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Fixed points of generalized α-ψ-contractions

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Abstract In this paper, we introduce generalized α-ψ-contractive mappings and multifunctions and give some results about fixed points of the mappings and multifunctions.

Keywords Fixed point · Generalized α-ψ-contractive mapping · Multifunction

1 Introduction

During the last few decades, there have appeared a lot of papers on fixed points of multifunctions with different methods (see for example [1–9]). One of the most interesting methods is due to Suzuki for fixed points of mappings and multifunctions (see [10] and [11]). Recently, Samet, Vetro and Vetro have introduced the notion of α-ψ-contractive type mappings [12]. Denote by Ψ the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ. It is known that $\psi(t) < t$ for all $t > 0$ and $\psi \in \Psi$ [12]. Also, there are a lot of sublinear mappings in Ψ [13]. Let (X, d) be a metric space and T a selfmap on X. Then T is called a α-ψ-contraction mapping whenever there exist $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ such that $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$ [12]. Also, we say that T is α-admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ [12]. Also, we say that X has the property (B) respect to α if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$ and $x_n \to x$, then $\alpha(x_n, x) \geq 1$ for all $n \geq 1$ [12].

Let (X, d) be a complete metric space and T a α-admissible α-ψ-contractive mapping on X. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. If T is continuous or X has the property (B) respect to α, then T has a fixed point ([12]; Theorems 2.1 and 2.2). Finally, we say that X has the property (H) whenever for each $x, y \in X$ there exists $z \in X$ such that $\alpha(x, z) \geq$

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1 and $\alpha(y, z) \geq 1$. If X has the property (H) in the Theorems 2.1 and 2.2, then T has a unique fixed point ([12]; Theorem 2.3). It is considerable that the results of Samet, Vetro and Vetro generalize similar ordered results in the literature (see the results of the third section in [12]). Now, by using the main idea of [14], we introduce a new notion. We say that T is a generalized α-ψ-contractive mapping whenever $\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y))$ for all $x, y \in X$, where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}\max\{d(x, Ty), d(y, Tx)\}\}$. Throughout the paper, we suppose that $\psi \in \Psi$ is sublinear and $\alpha : X \times X \to [0, \infty)$ is a mapping.

2 Main results

Now, we are ready to state and prove our main results.

Theorem 2.1 *Let (X, d) be a complete metric space and T a continuous generalized α-ψ-contractive and α-admissible selfmap on X. If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.*

Proof Take $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}_{n\geq 0}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_n = x_{n+1}$ for some $n \geq 0$, then $x^* = x_n$ is a fixed point for T. Assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Since T is α-admissible, we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$. But, we have

$$d(x_1, x_2) = d(Tx_0, Tx_1) \leq \alpha(x_0, x_1)d(Tx_0, Tx_1) \leq \psi(M(x_0, x_1)),$$

where

$$\begin{aligned} M(x_0, x_1) &= \max\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{1}{2}\max\{d(x_0, Tx_1), d(x_1, Tx_0)\}\} \\ &= \max\{d(x_0, x_1), d(x_1, x_2), \frac{1}{2}d(x_0, x_2)\}. \end{aligned}$$

Note that, $M(x_0, x_1) \neq d(x_1, x_2)$ because if $M(x_0, x_1) = d(x_1, x_2)$, then we have $d(x_1, x_2) \leq \psi(d(x_1, x_2)) < d(x_1, x_2)$ which is a contradiction. Thus,

$$M(x_0, x_1) = \max\left\{d(x_0, x_1), \frac{1}{2}d(x_0, x_2)\right\}.$$

If $M(x_0, x_1) = d(x_0, x_1)$, then $d(x_1, x_2) \leq \psi(d(x_0, x_1))$ and if $M(x_0, x_1) = \frac{1}{2}d(x_0, x_2)$, then

$$d(x_1, x_2) \leq \psi\left(\frac{d(x_0, x_2)}{2}\right) \leq \frac{\psi(d(x_0, x_1)) + \psi(d(x_1, x_2))}{2} < \frac{1}{2}\psi(d(x_0, x_1)) + \frac{1}{2}d(x_1, x_2)$$

because ψ is sublinear. Hence, $d(x_1, x_2) \leq \psi(d(x_0, x_1))$. Now by using induction, we obtain $d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1))$ for all n. Fix $\varepsilon > 0$ and choose $n(\varepsilon) \geq 1$ such that $\sum_{n\geq n(\varepsilon)} \psi^n(d(x_0, x_1)) < \varepsilon$. Let $m > n > n(\varepsilon)$. By using the triangular inequality, we obtain $d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k(d(x_0, x_1)) \leq \sum_{n\geq n(\varepsilon)} \psi^n(d(x_0, x_1)) < \varepsilon$. Thus, $\{x_n\}_{n\geq 0}$ is a Cauchy sequence. Therefore, there exists $x^* \in X$ such that $x_n \to x^*$. Thus, $x_{n+1} = Tx_n \to Tx^*$ and so x^* is a fixed point of T. □

Example 2.1 Let $X = [0, +\infty)$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Define the selfmap T on X by $Tx = x + 8$ whenever $0 \leq x \leq 1$ and $Tx = 9$ whenever $x > 1$, and

$$\alpha(x, y) = \begin{cases} 2 & x, y \in [0, 1]\, or\, x, y \in [8n, 8n + 1],\ for\, some\, n \geq 1, \\ 1 & x \in [0, 1]\, and\, y \in [8, 9], \\ 0 & otherwise. \end{cases}$$

If we define $\psi(t) = \frac{t}{2}$ for all $t \geq 0$, then it is easy to check that T is a generalized α-ψ-contractive mapping. In fact, for each $x, y \in [0, 1]$ with $x \leq y$ we have

$$2 \times |x - y| = \alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)) = \frac{1}{2} \times 8 = 4.$$

For $x \in [0, 1]$ and $y \in \left(1, \frac{81}{19}\right)$ we have

$$0 \times |x - 9| = \alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)) = \frac{1}{2} \times |y - 9|.$$

For $x \in [0, 1]$ and $y \geq \frac{81}{19}$ we have

$$0 \times |x - 9| = \alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)) = \frac{1}{2}|x - y|.$$

Let $x, y \in [8(n - 1), 8n]$ for some $n \geq 1$ and $x \leq y$. Then

$$2 \times 0 = \alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)) = \frac{1}{2} \times |x - 9|.$$

If $x \in [0, 1]$ and $y \in [8, 9]$, then

$$1 \times |x - 1| = \alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)) = \frac{1}{2} \max\{|x - y|, 8\}.$$

Also, for $x_0 = 0$ we have $\alpha(0, T0) = \alpha(0, 8) = 1$. Obviously T is continuous and so it remains to show that T is α-admissible. If $x, y \in [0, 1]$ or $x, y \in [8n, 8n + 1]$ for some $n \geq 1$, then $\alpha(x, y) = \alpha(Tx, Ty) = 2$. If $x \in [0, 1]$ and $y \in [8, 9]$, then $\alpha(x, y) = 1$ and $\alpha(Tx, Ty) = 2$. Hence, T is α-admissible. Now, note that T has the fixed point $x_0 = 9$. Finally, note that $\alpha(x, y)d(Tx, Ty) \nleq \psi(d(x, y))$ for all $x, y \in [0, 1]$. Thus, the last result is a generalization of Theorem 2.1 in [12].

Now, we state multifunction version of our results. For $A, B \in CB(X)$, let

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$. It is well known that H is a metric on $CB(X)$. Such a map H is called *Hausdorff metric* induced by d. Similar to mapping case, we say that the multifunction $T : X \to CB(X)$ on a metric space X is a generalized α-ψ-contraction whenever $\alpha(x, y)H(Tx, Ty) \leq \psi(M(x, y))$ for all $x, y \in X$, where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} \max\{d(x, Ty), d(y, Tx)\}\}$.

Theorem 2.2 *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, +\infty)$ a mapping, $\psi \in \Psi$ and $T : X \to CB(X)$ a generalized α-ψ-contractive multifunction such that $\alpha(x, y) \geq 1$ implies $\alpha(u, v) \geq 1$ for all $u \in Tx$ and $v \in Ty$. Suppose that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If X has the property (B) respect to α, then, T has a fixed point.*

Proof Take $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If $x_0 = x_1$, then x_0 is a fixed point for T. Let $x_0 \neq x_1$. Then,

$$H(Tx_0, Tx_1) \leq \alpha(x_0, x_1)H(Tx_0, Tx_1) \leq \psi(M(x_0, x_1)),$$

where $M(x_0, x_1) = \max\{d(x_0, x_1), d(x_1, Tx_1), \frac{1}{2}d(x_0, Tx_1)\}$ because $x_1 \in Tx_0$. Let $x_1 \notin Tx_1$. Then $M(x_0, x_1) \neq d(x_1, Tx_1)$. Thus, $M(x_0, x_1) = \max\{d(x_0, x_1), \frac{1}{2}d(x_0, Tx_1)\}$. If

$M(x_0, x_1) = d(x_0, x_1)$, then $d(x_1, Tx_1) \leq \psi(d(x_0, x_1))$. If $M(x_0, x_1) = \frac{1}{2}d(x_0, Tx_1)$, then $d(x_1, Tx_1) \leq \psi(\frac{1}{2}d(x_0, Tx_1)) \leq \frac{1}{2}\psi(d(x_0, x_1)) + \frac{1}{2}\psi(d(x_1, Tx_1))$ and so

$$d(x_1, Tx_1) < \frac{1}{2}\psi(d(x_0, x_1)) + \frac{1}{2}d(x_1, Tx_1) \Rightarrow d(x_1, Tx_1) \leq \psi(d(x_0, x_1))$$

because ψ is sublinear. Thus there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) \leq \psi(d(x_0, x_1))$. Let $x_1 \neq x_2$. Then, $H(Tx_1, Tx_2) \leq \alpha(x_1, x_2)H(Tx_1, Tx_2) \leq \psi(M(x_1, x_2))$ and $\alpha(x_1, x_2) \geq 1$, where $M(x_1, x_2) = \max\{d(x_1, x_2), d(x_2, Tx_2), \frac{1}{2}d(x_1, Tx_2)\}$ because $x_2 \in Tx_1$. Let $x_2 \notin Tx_2$. Thus, $M(x_1, x_2) = \max\{d(x_1, x_2), \frac{1}{2}d(x_1, Tx_2)\}$. If $M(x_1, x_2) = d(x_1, x_2)$, then $d(x_2, Tx_2) \leq \psi(d(x_1, x_2))$. If $M(x_1, x_2) = \frac{1}{2}d(x_1, Tx_2)$, then $d(x_2, Tx_2) < \frac{1}{2}\psi(d(x_1, x_2)) + \frac{1}{2}d(x_2, Tx_2)$ because ψ is sublinear. Thus, we get $d(x_2, Tx_2) \leq \psi(d(x_1, x_2))$. This implies that there exists $x_3 \in Tx_2$ such that $d(x_2, x_3) \leq \psi(d(x_1, x_2))$ and so $d(x_2, x_3) \leq \psi^2(d(x_0, x_1))$. By continuing this steps, we obtain a sequence $\{x_n\}_{n \geq 0}$ in X such that $x_{n+1} \in Tx_n, \alpha(x_n, x_{n+1}) \geq 1$ and $d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1))$ for all $n \geq 0$. Fix $\varepsilon > 0$ and choose $n(\varepsilon) \geq 1$ such that $\sum_{n \geq n(\varepsilon)} \psi^n(d(x_0, x_1)) < \varepsilon$. Let $m > n > n(\varepsilon)$. By using the triangular inequality, we obtain

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k(d(x_0, x_1)) \leq \sum_{n \geq n(\varepsilon)} \psi^n(d(x_0, x_1)) < \varepsilon.$$

Thus, $\{x_n\}_{n \geq 0}$ is a Cauchy sequence. Hence, there exists $x^* \in X$ such that $x_n \to x^*$. Note that, $\alpha(x_n, x^*) \geq 1$ for all n. Moreover, we have

$$H(Tx_n, Tx^*) \leq \alpha(x_n, x^*)H(Tx_n, Tx^*) \leq \psi(M(x_n, x^*))$$

for all n, where

$$M(x_n, x^*) = \max\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \frac{1}{2}\max\{d(x_n, Tx^*), d(x^*, Tx_n)\}\}.$$

If $M(x_n, x^*) = d(x_n, x^*)$, then $d(x_{n+1}, Tx^*) \leq H(Tx_n, Tx^*) \leq \psi(d(x_n, x^*))$. In the case $M(x_n, x^*) = d(x_n, Tx_n)$, we have

$$d(x_{n+1}, Tx^*) \leq H(Tx_n, Tx^*) \leq \psi(d(x_n, Tx_n)) \leq \psi(d(x_n, x_{n+1})).$$

If $M(x_n, x^*) = d(x^*, Tx^*)$, then $x^* \in Tx^*$. In fact, if $x^* \notin Tx^*$, then $d(x^*, Tx^*) > 0$ and so $d(x_{n+1}, Tx^*) \leq H(Tx_n, Tx^*) \leq \psi(d(x^*, Tx^*))$. Hence,

$$d(x^*, Tx^*) = \lim_{n \to +\infty} d(x_{n+1}, Tx^*) \leq \psi(d(x^*, Tx^*)) < d(x^*, Tx^*)$$

which is a contradiction. If $M(x_n, x^*) = \frac{1}{2}d(x_n, Tx^*)$, then

$$d(x_{n+1}, Tx^*) \leq H(Tx_n, Tx^*) \leq \psi\left(\frac{1}{2}d(x_n, Tx^*)\right) \leq \frac{1}{2}d(x_n, Tx^*).$$

If $M(x_n, x^*) = \frac{1}{2}d(x^*, Tx_n)$, then we have

$$d(x_{n+1}, Tx^*) \leq H(Tx_n, Tx^*) \leq \psi\left(\frac{1}{2}d(x^*, Tx_n)\right) \leq \frac{1}{2}\psi(d(x^*, x_{n+1})).$$

Since ψ is continuous at $t = 0$, we get $d(x^*, Tx^*) = 0$ and so $x^* \in Tx^*$. $\square$

Example 2.2 Let $X = [-2,-1] \cup \{0\} \cup [1, \frac{5}{2}]$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Define the multivalued $T : X \to CB(X)$ by

$$Tx = \begin{cases} [-\frac{x}{4}+2, \frac{5}{2}] & x \in [-2, -\frac{3}{2}) \\ \{0\} & x \in \{-1, 0, 1\} \cup (2, \frac{5}{2}] \cup [-\frac{3}{2}, -1) \\ \left[-\frac{3}{2}, -\frac{x}{4} - 1\right] & x \in (1, 2], \end{cases}$$

$\psi(t) = \frac{4t}{5}$ for all $t \geq 0$ and $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 2 & x, y \in \Big[-2, -\frac{3}{2}\Big) \text{ or } x, y \in (1, 2] \\ 1 & x \in \Big[-2, -\frac{3}{2}\Big) \cup (1, 2] \ \textit{and}\ y \in \{-1, 0, 1\} \cup \Big(2, \frac{5}{2}\Big] \cup \Big[-\frac{3}{2}, -1\Big) \\ \frac{3}{2} & x, y \in \{-1, 0, 1\} \cup \Big(2, \frac{5}{2}\Big] \cup \Big[-\frac{3}{2}, -1\Big) \\ \frac{1}{2} & x \in \Big[-2, -\frac{3}{2}\Big) \ \textit{and}\ y \in (1, 2] \end{cases}$$

with $\alpha(x, y) = \alpha(y, x)$ for all $x, y \in X$. One can check that (X, d) is a complete metric space, X has the property (B) respect to α and T is a closed and bounded valued generalized α-ψ-contractive multifunction on X. Note that, for $x_0 = -2$ we have $Tx_0 = \{\frac{5}{2}\}$ and $\alpha\Big(-2, \frac{5}{2}\Big) = 1$. Therefore, T satisfies the conditions Theorem 2.2.

Now by mixing our idea with the Suzuki's idea, we give the following result. We say that the multifunction $T : X \to CB(X)$ on a metric spaces X is a Suzuki-generalized α-ψ-contraction if $\theta(r)d(x, Tx) \leq d(x, y)$ implies $\alpha(x, y)H(Tx, Ty) \leq \psi(M(x, y))$ for all $x, y \in X$, where $M(x, y) = \max\{d(x, y), \frac{1}{2}\max\{d(x, Tx), d(y, Ty)\}, \frac{d(x,Ty)+d(y,Tx)}{2}\}$ and

$$\theta(r) = \begin{cases} \frac{1}{2} & 0 \leq r \leq \frac{1}{2}(\sqrt{5}-1), \\ \frac{1-r}{2r^2} & \frac{1}{2}(\sqrt{5}-1) \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+2r} & \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Finally, we say that X has the property (C) respect to α whenever for each sequence $\{x_n\}$ in X and $x \in X$ such that $\alpha(x_n, x) \geq 1$ for all n and $x_n \to x^* \in X$ we have $\alpha(x^*, x) \geq 1$.

Theorem 2.3 *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, +\infty)$ a mapping, $\psi \in \Psi$ and $T : X \to CB(X)$ a Suzuki-generalized α-ψ-contractive multifunction such that $\alpha(x, y) \geq 1$ implies $\alpha(u, v) \geq 1$ for all $u \in Tx$ and $v \in Ty$. Suppose that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If X has the property (C) respect to α, then T has a fixed point.*

Proof Take $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If $x_0 = x_1$, then x_0 is a fixed point of T. Let $x_1 \neq x_0$. Since $\theta(r) \leq 1$, $\theta(r)d(x_0, Tx_0) \leq d(x_0, Tx_0) \leq d(x_0, x_1)$ and so $d(x_1, Tx_1) \leq H(Tx_0, Tx_1) \leq \alpha(x_0, x_1)H(Tx_0, Tx_1) \leq \psi(M(x_0, x_1))$, where $M(x_0, x_1) = \max\{d(x_0, x_1), \frac{1}{2}d(x_1, Tx_1), \frac{1}{2}d(x_0, Tx_1)\}$ because $x_1 \in Tx_0$. Now, let $x_1 \notin Tx_1$. Then $M(x_0, x_1) \neq \frac{1}{2}d(x_1, Tx_1)$. If $M(x_0, x_1) = d(x_0, x_1)$, then $d(x_1, Tx_1) \leq H(Tx_0, Tx_1) \leq \psi(d(x_0, x_1))$. If $M(x_0, x_1) = \frac{1}{2}d(x_0, Tx_1)$, then

$$d(x_1, Tx_1) \leq H(Tx_0, Tx_1) \leq \psi(\frac{1}{2}d(x_0, Tx_1)) \leq \frac{1}{2}\psi(d(x_0, x_1)) + \frac{1}{2}\psi(d(x_1, Tx_1))$$

and so $d(x_1, Tx_1) < \frac{1}{2}\psi(d(x_0, x_1)) + \frac{1}{2}d(x_1, Tx_1)$. Hence, $d(x_1, Tx_1) \le \psi(d(x_0, x_1))$. Thus, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) \le \psi(d(x_0, x_1))$. By continuing this process, we obtain a sequence $\{x_n\}_{n\ge 0}$ in X such that $x_{n+1} \in Tx_n, \alpha(x_n, x_{n+1}) \ge 1$ and $d(x_n, x_{n+1}) \le \psi^n(d(x_0, x_1))$ for all $n \ge 0$. Fix $\varepsilon > 0$ and choose $n(\varepsilon) \ge 1$ such that $\sum_{n\ge n(\varepsilon)} \psi^n(d(x_0, x_1)) < \varepsilon$. Let $m > n > n(\varepsilon)$. By using the triangular inequality, we obtain

$$d(x_n, x_m) \le \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \le \sum_{k=n}^{m-1} \psi^k(d(x_0, x_1)) \le \sum_{n\ge n(\varepsilon)} \psi^n(d(x_0, x_1)) < \varepsilon.$$

Thus, $\{x_n\}_{n\ge 0}$ is a Cauchy sequence. Therefore, there exists $x^* \in X$ such that $x_n \to x^*$. By using the assumption, we get $\alpha(x_n, x^*) \ge 1$ for all n. Now, we show that $d(x^*, Tx) \le \psi(d(x^*, x))$ for all $x \in X \setminus \{x^*\}$ with $\alpha(x_n, x) \ge 1$ for all n. Suppose that $x \in X \setminus \{x^*\}$ with $\alpha(x_n, x) \ge 1$ for all n. Since $x_n \to x^*$, there exists $n_0 \in N$ such that $d(x_n, x^*) \le \frac{1}{3}d(x, x^*)$ for all $n \ge n_0$. Then,

$$\begin{aligned}\theta(r)d(x_n, Tx_n) &\le d(x_n, Tx_n) \le d(x_n, x_{n+1}) \le d(x_n, x^*) + d(x^*, x_{n+1})\\ &= d(x_n, x^*) + d(x^*, x_{n+1}) \le \frac{2}{3}d(x, x^*) = d(x, x^*) - \frac{1}{3}d(x, x^*)\\ &\le d(x, x^*) - d(x^*, x_n) \le d(x, x_n) \quad (*)\end{aligned}$$

and so $H(Tx_n, Tx) \le \alpha(x_n, x)H(Tx_n, Tx) \le \psi(M(x_n, x))$, where

$$M(x_n, x) = \max\left\{d(x_n, x), \frac{1}{2}\max\{d(x_n, Tx_n), d(x, Tx)\}, \frac{d(x, Tx_n) + d(x_n, Tx)}{2}\right\}.$$

Thus, $d(x_{n+1}, Tx) \le \psi(M(x_n, x))$ for all $n \ge n_0$. Therefore, if $M(x_n, x) = d(x_n, x)$ or $M(x_n, x) = \frac{1}{2}d(x_n, Tx_n)$, then by using (*) we have $d(x_{n+1}, Tx) \le \psi(d(x_n, x))$ and so $\lim_{n\to\infty} d(x_{n+1}, Tx) \le \lim_{n\to\infty} \psi(d(x_n, x)) \Rightarrow d(x^*, Tx) \le \psi(d(x^*, x))$. Now, note that if $M(x_n, x) = \frac{1}{2}d(x, Tx)$, then $d(x_{n+1}, Tx) \le \psi(\frac{1}{2}d(x, Tx)) \le \frac{1}{2}\psi(d(x, x_n)) + \frac{1}{2}d(x_n, Tx)$ and so $\lim_{n\to\infty} d(x_{n+1}, Tx) \le \frac{1}{2}\lim_{n\to\infty} \psi(d(x, x_n)) + \frac{1}{2}\lim_{n\to\infty} d(x_n, Tx)$. Hence, we obtain $d(x^*, Tx) \le \psi(d(x^*, x))$. If $M(x_n, x) = \frac{d(x,Tx_n)+d(x_n,Tx)}{2}$, then

$$d(x_{n+1}, Tx) \le \psi\left(\frac{d(x, Tx_n) + d(x_n, Tx)}{2}\right) \le \frac{1}{2}\psi(d(x, x_{n+1})) + \frac{1}{2}d(x_n, Tx)$$

and so $d(x^*, Tx) \le \psi(d(x^*, x))$. Therefore, we prove the claim. Again by using the assumption, we get $\alpha(x, x^*) \ge 1$ and so $H(Tx, Tx^*) \le \alpha(x, x^*)H(Tx, Tx^*)$. Now, we show that $H(Tx, Tx^*) \le \psi(M(x, x^*))$. If $x \ne x^*$, then we get three cases. First, suppose that $0 \le r \le \frac{1}{2}(\sqrt{5} - 1)$. Then, $\theta(r) = \frac{1}{2}$ and

$$d(x, Tx) \le d(x, x^*) + d(x^*, Tx) \le d(x, x^*) + \psi(d(x^*, x)) < 2d(x, x^*)$$

Hence, $\theta(r)d(x, Tx) \le d(x, x^*)$ and so $H(Tx, Tx^*) \le \psi(M(x, x^*))$. Now, suppose that $\frac{1}{2}(\sqrt{5} - 1) \le r \le \frac{1}{\sqrt{2}}$. Then, $\theta(r) = \frac{1-r}{2r^2}$. In this case, for each $n \ge 1$ there exists $y_n \in Tx$ such that $d(x^*, y_n) \le d(x^*, Tx) + \frac{1}{n}d(x^*, x)$. Thus,

$$\begin{aligned}d(x, Tx) \le d(x, y_n) \le d(x, x^*) + d(x^*, y_n) &\le d(x, x^*) + d(x^*, Tx) + \frac{1}{n}d(x^*, x)\\ &\le d(x, x^*) + \psi(d(x, x^*)) + \frac{1}{n}d(x^*, x) < \left(2 + \frac{1}{n}\right)d(x^*, x)\end{aligned}$$

for all $n \geq 1$ and so

$$\theta(r)d(x, Tx) = \frac{1}{2}d(x, Tx) \leq d(x^*, x) \Rightarrow H(Tx, Tx^*) \leq \psi(M(x, x^*)).$$

Finally, suppose that $\frac{1}{\sqrt{2}} \leq r < 1$. Then, $\theta(r) = \frac{1}{1+2r}$. For each $n \geq 1$, there exists $z_n \in Tx$ such that $d(x^*, z_n) \leq d(x^*, Tx) + \left(\frac{1}{4} + \frac{1}{n}\right) d(x^*, x)$. Hence,

$$\begin{aligned} d(x, Tx) &\leq d(x, z_n) \leq d(x, x^*) + d(x^*, z_n) \leq d(x, x^*) + d(x^*, Tx) + \left(\frac{1}{4} + \frac{1}{n}\right) d(x^*, x) \\ &\leq d(x, x^*) + \psi(d(x, x^*)) + \left(\frac{1}{4} + \frac{1}{n}\right) d(x^*, x) < \left(2 + \left(\frac{1}{4} + \frac{1}{n}\right)\right) d(x^*, x) \end{aligned}$$

for all n. Thus, $d(x, Tx) \leq \left(2 + \frac{1}{4}\right) d(x^*, x) = \frac{9}{4}d(x^*, x)$. This implies that

$$\theta(r)d(x, Tx) = \frac{1}{1+2r}d(x, Tx) \leq \frac{4}{9}d(x, Tx) \leq d(x^*, x)$$

and so $H(Tx, Tx^*) \leq \psi(M(x, x^*))$. Thus, $d(x^*, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) \leq \lim_{n\to\infty} H(Tx_n, Tx^*) \leq \lim_{n\to\infty} \psi(M(x_n, x^*))$. If $M(x_n, x^*) = d(x_n, x^*)$, then $d(x^*, Tx^*) \leq \lim_{n\to\infty} \psi(d(x_n, x^*)) = 0$ and so we get $d(x^*, Tx^*) = 0$. If $M(x_n, x^*) = \frac{1}{2}d(x_n, Tx_n)$, then

$$d(x^*, Tx^*) \leq \lim_{n\to\infty} \psi\left(\frac{1}{2}d(x_n, Tx_n)\right) \leq \lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 0.$$

If $M(x_n, x^*) = \frac{1}{2}d(x^*, Tx^*)$, then $d(x^*, Tx^*) \leq \lim_{n\to\infty} \psi(\frac{1}{2}d(x^*, Tx^*)) < \frac{1}{2}d(x^*, Tx^*)$ which is a contradiction. If $M(x_n, x^*) = \frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2}$, then

$$\begin{aligned} d(x^*, Tx^*) &\leq \lim_{n\to\infty} \psi\left(\frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2}\right) \leq \frac{1}{2}\lim_{n\to\infty} \psi(d(x_n, x^*)) \\ &+ \frac{1}{2}\lim_{n\to\infty} \psi(d(x^*, Tx^*)) + \frac{1}{2}\lim_{n\to\infty} \psi(d(x^*, x_{n+1})) \leq \frac{1}{2}\psi(d(x^*, Tx^*)). \end{aligned}$$

Therefore, $d(x^*, Tx^*) = 0$ and so $x^* \in Tx^*$. □

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