z, Tx) \le r \max\{d(z, x), d(x, Tx)\}$$

holds for all $x \ne z$. If $x = z$ then $d(z, Tz) \le r \max\{d(z, z), d(z, Tz)\}$ implies that $d(z, Tz) = 0$, that is, $z \in Tz$. Now we prove that $z \in Tz$, given that

$$d(z, Tx) \le r \max\{d(z, x), d(x, Tx)\}$$

holds for all $x \ne z$. For this we consider the case for $0 \le r \le s < 1/2$. Assume on contrary that $z \notin Tz$, we can choose $a \in Tz$ such that

$$d(a, z) < d(z, Tz) + \left(\frac{1}{2r} - 1\right) d(z, Tz)$$

that is

$$2rd(a, z) < d(z, Tz). \tag{8}$$

As $a \in Tz$ and $z \notin Tz$, so $a \ne z$, and hence we have

$$\psi(s)(d(z, Tz) + d(a, Tz)) \le d(z, Tz) \le d(z, a).$$

Thus

$$\psi(s)(d(z, Tz) + d(a, Tz)) \le d(z, a).$$

By (4), we have

$$\begin{aligned} H(Tz, Ta) &\le r \max\left\{d(z, a), d(z, Tz), d(a, Ta), \frac{d(z, Ta) + d(a, Tz)}{2}\right\} \\ &\le r \max\left\{d(z, a), d(a, Ta), \frac{d(z, a) + d(a, Ta)}{2}\right\} \\ &= r \max\{d(z, a), d(a, Ta)\}. \end{aligned} \tag{9}$$

Clearly, $d(a, Ta) \le H(Tz, Ta)$. By (9), we obtain $H(Tz, Ta) \le r \max \{d(z, a), H(Tz, Ta)\}$. Now $r < 1$ implies that

$$H(Tz, Ta) \le rd(z, a). \tag{10}$$

Hence $d(a, Ta) \le d(z, a)$. Now by (7), (9) and (10), we have

$$\begin{aligned} d(z, Tz) &\le d(z, Ta) + H(Tz, Ta) \le r \max\{d(z, a), d(a, Ta)\} + rd(z, a) \\ &= rd(z, a) + rd(z, a) = 2rd(z, a) < d(z, Tz), \end{aligned}$$

a contradiction. Hence $z \in Tz$. If $\frac{1}{2} \le r \le s < 1$ and $r \le s$, then first we show that

$$H(Tx, Tz) \le r \max\left\{d(x, z), d(x, Tx), d(z, Tz), \frac{d(x, Tz) + d(z, Tx)}{2}\right\} \tag{11}$$

for all $x \in X$ with $x \ne z$. Now for each $n \in N$, there exists $y_n \in Tx$ such that

$$d(z, y_n) < d(z, Tx) + \frac{1}{n}d(x, z).$$

So we have

$$\begin{aligned} d(x, Tx) + d(z, Tx) &\le d(x, y_n) + d(z, Tx) \le d(x, z) + d(z, y_n) + d(z, Tx) \\ &< d(x, z) + 2d(z, Tx) + \frac{1}{n} d(x, z). \end{aligned}$$

Hence by (7), we have

$$d(x, Tx) + d(z, Tx) < d(x, z) + 2r \max\{d(z, x), d(x, Tx)\} + \frac{1}{n} d(x, z). \tag{12}$$

If $\max\{d(z, x), d(x, Tx)\} = d(x, z)$, then by (12), we have

$$\begin{aligned} d(x, Tx) + d(z, Tx) &< d(x, z) + 2rd(z, x) + \frac{1}{n} d(x, z) \\ &< \left((1 + 2r) + \frac{1}{n}\right) d(x, z) \le \left((1 + 2s) + \frac{1}{n}\right) d(x, z), \end{aligned}$$

which implies that

$$\begin{aligned} \psi(s)(d(x, Tx) + d(z, Tx)) &= \frac{1 - s}{2}(d(x, Tx) + d(z, Tx)) \\ &\le \frac{1}{1 + 2s}(d(x, Tx) + d(z, Tx)) < \left(1 + \frac{1}{(1 + 2s)n}\right) d(x, z). \end{aligned}$$

On taking limit as n tends to ∞, we obtain that

$$\psi(s)[d(x, Tx) + d(z, Tx)] \le d(x, z).$$

Now by (4) with $y = z$, we get (11). If $\max\{d(z, x), d(x, Tx)\} = d(x, Tx)$, then by (8), we have

$$d(x, Tx) \le d(x, z) + d(z, Tx) \le d(x, z) + rd(x, Tx).$$

Hence

$$d(x, Tx) \le \frac{1}{1 - r} d(x, z).$$

Now by (12), we have

$$\begin{aligned} d(x, Tx) + d(z, Tx) &\le d(x, z) + 2rd(x, Tx) + \frac{1}{n} d(x, z) \\ &\le d(x, z) + \frac{2r}{1 - r} d(x, z) + \frac{1}{n} d(x, z) \le \frac{2}{1 - r} d(x, z) + \frac{1}{n} d(x, z). \end{aligned}$$

As $\frac{1}{2} \le r < 1$ and $r \le s$, so we have

$$\begin{aligned} \psi(s)(d(x, Tx) + d(z, Tx)) &= \frac{1 - s}{2}(d(x, Tx) + d(z, Tx)) \\ &\le \frac{1 - r}{2}(d(x, Tx) + d(z, Tx)) \le d(z, x) + \frac{1 - r}{2n} d(z, x), \end{aligned}$$

which on taking limit as n tends to ∞ gives that

$$\psi(s)(d(x, Tx) + d(z, Tx)) \le d(x, z).$$

We get (11). Now by (11) with $x = u_n$ and $y = z$, we have

$$\begin{aligned} d(u_{n+1}, Tz) &\leq H(Tu_n, Tz) \\ &\leq r \max\left\{ d(u_n, z), d(u_n, Tu_n), d(z, Tz), \frac{d(u_n, Tz) + d(z, Tu_n)}{2} \right\} \\ &\leq r \max\left\{ d(u_n, z),\ d(u_n, u_{n+1}), d(z, Tz), \frac{d(u_n, Tz) + d(z, u_{n+1})}{2} \right\}, \end{aligned}$$

which on taking limit as n tends to ∞ implies that

$$d(z, Tz) \leq r d(z, Tz).$$

As $r < 1$, so we have $d(z, Tz) = 0$, that is, $z \in Tz$. □

Remark 2.3 Let (X, d) be a complete metric space and $T : X \longrightarrow CB(X)$. We show that every (s, r)-contractive multivalued operator is (ψ, r)-contractive multivalued operators. We consider the case when $0 \leq r \leq s < \frac{1}{2}$. If $d(y, Tx) \leq sd(y, x)$ then we have

$$d(x, Tx) - d(y, x) \leq d(y, Tx) \leq sd(y, x),$$

which implies that

$$d(x, Tx) \leq (1 + s)d(y, x), \tag{13}$$

that is $\frac{1}{1+s} d(x, Tx) \leq d(y, x)$. As $\frac{1}{1+s} \leq 1$ and $\psi(s) \leq \frac{1}{2}$, so we have

$$2\psi(s)(d(x, Tx) + d(y, Tx)) \leq d(x, Tx) + d(y, Tx) \leq (1 + 2s)d(y, x) \leq 2d(x, y).$$

Hence

$$\psi(s)(d(x, Tx) + d(y, Tx)) \leq d(x, y).$$

If $\frac{1}{2} \leq r \leq s < 1$, then $1 - s \leq \frac{1}{2}$ and $\frac{1}{1+s} < 1$. Then we have

$$\begin{aligned} 2\psi(s)(d(x, Tx) + d(y, Tx)) &= (1 - s)(d(x, Tx) + d(y, Tx)) \\ &\leq \frac{1}{2} d(x, Tx) + \frac{1}{2} d(y, Tx) \leq \frac{1+s}{2} d(x, y) + \frac{s}{2} d(y, x) \\ &\leq \frac{1+2s}{2} d(x, y) \leq \frac{3}{2} d(x, y) \leq 2d(x, y). \end{aligned}$$

Thus

$$\psi(s)(d(x, Tx) + d(y, Tx)) \leq d(x, y).$$

Remark 2.4 Theorem 2.2 extends and generalizes results in [5,13,15,16,18].

Example 2.5 Let $X = \{0, 1, 2\}$ and d be the metric on X defined by:

$$d(0, 0) = d(1, 1) = d(2, 2) = 0,\ d(0, 1) = d(1, 0) = \frac{1}{4},$$
$$d(0, 2) = d(2, 0) = \frac{1}{3},\ d(2, 1) = d(1, 2) = \frac{1}{2}.$$

Define the mapping $T : X \longrightarrow CB(X)$ by

$$Tx = \begin{cases} \{0\}, & \text{when } x \neq 2 \\ \{0, 1\}, & \text{when } x = 2 \end{cases}.$$

Note that, for all $x, y \in X$, and any $s \in [0, 1)$, we have

$$\psi(s)(d(x, Tx) + d(y, Tx)) \leq d(x, y).$$

If $s = \frac{4}{5} > \frac{3}{4} = r$, then $\psi(s) = \frac{1}{10}$. Note that

$$H(Tx, Ty) \leq rM_T(x, y)$$

is satisfied for all $x, y \in X$. Thus, all the conditions of Theorem 2.24 are satisfied.

Example 2.6 Let $X = [0, 10]$ be a usual metric space. Define $T : X \to CB(X)$, where $Tx = [0, ke^{-\frac{1}{2}}x^2 + 1)]$, where $k \in (0, \frac{1}{20})$. Fix $x, y \in X$ such that $\psi(s)(d(x, Tx) + d(y, Tx)) \leq d(x, y)$. Note that

$$\begin{aligned} H(Tx, Ty) &= ke^{-\frac{1}{2}} \left|x^2 - y^2\right| = ke^{-\frac{1}{2}} |x - y| |x + y| \leq 20ke^{-\frac{1}{2}} \left|x^2 - y^2\right| \\ &\leq e^{-\frac{1}{2}} |x - y| = e^{-\frac{1}{2}} d(fx, fy) \leq rM_T(x, y) \end{aligned}$$

for all $x, y \in X$, where $M_T(x, y)$ is defined in (1) and $r = e^{-\frac{1}{2}}$. Then for any $0 < r < s < 1$ T is (ψ, r)-contractive multivalued mapping. Note that every $x \leq 10\sqrt{e} - 2e\sqrt{5(5e - \sqrt{e})}$ is such that $x \in Tx$.

Corollary 2.7 *Let (X, d) be a complete metric space and $T : X \longrightarrow CB(X)$ a multivalued mapping. Let ψ be the same as defined in Theorem 2.2 and $\psi_1(s) = \frac{\psi(s)}{2}$. If there exist $0 \leq r \leq s < 1$ such that*

$$\begin{aligned} &\psi_1(s)(d(x, Tx) + d(y, Tx)) \leq d(x, y) \text{ implies that} \\ &\qquad H(Tx, Ty) \leq rM_T(x, y) \end{aligned} \tag{14}$$

for all $x, y \in X$ whenever $x \neq y$. Then T has a fixed point.

Corollary 2.8 *Let (X, d) be a complete metric space and $T : X \longrightarrow CB(X)$ a multivalued mapping. Let ψ be the same as defined in Theorem 2.2 and $\psi_1(s) = \frac{\psi(s)}{2}$. If there exist $0 \leq r \leq s < 1$ such that*

$$\begin{aligned} &\psi_1(s)(d(x, Tx) + d(y, Tx)) \leq d(x, y) \text{ implies} \\ &\qquad H(Tx, Ty) \leq r \max\{d(x, y), d(x, Tx), d(y, Ty)\} \end{aligned}$$

for all $x, y \in X$ whenever $x \neq y$. Then T has a fixed point.

Remark 2.9 Let (X, d) be a complete metric space and $T : X \longrightarrow CB(X)$ a multivalued mapping. Let ψ be the same as defined in Theorem 2.2 and $\psi_1(s) = \frac{\psi(s)}{2}$. Suppose that there exists $0 \leq r \leq s < 1$ satisfying

$$\frac{1}{1 + r}d(x, Tx) \leq d(x, y) \leq \frac{1}{1 - s}d(x, Tx) \text{ implies} \tag{15}$$

$$H(Tx, Ty) \leq r \max\{d(x, y), d(x, Tx), d(y, Ty)\}. \tag{16}$$

Above contraction condition [18, Theorem 2.7] was employed to prove the existence of fixed points of T. Now if $0 \leq r \leq s < \frac{1}{2}$, then $4\psi_1(s) < 1$ and we have

$$\begin{aligned} 4\psi_1(s)(d(x,Tx)+d(y,Tx)) &\leq \frac{1}{1+s}d(x,Tx)+\frac{1}{1+s}d(y,Tx) \\ &\leq \frac{2}{1+s}d(x,Tx)+\frac{1}{1+s}d(y,x) \\ &\leq \frac{2(1+r)}{1+s}d(x,y)+\frac{1}{1+s}d(y,x) \\ &\leq 4d(x,y). \end{aligned}$$

Thus

$$\psi_1(s)(d(x,Tx)+d(y,Tx)) \leq d(x,y).$$

When $\frac{1}{2} \leq r \leq s < 1$. Then

$$\begin{aligned} 4\psi_1(s)(d(x,Tx)+d(y,Tx)) &\leq (1-s)d(x,Tx)+(1-s)d(y,Tx) \\ &\leq 2(1-s)d(x,Tx)+(1-s)d(y,x) \\ &\leq 2(1-s)d(x,Tx)+d(x,Tx) \\ &\leq (3-2s)d(x,Tx) \leq 2(1+r)d(x,y) \leq 4d(x,y). \end{aligned}$$

Hence we obtain

$$\psi_1(s)(d(x,Tx)+d(y,Tx)) \leq d(x,y).$$

Corollary 2.8 can be viewed as a generalization of results in [18, Theorem 2.7] which in turn generalize the results in [13, Theorem 1.6].

Corollary 2.10 *Let (X,d) be a complete metric space and $T : X \longrightarrow CB(X)$ a multivalued mapping. Let ψ be the same as defined in Theorem 2.2 and $\psi_1(s) = \frac{\psi(s)}{2}$. If there exist $0 \leq r \leq s < 1$ and $\alpha \in [0, \frac{1}{3})$ such that*

$$\begin{aligned} &\psi_1(s)(d(x,Tx)+d(y,Tx)) \leq d(x,y) \text{ implies} \\ &\qquad H(Tx,Ty) \leq \alpha[d(x,y)+d(x,Tx)+d(y,Ty)] \end{aligned}$$

for all $x, y \in X$ whenever $x \neq y$ and $r = 3\alpha$. Then T has a fixed point.

For single valued mappings, Theorem 2.2 reduces to the following corollary:

Corollary 2.11 *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ a single valued mapping. Let $\psi(s)$ be given as in Theorem 2.2. If there exist $0 \leq r \leq s < 1$ such that*

$$\begin{aligned} &\psi_1(s)(d(x,Tx)+d(y,Tx)) \leq d(x,y) \text{ implies} \\ &d(Tx,Ty) \leq r \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} \end{aligned}$$

for all $x, y \in X$ whenever $x \neq y$. Then T has a unique fixed point.

Proof Existence of fixed point follows from Theorem 2.2. We prove the uniqueness. If there exist $z_1 \neq z_2$ such that $z_1 = Tz_1$ and $z_2 = Tz_2$. Then

$$\begin{aligned} \psi(s)(d(z_1,Tz_1)+d(z_2,Tz_1)) &\leq d(z_1,Tz_1)+d(z_2,Tz_1) \\ &= d(z_1,z_1)+d(z_2,z_1) \leq d(z_1,z_2), \end{aligned}$$

which implies that

$$\psi(s)(d(z_1, Tz_1) + d(z_2, Tz_1)) \leq d(z_1, z_2).$$

It follows that

$$\begin{aligned} d(z_1, z_2) &= d(Tz_1, Tz_2) \\ &\leq r \max\{d(z_1, z_2), d(z_1, Tz_1), d(z_2, Tz_2), \frac{d(z_2, Tz_1) + d(z_1, Tz_2)}{2}\} \\ &\leq r \max\{d(z_1, z_2), d(z_1, z_1), d(z_2, z_2)\} \leq rd(z_1, z_2). \end{aligned}$$

Hence $d(z_1, z_2) = 0$, that is, $z_1 = z_2$. □

3 Characterization of metric completeness for multivalued mappings

Motivated by the work of Suzuki [24] we prove the characterization of metric space completeness for the class of (ψ, r)-contractive multivalued mappings.

Theorem 3.1 *Let (X, d) be a metric space then the following statements are equivalent:*

(a) (X, d) is complete;
(b) For each $r \in [0, 1)$ and $s \geq r$, every mapping $T : X \longrightarrow CB(X)$ such that $\psi(s)((d(x, Tx) + d(y, Tx)) \leq d(x, y)$ implies

$$H(Tx, Ty) \leq rM_T(x, y) \tag{17}$$

for all $x, y \in X$ has a fixed point.

Proof By Theorem (2.1) $(a) \Rightarrow (b)$. Now we prove that $(b) \Rightarrow (a)$. Suppose on contrary that (X, d) is not complete. That is there exists a Cauchy sequence $\{u_n\}$ which does not converge. Define a function $f : X \to [0, \infty)$ by $f(x) = \lim_{n\to\infty} d(x, u_n)$ for $x \in X$. Since $f(x) > 0$ and $\lim_{n\to\infty} f(u_n) = 0$ therefore for every $x \in X$ there exists $\upsilon \in \mathbb{N}$ such that $f(u_\upsilon) \leq \frac{\psi(s)r}{4+r+\psi(s)r} f(x)$. We put $T(x) = \{u_n : f(u_n) \leq \frac{\psi(s)r}{4+r+\psi(s)r} f(x)\}$. Define $g(x) = \sup_{y\in Tx} f(y)$, then $g(x) \leq \frac{\psi(s)r}{4+r+\psi(s)r} f(x)$ for all $x \in X$. Since $f(y) < f(x)$ for all $y \in Tx$, therefore T has no fixed point. By the definition of mapping f we have

$$f(x) - f(y) \leq d(x, y) \leq f(x) + f(y) \text{ for all } y \in Tx, \tag{18}$$

$$f(y) - f(x) \leq d(x, y) \leq f(x) + f(y) \text{ for all } y \in Tx. \tag{19}$$

This implies

$$f(x) - g(x) \leq d(x, Tx) \leq f(x) + g(x), \tag{20}$$

$$H(Tx, Ty) \leq g(x) + g(y). \tag{21}$$

Now fix $x, y \in X$ such that $\psi(s)((d(x, Tx) + d(y, Tx)) \leq d(x, y)$, we need to show that 17 holds. Observe that

$$\begin{cases} d(x, y) \geq \psi(s)((d(x, Tx) + d(y, Tx)) \geq \psi(s)(d(x, Tx) \\ \geq \psi(s)f(x) - g(x) \geq \psi(s)\left(1 - \frac{\psi(s)r}{4+r+\psi(s)r}\right) f(x) = \left(\frac{(4+r)\psi(s)}{4+r+\psi(s)r}\right) f(x). \end{cases} \tag{22}$$

Case (1) when $f(y) \geq f(x)$, then by

$$\begin{aligned}
H(Tx, Ty) &\leq g(x) + g(y) = \frac{4+r}{4}(g(x)+g(y)) - \frac{r}{4}(g(x)+g(y)) \\
&\leq \frac{4+r}{4}\frac{\psi(s)r}{4+r+\psi(s)r}(f(x)+f(y)) - \frac{r}{4}(g(x)+g(y)) + \frac{r}{4}(f(y)-f(x)) \\
&\leq \frac{r}{4}\frac{(4+r)\psi(s)}{4+r+\psi(s)r}(f(x)+f(y)) - \frac{r}{4}(g(x)+g(y)) \\
&\quad + \frac{r}{4}(f(y)-f(x)) + \frac{r}{4}(f(x)-f(y)) \\
&\leq \frac{r}{4}(f(x)+f(y)) - \frac{r}{4}(g(x)+g(y)) + \frac{r}{4}(f(y)-f(x)) + \frac{r}{4}(f(x)-f(y)) \\
&\leq \frac{r}{4}d(x, Tx) + \frac{r}{4}d(y, Ty) + \frac{r}{4}d(x, y) + \frac{r}{4}d(x, y) \\
&\leq \frac{r}{4}(4M_T(x, y)) = rM_T(x, y).
\end{aligned}$$

Case (2) when $f(y) < f(x)$, then

$$\begin{aligned}
H(Tx, Ty) &\leq g(x) + g(y) \leq \frac{\psi(s)r}{4+r+\psi(s)r}(f(x)+f(y)) \\
&= \frac{\psi(s)r}{4+r+\psi(s)r}f(x) + \frac{\psi(s)r}{4+r+\psi(s)r}f(y) \\
&\leq \frac{\psi(s)r}{4+r+\psi(s)r}f(x) + \frac{\psi(s)r}{4+r+\psi(s)r}f(x) \\
&\leq \frac{\psi(s)r}{4+r}d(x, Tx) + \frac{r}{4+r}d(x, y) \leq \frac{r}{4}d(x, Tx) + \frac{r}{4}d(x, y) \\
&\leq \frac{r}{4}(2M_T(x, y)) \leq rM_T(x, y).
\end{aligned}$$

Hence $\psi(s)((d(x, Tx) + d(y, Tx)) \leq d(x, y)$ implies

$$H(Tx, Ty) \leq rM_T(x, y)$$

for all $x, y \in X$. this implies that T has a fixed point, a contradiction. Hence X is complete and consequently $(b) \Rightarrow (a)$. □

4 Data dependence of the fixed point set

Let (X, d) be a metric space and and $T : X \longrightarrow P(X)$ (the collection of all the subsets of X) be a MWP operator. Define a multivalued operator $T^{\infty} : G(T) \rightarrow P(Fix(T))$ by

$$T^{\infty}(x, y) = \{z \in Fix(T) : \text{ there exists a sequence of successive approximations of } T \text{ starting from } (x, y) \text{ that converges to } z\}.$$

Further

$$G(T) = \{(x, y) : x \in X, y \in Tx\}$$

is called graph of multivalued mapping T. A selection for T is a single valued mapping $t : X \rightarrow X$ such that $tx \in Tx$ for all $x \in X$.

Definition 4.1 [20] Let (X, d) be a metric space and $T : X \longrightarrow P(X)$ a MWP operator. Then T is called c-multivalued weakly Picard (briefly c-MWP) operator if $c > 0$ and there exists a selection t^∞ of T^∞ such that

$$d(x, t^\infty(x, y)) \leq cd(x, y) \tag{23}$$

for all $(x, y) \in G(T)$.

One of the main result concerning c-MWP operators is the following:

Theorem 4.2 [20] *Let (X, d) be a metric space and $T_1, T_2 : X \to P(X)$ two multivalued operators. Suppose that:*

(i) T_i is a c_i-MWP operator for each $i \in \{1, 2\}$;
(ii) There exists $\lambda > 0$ such that $H(T_1x, T_2x) \leq \lambda$, for all $x \in X$.

Then

$$H(Fix(T_1), Fix(T_2)) \leq \lambda \max\{c_1, c_2\}.$$

Moţ and Petruşel [15] proved the following result.

Theorem 4.3 [15] *Let (X, d) be a metric space and $T_1, T_2 : X \to P(X)$ two multivalued operators. If*

(i) T_i is a b_i-KS multivalued operator for each $i \in \{1, 2\}$;
(ii) There exists $\lambda > 0$ such that $H(T_1x, T_2x) \leq \lambda$, for all $x \in X$.

Then:

(a) $Fix(T_i) \in CB(X)$, $i \in \{1, 2\}$;
(b) Each T_i is a MWP operator and

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\lambda}{1 - \max\{b_1, b_2\}}. \tag{24}$$

Recently Popescu [18] proved the following theorem.

Theorem 4.4 *Let (X, d) be a metric space and $T_1, T_2 : X \to P(X)$ two multivalued operators. If*

(i) T_i is an $(1, r_i)$-contractive multivalued operator for each $i \in \{1, 2\}$;
(ii) There exists $\lambda > 0$ such that $H(T_1x, T_2x) \leq \lambda$, for all $x \in X$.

Then:

(a) $Fix(T_i) \in CB(X)$, $i \in \{1, 2\}$;
(b) Each T_i is a MWP operator and

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\lambda}{1 - \max\{r_1, r_2\}}. \tag{25}$$

Now we prove the following result for (ψ, r)-contractive multivalued operators.

Theorem 4.5 *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P(X)$ two multivalued operators. If*

(i) T_i is (ψ, r_i)-contractive multivalued operators for each $i \in \{1, 2\}$;
(ii) There exists $\lambda > 0$ such that $H(T_1x, T_2x) \leq \lambda$, for all $x \in X$.

Then:

(a) $Fix(T_i) \in CB(X)$, $i \in \{1, 2\}$;
(b) Each T_i is a MWP operator and

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\lambda}{1 - \max\{r_1, r_2\}}. \quad (26)$$

Proof From Theorem 2.2, $Fix(T_i)$ is nonempty for each $i \in \{1, 2\}$. Let $x_n \in Fix(T_1)$ be such that $x_n \to z$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} d(x_n, z) = 0. \quad (27)$$

Note that

$$\psi(s)(d(x_n, T_1x_n) + d(z, T_1x_n)) = \psi(s)d(z, x_n) \leq d(z, x_n).$$

Thus

$$\begin{aligned} d(z, T_1z) &\leq d(z, x_n) + d(x_n, T_1z) \leq d(z, x_n) + H(T_1x_n, T_1z) \\ &\leq d(z, x_n) + r_1 \max\left\{p(z, x_n), p(z, T_1z), p(T_1x_n, x_n), \frac{p(z, T_1x_n) + p(x_n, T_1z)}{2}\right\} \\ &\leq d(z, x_n) + r_1 d(z, x_n). \end{aligned}$$

Taking limit as $n \to \infty$, we obtain that $d(z, T_1z) = 0$, that is, $z \in T_1z$. Hence $Fix(T_1)$ is closed. In the same way, we can prove that $Fix(T_2)$ is closed. Using arguments as in proof of the Theorem 2.2, each T_i is a MWP operator. To prove

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\lambda}{1 - \max\{r_1, r_2\}}.$$

(C_1) *A "Classical" proof:* Let $a > 1$. Then for an arbitrary $x_0 \in Fix(T_1)$, there exists $x_1 \in T_2x_0$ such that

$$d(x_0, x_1) \leq aH(T_1x_0, T_2x_0).$$

As $x_1 \in T_2x_0$, so there exists $x_2 \in T_2x_1$ such that

$$\psi(s)(d(x_0, T_2x_0) + d(x_1, T_2x_0)) \leq \psi(s)d(x_0, x_1) \leq d(x_0, x_1),$$

which implies that

$$\begin{aligned} d(x_1, x_2) &\leq aH(T_2x_0, T_2x_1) \\ &\leq ar_2 \max\left\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2}\right\} \\ &\leq ar_2 d(x_0, x_1). \end{aligned}$$

Continuing this way, we can obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in T_2x_n$ and

$$d(x_n, x_{n+1}) \leq ar_2 d(x_n, x_{n+1}) \leq \ldots \leq (ar_2)^n d(x_0, x_1).$$

Thus

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \\ &\leq (ar_2)^n d(x_0, x_1) + \ldots + (ar_2)^{n+p-1} d(x_0, x_1) \leq \frac{(ar_2)^n}{1 - ar_2} d(x_0, x_1). \quad (28) \end{aligned}$$

Chose $1 < a < \min\left\{\frac{1}{r_1}, \frac{1}{r_2}\right\}$. This implies that $\{x_n\}$ is Cauchy sequence in X. Then there exists u in X such that $x_n \to u$ as $n \to \infty$. Following arguments similar to those given in Theorem 2.2, it follows that $u \in T_2u$. By (28), we obtain that

$$d(x_n, u) \leq \frac{(ar_2)^n}{1 - ar_2} d(x_0, x_1) \tag{29}$$

Thus, in particular

$$d(x_0, u) \leq \frac{1}{1 - ar_2} d(x_0, x_1) \leq \frac{\lambda}{1 - ar_2}. \tag{30}$$

In a similar way, we conclude that for each $z_0 \in Fix(T_2)$, there is an $x \in Fix(T_1)$ such that

$$d(z_0, x) \leq \frac{1}{1 - ar_1} d(z_0, z_1) \leq \frac{\lambda}{1 - ar_1}. \tag{31}$$

By (30) and (31), we obtain that

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\lambda}{1 - \max\{ar_1, ar_2\}}.$$

Letting $a \searrow 1$ we get the conclusion.

(C_2) *Proof based on MWP operator technique:* Suppose that T is a (ψ, r)-contractive multi-valued operators. Now we show that T is c-MWP operator with $c = \frac{1}{1-r}$. Then the conclusion will follow from Theorem 4.2. Let $a > 1$, $x \in X$ and $y \in Tx$ be arbitrary chosen. By a similar approach to (C_1), we obtain a sequence of successive approximations $\{x_n\}$ starting from $(x = x_0, y = x_1) \in G(T)$ such that

$$d(x_n, x_{n+p}) \leq \frac{(ar_2)^n}{1 - ar_2} d(x_0, x_1),$$

for each $n \in \mathbb{N}$ and $p \to +\infty$ in the above estimation we get that $d(x_n, u) \leq \frac{(ar_2)^n}{1-ar_2} d(x_0, x_1)$, for each $n \in \mathbb{N}$. For $n = 0$ we obtain that $d(x, u) \leq \frac{1}{1-ar_2} d(x, y)$. Letting $a \searrow 1$ we obtain $d(x, u) \leq \frac{1}{1-r} d(x, y)$. Thus T is a $\frac{1}{1-r}$-MWP operator. □

5 Application in dynamic programming

A dynamic process consists of a state space (a set of initial states, actions and transitions) and a decision space (set of possible input and output actions). We assume U and V are Banach spaces where $W \subseteq U$ is state space and $D \subseteq V$ is decision space. Now define the mappings as

$$\tau : W \times D \longrightarrow W,\ g : W \times D \longrightarrow \mathbb{R}, G : W \times D \times \mathbb{R} \longrightarrow \mathbb{R},$$

where $\mathbb{R}$ is the field of real numbers. Dynamic programming provides tools for mathematical optimization and computer programing as well. It is well known that the problem of dynamic programming related to multistage process reduces to the problem of solving the functional equation:

$$q(x) := \sup_{y \in D}\{g(x, y) + G(x, y, q(\tau(x, y)))\},\ x \in W. \tag{32}$$

For the detailed background of the problem (see [1–4,17,22]). Let $B(W)$ be the set of all bounded real-valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x\in W} |hx|$. Then $(B(W), \|\cdot\|)$ is a Banach space endowed with the metric d defined by

$$d(h,k) = \sup_{x\in W} |hx - kx| \tag{33}$$

where $h, k \in B(W)$. Suppose that the following conditions hold:

$(DT-1)$ functions G and g are bounded.

$(DT-2)$ For $h, k \in B(W)$ and $x, z \in W$, define T by

$$T(hx) := \sup_{y\in D}\{g(x,y) + G(x,y,h(\tau(x,y)))\}. \tag{34}$$

Moreover, there exist $0 \le r \le s < 1$ such that

$$|G(x,y,hz) - G(x,y,kz)| \le rM_T(hz,kz)$$

for all $h, k \in B(W)$ and $x, z \in W$, where

$$M_T(hz,kz) = \max\left\{d(hz,kz), d(hz,Thz), d(kz,Tkz), \frac{d(hz,Tkz)+d(kz,Thz)}{2}\right\}.$$

Theorem 5.1 *If conditions* $(DT-1)$ *and* $(DT-2)$ *are satisfied, then the functional Eq.* (32) *has a unique bounded solution.*

Proof Note that $(B(W), d)$ is a complete metric space and T is a self map of $B(W)$. Let λ be an arbitrary positive number and $h_1, h_2 \in B(W)$. Choose $x \in W$ and $y_1, y_2 \in D$ such that

$$Th_1x < g(x,y_1) + G(x,y_1,h_1(\tau(x,y_1))) + \lambda, \tag{35}$$
$$Th_2x < g(x,y_2) + G(x,y_2,h_2(\tau(x,y_2))) + \lambda, \tag{36}$$
$$Th_1x \ge g(x,y_2) + G(x,y_2,h_1(\tau(x,y_2))), \tag{37}$$
$$Th_2x \ge g(x,y_1) + G(x,y_1,h_2(\tau(x,y_1))). \tag{38}$$

By (35) and (38), we have

$$\begin{aligned} Th_1x - Th_2x &< G(x,y_1,h_1(\tau(x,y_1))) - G(x,y_1,h_2(\tau(x,y_2))) + \lambda \\ &\le |G(x,y_1,h_1(\tau(x,y_1))) - G(x,y_1,h_2(\tau(x,y_2)))| + \lambda \\ &\le rM_T(h_1x,h_2x) + \lambda. \end{aligned}$$

That is

$$Th_1x - Th_2x \le rM_T(h_1x,h_2x) + \lambda. \tag{39}$$

By (36) and (37), we obtain

$$Th_2x - Th_1x \le rM_T(h_1x,h_2x) + \lambda. \tag{40}$$

Finally, by (39) and (40), we have

$$|Th_1x - Th_2x| \le rM_T(h_1x,h_2x) + \lambda, \tag{41}$$

that is

$$d(Th_1,Th_2) \le rM_T(h_1x,h_2x).$$

As above inequality is true for any $x \in W$, ψ and $\lambda > 0$, so

$$\psi(s)(d(h, Th) + d(k, Th)) \leq d(h, k) \text{ implies } d(Th_1, Th_2) \leq rM_T(h_1x, h_2x).$$

Thus all the conditions of Corollary 2.11 are satisfied for the mapping T. So functional Eq. 32 has a unique bounded solution. □

Example 5.2 Let $U = V = \mathbb{R}$, $W = [0, 20]$ and $D = [0, 10]$. Consider the functional equation

$$q(x) = \max_{y \in D} \left\{ 2x^2y + \frac{x}{30 + 30x^2y} \cos(q(\frac{x+y}{2})) \right\}, \; x \in W. \tag{42}$$

For each $h, k \in B(W)$, define the functional

$$Th(x) = \max_{y \in D} \left\{ 2x^2y + \frac{x}{30 + 30x^2y} \cos(h(\frac{x+y}{2})) \right\}, \; x \in W.$$

Suppose that $g(x, y) = 2x^2y$, $G(x, y, z) = \frac{x}{2+2x^2y} \cos(hz)$. Clearly g, G are bounded and

$$\begin{aligned} |G(x, y, hz) - G(x, y, kz)| &= \left| \frac{x}{30 + 30x^2y} \cos(hz) - \frac{x}{30 + 30x^2y} \cos(kz) \right| \\ &\leq \frac{x}{30 + 30x^2y} |\cos(hz) - \cos(kz)| \leq \frac{x}{30 + 30x^2y} |hz - kz| \\ &\leq \frac{2}{3} \sup_{z \in W} |hz - kz| = \frac{2}{3} d(h, k) \leq rM_T(h, y) \end{aligned}$$

for all $\in B(W)$, where $r = \frac{2}{3}$. Hence all the conditions of Theorem 5.1 are satisfiend and consequently functional Eq. (42) has a unique and bounded solution.

6 Homotopy result

Following is the local fixed point result for (ψ, r)-contractive multivalued mappings.

Theorem 6.1 *Let (X, d) be a complete metric space, $x_0 \in X$ and $a > 0$. Suppose that $T : B(x_0, a) \to CB(X)$ be (ψ, r)-contractive multivalued mappings and $d(x_0, Tx_0) < (1 - s)a$. Then T has a fixed point in $B(x_0, a)$.*

Proof Let $0 < a_1 < a$ be such that $\widetilde{B}(x_0, a_1) \subset B(x_0, a)$ and $d(x_0, Tx_0) < (1 - s)a_1 < (1 - s)a$. Let $x_1 \in Tx_0$ be such that $d(x_0, x_1) < (1 - s)a_1$. Then for $h = \frac{1}{\sqrt{r}} > 1$ and $x_1 \in Tx_0$ there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq hH(Tx_0, Tx_1)$$

Since $\psi(s)(d(x_0, Tx_0) + d(x_1, Tx_0)) = \psi(s)d(x_0, Tx_0) \leq \psi(s)d(x_0, x_1) \leq d(x_0, x_1)$, therefore we obtain

$$\begin{aligned} d(x_1, x_2) &\leq hH(Tx_0, Tx_1) = \frac{1}{\sqrt{r}} H(Tx_0, Tx_1) \leq \sqrt{r} M_T(x_0, x_1) \\ &\leq \sqrt{r} \max \left\{ d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_0, Tx_1) + d(Tx_0, x_1)}{2} \right\} \end{aligned}$$

$$\leq \sqrt{r}\max\left\{d(x_0,x_1), d(x_0,x_1), d(x_1,x_2), \frac{d(x_0,x_1)+d(x_1,x_2)}{2}\right\}$$
$$\leq \sqrt{r}\max\left\{d(x_0,x_1), d(x_0,x_1), d(x_1,x_2), \frac{d(x_0,x_1)+d(x_1,x_2)}{2}\right\}$$
$$\leq \sqrt{r}d(x_0,x_1) < \sqrt{r}(1-s)a_1 \leq \sqrt{r}(1-r)a_1.$$

Also, we have $x_2 \in B(x_0, a)$ because

$$d(x_0,x_2) \leq d(x_0,x_1) + d(x_1,x_2) < (1-s)a_1 + \sqrt{r}(1-r)a_1$$
$$\leq (1-r)a_1 + \sqrt{r}(1-r)a_1 = (1-r)(1+\sqrt{r})a_1.$$

In this way, we obtain inductively a sequence $(x_n)_{n\in\mathbb{N}}$ satisfying (i) $x_n \in B(x_0, a)$; for each $n \in \mathbb{N}$, (ii) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$, (iii) $d(x_n, x_{n+1}) \leq (\sqrt{r})^n(1-r)s$ for each $n \in \mathbb{N}$. From (iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy and hence, it converges to a certain $u \in B(x_0, a)$. Following similar arguments to those given in Theorem 2.2, we obtain $u \in Tu$. □

Now we present a homotopy result for (ψ, r)-contractive multivalued mappings.

Theorem 6.2 *Let (X, d) be a complete metric space and U an open subset of X. Let $G : \overline{U}\times[0,1] \to P(X)$ be a multivalued operator such that the following conditions are satisfied:*

- *h-1 $x \notin G(x,t)$, for each $x \in \partial U$ (boundary of U) and each $t \in [0,1]$;*
- *h-2 $G(.,t) : \overline{U} \to P(X)$ is a (ψ, r)-contractive multivalued mappings for each $t \in [0,1]$;*
- *h-3 there exists a continuous increasing function $\rho : [0,1] \to \mathbb{R}$ such that*

$$H(G(x,t), G(x,s)) \leq |\rho(t)-\rho(s)| \text{ for all } t,s \in [0,1] \text{ and each } x \in \overline{U};$$

- *h-4 $G : \overline{U}\times[0,1] \to P(X)$ is closed.*

Then $G(.,0)$ has a fixed point if and only if $G(.,1)$ has a fixed point.

Proof Let $G(.,0)$ has a fixed point z, then (h-1) implies that $z \in U$. Define

$$\Delta = \{(t,x) \in [0,1]\times U \mid x \in G(x,t)\}.$$

Since $(0,z) \in \Delta$ therefore $\Delta \neq \emptyset$, as. Now we define a partial order on Δ, that is

$$(t,x) \leq (s,y) \text{ if and only if } t \leq s \text{ and } d(x,y) \leq \frac{2}{1-r}[\rho(s)-\rho(t)]$$

where $0 \leq r < 1$. Let M be a totally ordered subset of Δ and $t^* := \sup\{t \mid (t,x) \in M\}$. Consider a sequence $(t_n, x_n)_{n\in\mathbb{N}} \subset M$ such that $(t_n, x_n) \leq (t_{n+1}, x_{n+1})$ and $t_n \to t^*$ as $n \to \infty$. Then

$$d(x_m,x_n) \leq \frac{2}{1-r}[\rho(t_m)-\rho(t_n)], \text{ for each } m,n \in \mathbb{N}, m > n.$$

Taking limit as $m, n \to \infty$, we obtain $d(x_m, x_n) \to 0$. Thus $(x_n)_{n\in\mathbb{N}}$ is Cauchy sequence which converges to (say) x^* in X. As $x_n \in G(x_n, t_n)$, $n \in \mathbb{N}$ and G is closed, so $x^* \in G(x^*, t^*)$. Also, from (h-1) we have $x^* \in U$. Hence $(t^*, x^*) \in \Delta$. Since M is totally ordered, therefore $(t,x) \leq (t^*, x^*)$, for each $(t,x) \in M$. That is, (t^*, x^*) is an upper bound of M. By Zorn's Lemma Δ have a maximal element $(t_0, x_0) \in \Delta$. We claim that $t_0 = 1$. Suppose that

$t_0 < 1$. Choose $a > 0$ and $t \in (t_0, 1]$ such that $B(x_0, a) \subset U$ and $a = \frac{2}{1-r}[\rho(t) - \rho(t_0)]$. Note that

$$d(x_0, G(x_0, t)) \leq d(x_0, G(x_0, t_0)) + H(G(x_0, t_0), G(x_0, t)) \\ \leq [\rho(t) - \rho(t_0)] = \frac{(1-r)a}{2} < (1-r)a.$$

Thus $G(., t) : B(x_0, a) \to CL(X)$ satisfies, for all $t \in [0, 1]$, the assumptions of Theorem 6.1. Hence, for all $t \in [0, 1]$, there exists $x \in B(x_0, a)$ such that $x \in G(x, t)$ which implies that $(t, x) \in \Delta$. Now

$$d(x_0, x) \leq a = \frac{2}{1-r}[\rho(t) - \rho(t_0)],$$

gives $(t_0, x_0) < (t, x)$, a contradiction to the maximality of (t_0, x_0). Conversely if $G(., 1)$ has a fixed point, then by a similar approach we obtain that $G(., 0)$ has a fixed point.

Acknowledgments The authors are thankful to the editor and reviewers for their valuable suggestions and comments for the improvement of this paper.

References

1. Baskaran, R., Subrahmanyam, P.V.: A note on the solution of a class of functional equations. Appl. Anal. **22**, 235–241 (1986)
2. Bellman, R.: Methods of Nonlinear Analysis. Vol. II, vol. 61 of Mathematics in Science and Engineering. Academic Press, New York (1973)
3. Bellman, R., Lee, E.S.: Functional equations in dynamic programming. Aequationes Mathematicae **17**, 1–18 (1978)
4. Bhakta, T.C., Mitra, S.: Some existence theorems for functional equations arising in dynamic programming. J. Math. Anal. Appl. **98**, 348–362 (1984)
5. Ćirić, Lj: Fixed points for generalized multivalued contractions. Matematički Vesnik **9**, 265–272 (1972)
6. Ćirić, Lj: multivalued nonlinear contraction mappings. Nonlinear Anal. **71**, 2716–2723 (2009)
7. Ćirić, Lj., Abbas, M., Rajović, M., Ali, B.: Suzuki type fixed point theorems for generalized multivalued mappings on a set endowed with two $b-$metrics. Appl. Math. Comput. **219**, 1712–1723 (2012)
8. Covitz, H., Nadler Jr, S.B.: Multivalued contraction mappings in generalized metric spaces. Israel J. Math. **8**, 5–11 (1970)
9. Daffer, P.Z., Kaneko, H.: Fixed points of generalized contractive multivalued mappings. J. Math. Anal. Appl. **192**, 655–666 (1995)
10. Dorič, D., Lazovič, R.: Some Suzuki type fixed point theorems for generalized multivalued mappings and applications. Fixed Point Theory Appl. **40**, 1687–1812 (2011)
11. Dhompongsa, S., Yingtaweesittikul, H.: Fixed points for multivalued mappings and the metric completeness. Fixed Point Theory Appl. 15 (2009)
12. Kikkawa, M., Suzuki, T.: Some similarity between contractions and Kannan mappings. Fixed Point Theory Appl. 8 (2008)
13. Kikkawa, M., Suzuki, T.: Three fixed point theorems for generalized contractions with constants in complete metric spaces. Nonlinear Anal. **69**, 2942–2949 (2008)
14. Mizoguchi, N., Takahashi, W.: Fixed point theorems for multivalued mappings on complete metric spaces. J. Math. Anal. Appl. **141**, 177–188 (1989)
15. Moţ, G., Petruşel, A.: Fixed point theory for a new type of contractive multivalued operators. Nonlinear Anal. **70**, 3371–3377 (2009)
16. Nadler Jr, S.B.: Multivalued contraction mappings. Pac. J. Math. **30**, 475–488 (1969)
17. Pathak, H.K., Cho, Y.J., Kang, S.M., Lee, B.S.: Fixed point theorems for compatible mappings of type P and applications to dynamic programming. Le Matematiche **50**, 15–33 (1995)
18. Popescu, O.: A new type of contractive multivalued operators. Bull. Sci. Math. 15. doi:10.1016/j.bulsci.2012.07.001
19. Reich, S.: Fixed points of contractive functions. Bollettino della Unione Matematica Italiana **5**, 26–42 (1972)

20. Rus, I.A., Petruşel, A., Sîntămărian, A.: Data dependence of the fixed point set of some multivalued weakly Picard operators. Nonlinear Anal. **52**, 1947–1959 (2003)
21. Semenov, P.V.: Fixed points of multivalued contractions. Funct. Anal Appl. **36**, 159–161 (2002)
22. Singh, S.L., Mishra, S.N.: On a Ljubomir Ćirić fixed point theorem for nonexpansive type maps with applications. Indian J. Pure Appl. Math. **33**, 531–542 (2002)
23. Singh, S.L., Mishra, S.N.: Coincidence theorems for certain classes of hybrid contractions. Fixed Point Theory Appl 14 (2010)
24. Suzuki, T.: A generalized Banach contraction principle that characterizes metric completeness. Proc. Am. Math. Soc. **136**(5), 1861–1869 (2008)