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Resolution of singularities of threefolds in mixed characteristic: case of small multiplicity

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Dedicated to Heisuke Hironaka on the occasion of his 80th birthday

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Abstract We prove Local Uniformization for arbitrary excellent hypersurface threefolds of multiplicity smaller than the residue characteristic. This article is part of the authors' Resolution of Singularities program for arithmetic varieties of dimension three. The proof builds upon Hironaka's characteristic polyhedron and invariants.

Keywords Arithmetic varieties · Hironaka · Resolution of singularities · Blowing up · Local uniformization

Mathematics Subject Classification 13A18 · 14B05 · 14E15

1 Introduction

This article is part of the authors' program whose purpose is to prove the following conjecture on Resolution of Singularities of threefolds in mixed characteristic. The conjecture is a special case of Grothendieck's Resolution conjecture for quasi-excellent schemes.

Conjecture 1.1 *Let C be an integral regular excellent curve with function field F. Let S/F be a reduced algebraic projective surface and $\mathcal{X}$ be a flat projective C-scheme with generic fiber $\mathcal{X}_F = S$. There exists a birational projective C-morphism $\pi : \mathcal{Y} \to \mathcal{X}$ such that*

(i) *$\mathcal{Y}$ is everywhere regular.*
(ii) *$\pi^{-1}(Reg\mathcal{X}) \to Reg\mathcal{X}$ is an isomorphism.*

Let us point out that the equicharacteristic techniques designed in [12] extend to the situation described in the above conjecture. In particular, [12] theorem 3.3 extends and reduces Conjecture 1.1 to the following variant:

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Conjecture 1.2 *Let A be an excellent DVR with quotient field F and residue characteristic $p > 0$. Let (S, M, k) be a regular local ring of dimension three dominating A, essentially of finite type over A with $K := QF(S)$ of transcendence degree two over F. Let finally L/K be a finite field extension and v be a valuation of L. Assume:*

(i) *L/K is cyclic Galois or purely inseparable of degree p.*
(ii) *v has rank one and is centered in S.*

Then there exists a regular local ring T essentially of finite type over A with $QF(T) = L$ such that v is centered in T.

Applying embedded resolution techniques for surfaces, it can be assumed that such a v is centered in a local model of L of the form $B = (S[X]/(h))_{(M,X)}$ with h monic of degree p; more precisely, $h = X^p - g^{p-1}X - f$, $f, g \in M$ and ($g = 0$ if char$A = 0$). In particular the Local Uniformization statement of Conjecture 1.2 only involves certain hypersurface singularities (Spec B, x), of multiplicity $m(x) \leq p = \mathrm{char} k$, and embedded in an excellent fourfold ($Z =$ Spec $S[X], x$). We prove here:

Main Theorem 1.3 *Let $(R, \mathfrak{M}, k = k(x) := \frac{R}{\mathfrak{M}})$ be an excellent regular local ring of dimension four, $(Z, x) := (\mathrm{Spec} R, \mathfrak{M})$ and $(X, x) := (\mathrm{Spec} R/(h), x)$ be a reduced hypersurface. Assume that the multiplicity $m(x)$ of (X, x) satisfies $m(x) < p :=$ char$k(x)$. Let v be a valuation of $K(X)$ centered at x. Then there exists a finite sequence of local blowing ups*

$$(X, x) =: (X_0, x_0) \longleftarrow (X_1, x_1) \longleftarrow \cdots \longleftarrow (X_n, x_n),$$

where $x_i \in X_i$, $0 \leq i \leq n$ is the center of v, each blowing up center $Y_i \subset X_i$ is permissible at x_i (in Hironaka's sense), such that x_n is regular.

The proof of theorem 1.3 builds upon classical Resolution of Singularities techniques. We use systematically the Hironaka characteristic polyhedron and Hironaka's invariants: the multiplicity $m(x)$ and τ-number $\tau(x)$ for the hypersurface singularity $(X, x) := (\mathrm{Spec} R/(h), x)$.

Since we are working without any ground field (at least when R is not equicharacteristic), the Tschirnhausen trick (killing the degree $(m(x) - 1)$-term in the equation) cannot be directly applied even though $m(x) < p$. Rather, we use it for the initial face of the Hironaka polyhedron (Theorem 3.5) to define well behaved invariants. The pair $(m(x), \tau(x))$ is then further on completed to a 6-tuple $\iota(x)$ defined in **5.2**.

The main technical part is concentrated in Sect. 7. We consider projections to a two dimensional space to define a refined invariant in Sects. **7.3** and **7.4.1**. Controlling the transformation law for this refined invariant under blowing up is much harder but leads essentially to the same formulæ as for the characteristic polygon of a surface singularity. The proof of the Main Theorem follows rather easily from these computations (Sect. **7.4.5**).

It is worth pointing out that these techniques are global in nature and it is to be expected that Theorem 1.3 can be extended to a global version, i.e. without referring to a given valuation v and using global blowing up centers. We use the valuation only at a few specific places (mostly in Sect. 6) to make the argument quicker.

This article is organized as follows: Sect. 2 states the reduction of the Main Theorem to the case $\tau(x) = 1$, immediate from [11]. This means that the initial form $\mathrm{in}_x(h)$ can be written

$$\mathrm{in}_x(h) = \lambda Y^{m(x)}, \ \lambda \in k(x), \lambda \neq 0, \ y \text{ a regular parameter of } R, \ Y := \mathrm{in}_x(y).$$

Section 3 first recollects known material from [16,17] about characteristic polyhedra and associated invariants (Definition 3.2). Special coordinates $(z, u) := (z, u_1, u_2, u_3)$ on R are said to be *fully prepared* if they compute the Hironaka characteristic polyhedron $\Delta(h; u)$ *and* if the δ-initial $\mathrm{in}_\delta(h)$ of h is Tschirnhausen transformed, i.e. has no term of degree $m(x) - 1$

(Theorem 3.5 and Definition 3.5.3). The form $\mathrm{in}_\delta(h)$ is defined in **3.2**(iii) and is the sum of the initial forms of all those terms in h contributing to the face of minimal order at x of $\Delta(h; u)$. In such special coordinates, invariants $d_1, d_2, d_3, \epsilon(x)$ (each of them is a nonnegative rational number) can be computed from the polyhedron $\Delta(h; u)$ and the δ-initial $\mathrm{in}_\delta(h)$ (Definition **3.7**). Theorem 3.9 proves the Main Theorem when $\epsilon(x) = 0$, in which case only combinatorial blowing ups are used.

When $\epsilon(x) > 0$, some preparations are required in order to get the locus

$$\Sigma := \{y \in X : \ m(y) = m(x),\ \tau(y) = \tau(x),\ \epsilon(y) > 0\}$$

Zariski closed and of dimension at most one (Theorems 3.10 and 3.11). Section 4 prepares $\mathrm{Spec} R$ and constructs an equicharacteristic p normal crossings divisor

$$E \subseteq \mathrm{div}(u_1 u_2 u_3) \subset \mathrm{Spec}\, R$$

which contains Σ ($E = E_n$ in Proposition 4.1).

Section 5 then provides some further invariants build up from the ideal of coefficients of $\mathrm{in}_\delta(h)$ once this preparation is achieved: a refined directrix $\mathcal{V} \subseteq \langle U_1, U_2, U_3\rangle$ (Definition **5.1**) and associated refined numerical invariant $\iota(x)$ (Definition 5.2):

$$\iota(x) := (m(x), -\tau(x), \epsilon(x), -\rho(x), -t(x), -e(x)).$$

Section 6 introduces the notion of ϵ-permissible blowing up centers (Definition 6.1). For curves, being ϵ-permissible is stronger than being Hironaka permissible (Proposition 6.2); blowing up along an ϵ-permissible center does not increase the invariant $\iota(x)$ (Proposition 6.3). Furthermore, $\iota(x)$ can be decreased by blowing up along ϵ-permissible centers except possibly when $\mathcal{V} = \langle U_3\rangle$ *and* (either $\mathrm{div}(u_3) \subseteq E$ or $E \subseteq \mathrm{div}(u_1 u_2)$) (Propositions 6.4 and 6.5).

Section 7 proves the same result in these remaining cases (Theorem 7.1), thus concluding the proof of the Main Theorem. We now project to the (u_1, u_2)-space and define *well prepared* coordinates by minimizing the induced image of $\Delta(h; u)$ by this projection (this requires choosing special coordinates (z, u_3)). There are further associated invariants $\beta(u, z), C(u, z), \gamma(u, z)$ defined in **7.4.1**. The behaviour of these invariants by blowing up ϵ-permissible curves and closed points are studied respectively in Propositions 7.4.2 and 7.4.3. Section **7.4.5** contains the proof of theorem 7.1 and is basically a consequence of the former computations.

The notation and assumption in the Main Theorem will be kept all along this article. The proof will be made by induction on the multiplicity $m(x) = \mathrm{ord}_x(h)$ of $x \in X$. Since it is assumed that $m(x) < p$, (X, x) is already regular if $p = 2$, so we assume $p \geq 3$ from now on. The formal completion of R with respect to $\mathfrak{M}$ is denoted by $\widehat{R}$.

2 Basic invariants

Two basic invariants are attached to the hypersurface singularity $(X, x) = (\mathrm{Spec} R/(h), x)$. The first invariant is its multiplicity $m(x)$ (or m for short) of (X, x). The second invariant is $\tau(x)$ (or τ for short), which is the dimension of the smallest $k(x)$-vector subspace $\mathcal{T}$ of $\frac{\mathfrak{M}}{\mathfrak{M}^2}$ such that $\mathrm{in}_x(h) \in k(x)[\mathcal{T}]$ [16], Ch.2, Lemma 10. This vector space is called the *directrix* of $\mathrm{in}_x(h)$.

Proving the Main Theorem in the cases $\dim(Z) - \tau(x) \in \{0, 1, 2\}$, i.e. $\tau(x) \in \{2, 3, 4\}$ is done in [11]. So from now on, we assume that $\tau(x) = 1$. Equivalently:

$$\mathrm{in}_x(h) = \lambda Y^m,\ \lambda \in k(x), \lambda \neq 0,\ y \text{ a regular parameter of } R,\ Y := \mathrm{in}_x(y).$$

3 Characteristic polyhedron

Definition 3.1 (i) An F-subset $\Delta \subset \mathbb{R}_+^d$ is a closed complex subset of $\mathbb{R}_+^d$ such that $v \in \Delta$ implies $v + \mathbb{R}_+^d \subset \Delta$.

(ii) A point $v \in \Delta$ is called a vertex if there is a positive linear form L on $\mathbb{R}^d$ (i.e. has strictly positive coefficients) such that

$$\{v\} = \Delta \cap \{A \in \mathbb{R}^d | L(A) = 1\}.$$

(iii) The essential boundary $\partial\Delta$ of an F-subset Δ is the subset of Δ consisting of those $v \in \Delta$ such that $v \notin v' + \mathbb{R}_+^d$ with $v' \in \Delta$ unless $v' = v$. We write $\Delta^+ = \Delta - \partial\Delta$.

For the next definition and proposition, we will forget the hypothesis $\dim(R) = 4$: we will have to use the notions defined there for different regular rings of dimension at most three. Given a r.s.p. $(y, u_1, u_2, \ldots, u_d) =: (y, u)$ of a regular local ring R and $f \in R$, there exists a *finite* sum expansion

$$f = \sum_{A,b} C_{A,b} y^b u^A, \quad b \in \mathbb{N},\ A \in \mathbb{N}^d. \tag{1}$$

where each $C_{A,b}$ is a unit in R. This follows easily from the facts that R is Noetherian and the map $R \subseteq \widehat{R}$ faithfully flat. We regard u as "fixed" parameters and y as "varying", which is reflected in the indexing below. Assume furthermore that

$$h \in \mathfrak{M},\ h \notin (u_1, \ldots, u_d). \tag{2}$$

We let $\overline{R} := R/(u_1, \ldots, u_d)$, $\overline{h} \in \overline{R}$ be the image of h and "ord" be the valuation of the discrete valuation ring $\overline{R}$. We extend our conventions by letting now

$$m := \mathrm{ord}\overline{h} \geq 1. \tag{3}$$

Assumption (2) and notation (3) are maintained all along this article. Our original concern is for $\tau(x) = 1$, say $\mathrm{in}_x(h) = \lambda Y^m$, $0 \neq \lambda \in k(x)$ which fits into these conventions provided $Y = \mathrm{in}_x(y)$.

Definition 3.2 (i) The polyhedron $\Delta(h; u; y) \subset \mathbb{R}_{\geq 0}^d$ is defined as the smallest F-subset containing all points of

$$S(h) := \left\{ v = \frac{A}{m-b} | 0 \leq b < m \right\}.$$

The characteristic polyhedron $\Delta(h; u) \subset \mathbb{R}_{\geq 0}^d$ is defined by the formula

$$\Delta(h; u) := \bigcap_{(\widehat{y}, u_1, \ldots, u_d)} \Delta(h; u; \widehat{y}), \tag{4}$$

where the intersection runs over all r.s.p.'s of $\widehat{R}$ of the form $(\widehat{y}, u_1, \ldots, u_d)$.

(ii) For $v \in \partial\Delta(h; u; y)$, the v-initial of h is defined as

$$\mathrm{in}_v(h) := \sum_{A,b} \overline{C_{A,b}} Y^b U^A \in k[U, Y] = k[U_1, U_2, \ldots, U_d, Y],$$

where $\overline{C_{A,b}} \in k$ is the residue of $C_{A,b}$ and the sum ranges over such (A, b) that

$$C_{A,b} \neq 0,\ (b \leq m,\ A = 0) \text{ or } \left(b < m \text{ and } v = \frac{A}{m-b} \right).$$

(iii) For $A \in \mathbb{N}^d$, $let |A| := a_1 + \cdots + a_d$. We put

$$\delta(h, u, y) := \min \left\{ \frac{|A|}{m - b} \ : \ C_{A,b} \neq 0, b < m \right\}.$$

This is in fact an invariant of the polyhedron $\Delta(h; u; y)$ since

$$\delta(h, u, y) = \min\{|v| \ : \ v \in \Delta(h; u; y)\}.$$

The δ-initial of h is defined as

$$\text{in}_{\delta,u,y}(h) := \sum_{A,b} \overline{C_{A,b}} Y^b U^A \in k(x)[U, Y] = k(x)[U_1, U_2, \ldots, U_d, Y],$$

where the sum ranges over such (A, b) that

$$C_{A,b} \neq 0, \ (b \leq m, A = 0) \text{ or } \left(b < m \text{ and } \frac{|A|}{m - b} = \delta(h, u, y) \right).$$

(iv) More generally, let

$$L : (x_1, x_2, \ldots, x_d) \mapsto L(x_1, x_2, \ldots, x_d) = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_d x_d, \ \lambda_1, \lambda_2, \ldots, \lambda_d \in \mathbb{Q}_{\geq 0},$$

be a nonzero nonnegative linear form on $\mathbb{R}^d$. We define

$$l(h, u, y) := \min\{L(A) | A \in \Delta(h; u; y)\} \geq 0.$$

We define a monomial valuation $v_{L,h,u,y}$ on R by setting

$$I_\lambda := (\{y^b u^A | l(h, u, y)b + L(A) \geq \lambda\}) \subseteq R,$$

for $\lambda \geq 0$ and $v_{L,h,u,y}(g) := \min\{\lambda \in \mathbb{Q} | g \in I_\lambda\}$ for any nonzero $g \in R$.

Proposition 3.3 *Let L be a nonzero nonnegative linear form as above, and let*

$$I := \{i | \lambda_i > 0\}, \ I' := \{i | \lambda_i = 0\} = \{1, \ldots, d\} \backslash I.$$

The graded algebra $gr_{v_{L,h,u,y}}(R)$ of R w.r.t. $v_{L,h,u,y}$ is given by

(i) *if $l(h, u, y) \neq 0$, then*

$$gr_{v_{L,h,u,y}}(R) = \frac{R}{(y, \{u_i\}_{i \in I})}[Y, \{U_i\}_{i \in I}];$$

(ii) *if $l(h, u, y) = 0$, then*

$$gr_{v_{L,h,u,y}}(R) = \frac{R}{(\{u_i\}_{i \in I})}[\{U_i\}_{i \in I}].$$

In particular, we have $gr_{v_{L,h,u,y}}(R) \simeq k[Y, U_1, U_2, \ldots, U_d]$ whenever L is positive.

The above proposition is obvious. One also checks easily the following:

Remark 3.3.1 Let v be a vertex of $\Delta(h, u, y)$. We have:

(i) $\text{in}_v(h)$ is independent of the presentation **3.1** (1),
(ii) $\text{in}_v(h) \neq \text{in}_x(h)$,

(iii) L being a positive linear form L on $\mathbb{R}^d$ such that $\{v\} = \Delta \cap \{A \in \mathbb{R}^d | L(A) = 1\}$, (cf. **3.1**(ii)), then

$$\mathrm{in}_v(h) = \mathrm{in}_{v_{L,h,u,y}}(h) \in \mathrm{gr}_{v_{L,h,u,y}}(R) = k(x)[U_1, U_2, \ldots, U_d, Y].$$

When there is no ambiguity, we will write $\mathrm{gr}_\delta(R)$ and $\mathrm{in}_\delta(h) \in \mathrm{gr}_\delta(R)$ instead of respectively $\mathrm{gr}_{v_{L,h,u,y}}(R)$ and $\mathrm{in}_{v_{L,h,u,y}}(h)$, where $L(x_1, x_2, \ldots, x_d) = x_1 + x_2 + \cdots + x_d$.

Remark 3.4 With notations as above, we have: $\delta(h, u, y) \in \frac{1}{m!}\mathbb{N}$ and $\delta(h, u, y) > 1$ if ($m = \mathrm{ord}_x(h)$ and $\langle \mathrm{in}_x(h) \rangle = \langle (\mathrm{in}_x(y))^m \rangle$).

Assumption 3.4.1 We now apply these constructions to the case $R := \mathcal{O}_{Z,x}$, $\dim(R) = 4$; the element $h \in R$ verifies assumptions **3.1**(2)(3) with $m = \mathrm{ord}_x(h) < p = \mathrm{char} k(x)$ and $\langle \mathrm{in}_x(h) \rangle = \langle Y^m \rangle$. In addition, $X = \mathrm{Spec}(R/(h))$ is reduced.

Theorem 3.5 *Given* $(y, u_1, u_2, u_3) =: (y, u)$ *as above, there exists* $z \in R$, $z \equiv y$ *mod*(u_1, u_2, u_3) *such that*

$$\Delta(h, u, z) = \Delta(h, u) \neq \varnothing, \tag{5}$$

$$in_{\delta,u,z}(h) = \sum_{A,b,b \neq m(x)-1} \overline{C_{A,b}} z^b U^A. \tag{6}$$

Proof Suppose $\Delta(h, u) = \varnothing$, then, in $\widehat{R}$, we should have $h = \gamma z^m$, γ invertible in $\widehat{R}$ and $z \in \widehat{R}$ a local parameter: h should be nonreduced in $\widehat{R}$. By excellence, h should be nonreduced in R, in contradiction with the hypothesis X reduced.

Since $\Delta(h, u) \neq \varnothing, \Delta(h, u)$ may be defined by a finite number n of inequalities $L_i(x_1, x_2, x_3) \geq 1, 1 \leq i \leq n$, with $L_i(x_1, x_2, x_3) = a_{i,1}x_1 + a_{i,2}x_2 + a_{i,3}x_3, a_{i,1}, a_{i,2}, a_{i,3} \in \mathbb{Q}_{\geq 0}$, $L_i \neq 0$. In a few words:

$$\Delta(h, u) = \{(x_1, x_2, x_3) | L_i(x_1, x_2, x_3) \geq 1,\ 1 \leq i \leq n\}.$$

We choose $L_1(x_1, x_2, x_3) = \frac{1}{\delta(h,u)}(x_1 + x_2 + x_3)$, with $\delta(h, u) := \min\{|v| : v \in \Delta(h, u)\}$.

Suppose (5) does not hold for (y, u). Then, with notations as in **3.2**(iv), some L_i, $1 \leq i \leq n$ satisfies

$$L_i(\Delta(h, u, y)) = [l_i(h, u, y), \infty[\not\subset [1, +\infty[\Leftrightarrow l_i(h, u, y) < 1.$$

We skip the index i of L_i and of $l_i(h, u, y)$ to simplify the notations. Following **3.2**(iv), we define the initial form of h with respect to L, u, y: in the case $l(h, u, y) > 0$

$$\mathrm{in}_{L,u,y}(h) := \sum_{A,b} \overline{C_{A,b}} Y^b U^A \in \mathrm{gr}_{L,u,y}(R), \tag{7}$$

with $bl(h, u, y) + L(A) = ml(h, u, y)$, $\overline{C_{A,b}} \in \frac{R}{(u_i)_{a_i>0} + (y)}$. In the case $l(h, u, y) = 0$, we have

$$\mathrm{in}_{L,u,y}(h) := \sum_{A,b} \overline{C_{A,b} y^b} U^A \in \mathrm{gr}_{L,u,y}(R), L(A) = 0,\ \overline{C_{A,b} y^b} \in \frac{R}{(u_i)_{a_i>0}}. \tag{8}$$

Claim 3.5.1 In (7) (resp. (8)), there exists A with $\overline{C_{A,m-1}} \neq 0 \in \frac{R}{(u_i)_{a_i>0}}$ (resp. $\overline{C_{A,m-1} y^{m-1}} \neq 0 \in \frac{R}{(u_i)_{a_i>0}}$).

Indeed, in the face with equation $L(x_1, x_2, x_3) = l(h, u, y)$ of $\Delta(h, u, y)$ there is at least a vertex v which is solvable [17, (3.8)]. Then $in_v(h)$ is collinear to an mth-power: $C_{v,m-1} \notin \mathfrak{M}$ since $m < p$ and the claim is proved. Take $A = v$ and let

$$y_1 = y + \frac{1}{m} C_{0,m}^{-1} \sum_A C_{A,m-1} u^A \in R. \tag{9}$$

Note that, for any A with $C_{A,m-1} \neq 0$, $A \in \Delta(h, u, y)$. So, for any i, $1 \le i \le n$, and any A with $C_{A,m-1} \neq 0$, $L(A) \ge l(h, u, y)$. So if in the expansion of **3.2**(1) we set

$$y = y_1 - \frac{1}{m} C_{0,m}^{-1} \sum_A C_{A,m-1} u^A,$$

we get a new expansion

$$h = \sum_{A,b} D_{A,b} y_1^b u^A, \ D_{A,b} \in R^\times \cup \{0\}, \ b \in \mathbb{N}, \ A \in \mathbb{N}^3, \tag{10}$$

and $D_{A,b} \neq 0 \ \Rightarrow \ L(A) + bl_i(h, u, y) \ge m$. So

$$l(h, u, y_1) \ge l(h, u, y), \ 1 \le i \le n.$$

Suppose $l(h, u, y_1) = l(h, u, y)$. Then $v_{L,u,y_1}(y) = v_{L,u,y_1}(y_1) = l(h, u, y)$,

$$Y := \text{in}_{L,u,y_1}(y) = Y_1 - \text{in}_{L,u,y_1}\left(\frac{1}{m} C_{0,m}^{-1} \sum_A C_{A,m-1} u^A\right), \ Y_1 := \text{in}_{L,u,y_1}(y_1),$$

$$\text{in}_{L,u,y_1}(h) = \sum_{A,b,L(A)+l(h,u,y)b=l(h,u,y)m} \text{in}_{L,u,y_1}(D_{A,b}) Y_1^b (\text{in}_{L,u,y_1}(u^A)), \tag{11}$$

$$\text{in}_{L,u,y_1}(h) = \sum_{A,b,L(A)+l(h,u,y)b=l(h,u,y)m} \text{in}_{L,u,y_1}(C_{A,b}) \left(Y_1 - \text{in}_{L,u,y_1}\left(\frac{1}{m} C_{0,m}^{-1} \sum_A C_{A,m-1} u^A\right)\right)^b, \tag{12}$$

where (11) is the expansion of (12). In (11), the terms with $b = m - 1$ are all zero; in (10), $D_{A,m-1} \neq 0$ implies $L(A) + l(h, u, y_1)b > ml(h, u, y_1)$. By the claim, $l(h, u, y_1) = 1$: a contradiction, hence

$$l(h, u, y_1) > l(h, u, y).$$

Note that (9) is independent of the linear form L, so

$$l_i(h, u, y_1) > l_i(h, u, y), \ \text{for all } i \ 1 \le i \le n.$$

By induction on the $l_i(h, u, y)$'s, we get (5). If (5) holds but not (6) for h, u, y, we make the change of variable (7) and get (6).

Proposition 3.5.2 *With notations as above, assume furthermore that* (z, u) *and* (z', u') *satisfy* (5) *of the previous theorem. Then* $\delta(h, u, z) = \delta(h, u', z')$.

Proof This is obvious if $u_i = u'_i$, $1 \le i \le 3$. On the other hand, the condition

$$\text{in}_\delta(h) \neq \lambda(Z - \Phi(U_1, U_2, U_3))^m \quad \text{for every } \Phi \in k(x)[U_1, U_2, U_3] \tag{13}$$

is preserved if $z = z'$ since $\delta(h, u, z) > 1$ (Remark 3.4). In particular we have

$$\delta(h, u, z) = \delta(h, u', z) \leq \delta(h, u', z')$$

and we conclude by symmetry that $\delta(h, u, z) = \delta(h, u', z')$.

Definition 3.5.3 When $z \in R$ is such that (5) (resp. (6) holds), we say that (z, u) is prepared (resp. δ-prepared). If both of (5) and (6) hold for $z \in R$, we say that (z, u) is fully prepared. If there is no ambiguity on $u = (u_1, u_2, u_3)$, we simply say that z is prepared, δ-prepared, or fully-prepared.

If (z, u) is prepared, the invariant $\delta(h, u, z)$ will be henceforth denoted by $\delta(x)$.

Theorem 3.6 *Let $(y, u_1, u_2, u_3) =: (y, u)$ be as before and E be a normal crossings divisor $E \subseteq div(u_1u_2u_3) \subset Spec(R)$. For a component $div(u_i)$ of E, we define*

$$d_i(u, y) := inf\{x_i | (x_1, x_2, x_3) \in \Delta(h, u, y)\},$$

$$d_i(u) := inf\{x_i | (x_1, x_2, x_3) \in \Delta(h, u)\}.$$

(i) *Let $z \in R$ be such that $(z, u) = (z, u_1, u_2, u_3)$ is prepared. Then*

$$d_i(u) \geq 1 \Leftrightarrow V(z, u_i) \textit{ is a permissible blowing up center of } Spec(R/(h)). \tag{14}$$

(ii) *We have $d_i(u) > 0$ if and only if $Y_i := V(h, u_i)_{\mathrm{red}}$ is a regular surface.*

(iii) *Assume that $d_i(u) > 0$. Then $d_i(u) = \delta(\eta_i)$, where $\eta_i \in Z$ is the generic point of $Y_i = V(h, u_i)_{\mathrm{red}}$ as above; in particular, $d_i(u)$ is independent of the choice of a prepared $(z, u) = (z, u_1, u_2, u_3)$ containing u_i.*

Proof of (ii). Take $z \in R$ such that $\Delta(h, u) = \Delta(h, u, z)$, so $h = z^m \mathrm{mod}(u_i)$ iff $d_i(u) > 0$ iff $Y := V(z, u_i) = V(h, u_i)_{red}$. We get

$$d_i(u) > 0 \Leftrightarrow (V(h, u_i))_{red} \text{ is regular at } x.$$

Proof of (i). In that hypersurface case, $Y := V(z, u_i) \in \mathrm{Spec}R$ permissible means $h \in (z, u_i)^m$ which is equivalent to $d_i(u) \geq 1$.

Proof of (iii). Let η be the generic point of $Y = V(h, u_i)_{red} = V(z, u_i)$. The following equivalence is straightforward

$$d_i(u) = 1 \Leftrightarrow V(z, u_i) \text{ is a permissible center of } \mathrm{Spec}(R/(h)) \text{ and } \tau(\eta) = 2.$$

If $0 < d_i(u) < 1$, we obviously have $d_i(u) = d_i(u, y)$ for every r.s.p. (y, u_1, u_2, u_3) (with $\langle \mathrm{in}_x(h) \rangle = \langle Y^m \rangle$).

We turn to the case:

$$d_i(u) > 1 \Leftrightarrow V(z, u_i) \text{ is a permissible center of } \mathrm{Spec}(R/(h)) \text{ and } \tau(\eta_i) = 1.$$

We claim that $d_i(u) = \delta(\eta_i)$. As $V(z, u_i) = V(h, u_i)_{\mathrm{red}}$, this will prove the invariance of $d_i(u)$. We take $i = 1$ and write η for η_1 in the following lemma.

Lemma 3.6.1 *Let (y, u_1, u_2, u_3) be as above and assume that $d_1(u, y) > 1$. Then*

$$d_1(u, y) = \delta(h, u_1, y), \tag{15}$$

where the right hand side is computed w.r.t. the datum $(h) \subset R_{(y, u_1)}$. If furthermore (y, u) is prepared, then

$$d_1(u) = \delta(\eta). \tag{16}$$

Proof By **3.1** (1), we have a finite expansion

$$h = \sum_{A,b} C_{A,b} y^b u^A, \ C_{A,b} \in R^\times, \ b \in \mathbb{N}, \ A \in \mathbb{N}^3$$

that we rewrite as:

$$h = \sum_{a_1,b} \left(\sum_{a_2,a_3} C_{A,b} u_2^{a_2} u_3^{a_3} \right) y^b u_1^{a_1}, \ A = (a_1, a_2, a_3) \in \mathbb{N}^3.$$

As $d_1(u, y) > 1$, we have $b + a_1 > m$ for all $b < m$. Note that every $\sum_{a_2,a_3} C_{A,b} u_2^{a_2} u_3^{a_3}$ is invertible in $R_{(y,u_1)} = \mathcal{O}_{Z,\eta}$. Then $\delta(h, u_1, y) = \inf\{\frac{a_1}{m-b} | b < m\} = d_1(u, y)$: this is (15).

Now suppose that y, u_1, u_2, u_3 is prepared. We claim that y, u_1 is prepared w.r.t. the datum $(h) \subset R_{(y,u_1)}$. If not, then

$$in_\delta(h) = \sum_{a_1+bd_1=d_1 m} \left(\sum_{a_2,a_3} \overline{C_{A,b} u_2^{a_2} u_3^{a_3}} \right) Y^b u_1^{a_1} \in \mathrm{gr}_\delta \left(R_{(y,u_1)} \right),$$

is proportional to an mth-power. So there exists some a_1 with

$$a_1 + (m-1)d_1 = d_1 m, \quad \text{and} \quad \sum_{a_2,a_3} C_{A,b} u_2^{a_2} u_3^{a_3} \neq 0,$$

so there exists some A with $a_1 + (m-1)d_1 = d_1 m$ and $C_{A,m-1} \neq 0$. Then, as in **3.5** (9), we change the variable y by $y_1 = y + \frac{1}{m} C_{0,m}^{-1} \sum_A C_{A,m-1} u^A \in R$, we get a new expansion

$$h = \sum_{A,b} D_{A,b} y_1^b u^A, \ D_{A,b} \in R^\times \cup \{0\}, \ b \in \mathbb{N}, \ A \in \mathbb{N}^3.$$

Now $D_{A,b} \neq 0$ implies $\frac{L_i(A)}{l_i(h,u,y)} + b \geq m$ for each linear form L_i such that

$$\Delta(h, u) = \{(x_1, x_2, x_3) | L_i(x_1, x_2, x_3) \geq 1, \ 1 \leq i \leq n\}.$$

This holds in particular for the linear form $L(x_1, x_2, x_3) = \frac{1}{d_1} x_1$. Since y is supposed to be nonprepared for u_1, the unique vertex $d_1(u_1, y)$ of $\Delta(h, u_1, y) \subset \mathbb{R}^+$ does not belong to $\Delta(h, u_1, y_1)$. We get $d_1(u_1, y_1) > d_1(u, y)$, a contradiction with the fact that $\Delta(h, u, y)$ was minimal.

Definition and Notation 3.7 *Given* $(y, u_1, u_2, u_3) =: (y, u)$*, h reduced, with assumptions* **3.1***(2)(3) and a normal crossings divisor* $E \subset div(u_1 u_2 u_3) \subset Spec(R)$*, we let* $d_i := d_i(u)$ *for each irreducible component* $div(u_i)$ *of* E*. We let* $d_i := 0$ *whenever* $div(u_i)$ *is not an irreducible component of* E.

We define $\epsilon(x, E) \in \mathbb{Q}_{\geq 0}$ *(or* $\epsilon(x)$ *for short) by:*

$$\epsilon(x, E) = \delta(x) - \sum_{div(u_i) \subset E} d_i.$$

These invariants appear in [13] Ch.1 (II.3.3) in an equal characteristic context. The following remarks are obvious from the definitions.

Remark 3.8 We have

(i) $\epsilon(x, E) \in \frac{1}{m!}\mathbb{N}$,
(ii) if $\epsilon(x, E) = 0$, $\Delta(h, u)$ has only one vertex: the point $v = (d_1, d_2, d_3)$.

Theorem 3.9 *Given* $(y, u_1, u_2, u_3) =: (y, u)$ *and a divisor* $E \subset div(u_1u_2u_3) \subset Spec(R)$ *as above, assume that* $\epsilon(x, E) = 0$. *There exists a finite sequence of local blowing ups*

$$(X, x) =: (X_0, x_0) \longleftarrow (X_1, x_1) \longleftarrow \cdots \longleftarrow (X_n, x_n),$$

where $x_0 = x$, $x_i \in X_i$, $0 \leq i \leq n$ *is the center of* v, *each blowing up center* $Y_i \subset X_i$ *is permissible in Hironaka's sense, such that* $m(x_n) < m(x)$.

Proof See the connection with [13, Ch.1 (II.4.6)]. Let $z \in R$ be such that $\Delta(h, u, z) = \Delta(h, u)$. Then $\delta(x) = \sum_{1 \leq i \leq 3} d_i$. Let $\mathcal{I} \subset \{1, 2, 3\}$ satisfy the two following conditions:

P (for permissibility): $\sum_{i \in \mathcal{I}} d_i \geq 1$,

M (for maximality): $|\mathcal{I}|$ minimal for P, i.e. the dimension of $\mathrm{V}(z, \langle u_i, i \in \mathcal{I}\rangle)$ is maximal for P.

Note that $\mathcal{I} \subset \{1, 2, 3\}$ is not unique in general. Then we choose $\mathcal{I}$ with PM and we blow up Z along $\mathrm{V}(z, \langle u_i, i \in \mathcal{I}\rangle)$. Let $e : Z' \longrightarrow Z$ denote this blowing up, X' be the strict transform of X, $x' \in X'$ be a point above x, $E' \subset Z'$ be the reduced inverse image of E.

We claim that for either $(m(x'), -\tau(x')) <_{lex} (m(x), -\tau(x))$ or

$$\big((m(x'), \tau(x')) = (m(x), \tau(x)) = (m(x), 1) \text{ and } \epsilon(x') = 0 \text{ and } \delta(x') < \delta(x)\big).$$

Since $\delta(h, u, y) \in \frac{1}{m!}\mathbb{N}$, a descending induction on $\delta(x)$ reduces to $m(x') < m(x)$ or $(m(x') = m(x), \tau(x') \geq 2)$. As stated in Sect. 2, this completes the proof.

Proof of Claim We only treat the case $\mathcal{I} = \{1, 2, 3\}$, the other cases being similar, if somewhat simpler. By PM, $\mathcal{I} = \{1, 2, 3\}$ means

$$d_i > 0, \ d_i + d_j < 1 \text{ when } i \neq j, \ 1 \leq i, j \leq 3. \tag{17}$$

By [18, thm. 3, p. 331], if $m(x') = m(x)$, then x' lies on the strict transform of $z = 0$. The variables u_1, u_2, u_3 play symmetric roles; so after reordering, it can be assumed that x' belongs to the affine chart $\mathrm{Spec} R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$. Let

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3/u_1).$$

Let

$$h = C_{0,m} z^m + \sum_{\frac{|A|}{\delta(x)} + b \geq m} C_{A,b} z^b u_1^{a_1} u_2{}^{a_2} u_3{}^{a_3},$$

be an expansion **3.1**(1) of h with (z, u) fully prepared (Theorem 3.5 and Definition 3.5.3),

$$h' := h/u_1^m = C_{0,m} z'^m + \sum_{\frac{|A|}{\delta(x)} + b = m} C_{A,b} z'^b u_1^{(m-b)(\delta(x)-1)} u'_2{}^{a_2} u'_3{}^{a_3} + h'_1 \tag{18}$$

where $h'_1 \in I'_{\delta(x)}{}^{+} := (z'^{m+1}, z'^b u_1^{(m-b)(\delta(x)-1)+1}, 0 \leq b \leq m)$.

Since $\epsilon(x) = 0$, $C_{A,b}$ invertible in (18) implies $a_2 = (m-b)d_2$, $a_3 = (m-b)d_3$. Note also that $\delta(x) - 1 = d_1 + d_2 + d_3 - 1 < d_1$ by (17). As $d_1 + d_2 < 1$ and $d_1 + d_3 < 1$, $m(x') = m(x)$ implies $x' = (z', u'_1, u'_2, u'_3)$. The coordinate change $(z, u_1, u_2, u_3) \mapsto (z', u'_1, u'_2, u'_3)$ is a monomial substitution, so $\Delta(h', u', z')$ is again minimal. With natural notations, we get $(d'_1, d'_2, d'_3) = (\delta(x) - 1, d_2, d_3)$, $\epsilon(x') = 0$ and

$$\delta(x') = d'_1 + d'_2 + d'_3 = \delta(x) - 1 + d_2 + d_3 < \delta(x)$$

provided $\langle \mathrm{in}_{x'}(h')\rangle = \langle Z'^2\rangle$, i.e. $\tau(x') = \tau(x) = 1$.

Theorem 3.10 *Given* $(y, u_1, u_2, u_3) =: (y, u)$ *and a divisor* $E \subset div(u_1u_2u_3) \subset Spec(R)$ *as above, assume that* E *is equicharacteristic* $p = \ chark(x)$ *and*

$$Sing_{m(x)}(X) \subset E.$$

Then the set $\{y \in X|m(y) = m(x), \epsilon(y) > 0 \ \ and \ \tau(y) = 1\}$ *is locally closed.*

Proof It is well known that the set

$$\{y \in X|m(y) = m(x), \tau(y) \geq 2\} \subseteq E$$

is locally closed. Suppose $\epsilon(x) = 0$ for some closed point $x \in E$. We choose a r.s.p. (z, u_1, u_2, u_3) of R at x which is fully prepared. There is a finite expansion

$$h = C_{0,m}z^m + \sum_{\frac{|A|}{\delta(x)} \geq b} C_{A,b}z^{m-b}u_1^{a_1}u_2{}^{a_2}u_3{}^{a_3} \tag{19}$$

with each $C_{A,b}$ invertible in R and $a_i \geq bd_i$, $i = 1, 2, 3$. Since $\epsilon(x) = 0$, there exists $C_{A,b}$ such that $a_i = bd_i$, $i = 1, 2, 3$, with $b \geq 2$ by full preparedness. Then the locus

$$\{y \in X|m(y) = m(x), \epsilon(y) = 0 \ \text{ or } \tau(y) \geq 2\}$$

contains the intersection of $\text{Sing}_{m(x)}(X)$ with the complement of the hypersurface $V(C_{A,b})$.

Theorem 3.11 *With assumptions as in* **3.10**, *assume furthermore* $\epsilon(x) > 0$ *and let* F *be an irreducible component of* E *with* $x \in F$. *Then*

$$dim(\{y \in X|m(y) = m(x), \epsilon(y) > 0 \ \ and \ \tau(y) = 1\} \cap F) \leq 1.$$

Proof Say $\text{div}(u_1)$ is the given component. If $d_1 < 1$, then $\dim(\{y \in X|m(y) = m(x)\} \cap F) \leq 1$ and the result is clear.

Assume now $d_1 > 1$ and pick a fully prepared (y, u_1) w.r.t. to datum $h \subset R_{y,u_1}$. There exists a nonempty Zariski open set $\Omega \subseteq F$ such that for $y \in \Omega$ there is an expansion

$$h = \gamma_0 z^m + \sum_{1 \leq i \leq m} \gamma_i z^{m-i}u_1^{a_i},$$

with $a_i \geq id_1$, $\gamma_i \in \mathcal{O}_{X,y}$ for $i \geq 1$. By definition of d_1, some $i \geq 2$ satisfies ($a_i = id_1$ and $\gamma_i \notin (u_1)$). By full preparedness, we have $\gamma_1 \in (u_1)$ if $a_1 = d_1$. Let Ω' be the intersection of Ω with complement of the proper closed subset $V(\gamma_i)$, so

$$\Omega' \subseteq \{y \in X|m(y) = m(x), \epsilon(y) = 0\}.$$

Assume finally $d_1 = 1$. The same construction now yields

$$\Omega' \subseteq \{y \in X|m(y) = m(x), \tau(y) \geq 2\}.$$

4 Construction of the divisor E

In this section, we reach the assumptions of **3.10**. We show they are stable under a class of local permissible blowing ups which we will prove ahead are sufficient to prove Theorem 1.3. We stick to Assumptions 3.4.1.

Proposition 4.1 *With assumptions as above, there exists a finite sequence of local blowing ups*

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n),$$

where $x_0 = x$, $x_i \in X_i$ (X_i denoting the strict transform of X), $0 \leq i \leq n$, is the center of v, each blowing up center $Y_i \subset X_i$ is permissible for X_i in Hironaka's sense, such that one the following properties holds:

(i) $(m(x_n), -\tau(x_n)) <_{lex} (m(x), -\tau(x))$, *or*
(ii) $(m(x_n), -\tau(x_n)) = (m(x), -\tau(x))$ *and there exists a (reduced) normal crossings divisor $E_n \subset (Z_n, x_n)$ of equicharacteristic $p = \text{char}k(x_n) = \text{char}k(x)$ such that*

$$C_{x_n}(E_n) \perp C_{x_n}(X_n)_{red}, \tag{20}$$
$$S_n := Sing_{m(x_n)} \subset E_n, \tag{21}$$

where C_{x_n} denotes the tangent cone and $Sing_{m(x_n)}(X_n)$ is the stalk at x_n of the set of multiplicity $m(x)$.

Proof We begin with the following lemma.

Lemma 4.2 *With assumptions as above, assume furthermore that there exists a normal crossings divisor $E \subset (Z, x)$ such that*

$$C_x(E) \perp C_x(X)_{red}. \tag{22}$$

Then for any local blowing up:

$$\pi : (Z', x') \longrightarrow (Z, x)$$

of center $Y \subset X$, permissible for (X, x) and at normal crossing with E, we have $(m(x'), \tau(x')) \leq (m(x), \tau(x))$, where $x' \in X'$ is the center of v; if equality holds, then

$$C_{x'}(E') \perp C_{x'}(X')_{red},$$

where $E' := \pi^{-1}(E)_{red}$, X' the strict transform of X.

Proof The normal crossing assumption implies that we can choose a r.s.p. (v_1, v_2, v_3, v_4) of $R := \mathcal{O}_{Z,x}$ such that $Y = V(v_1, \ldots, v_e)$ and $E \subseteq \text{div}(v_1 \cdots v_4)$. By permissibility, we have $h \in (v_1, \ldots, v_e)^{m(x)}$. Assumption (22) means that $\langle \text{in}_x(h) \rangle = \langle Z^d \rangle$, where $Z \notin \langle \text{in}_x(v_j), \ \text{div}(u_j) \subseteq E \rangle$. Changing generators of the ideal of Y, we relabel parameters as (z, u_1, u_2, u_3) with

$$E = \text{div}(u_1 \cdots u_d) \subset \text{div}(u_1 u_2 u_3), \ \ I(Y) = (z, \{u_i, \ i \in A\}) \text{ for some } A \subseteq \{1, 2, 3\}. \tag{23}$$

If $m(x') = m(x)$, x' belongs to the strict transform of div(z) by (22). Let $i \in A$ such that u_i generates the ideal of the exceptional divisor of π in a neighbourhood of x' and let $z' = z/u_i$. A local equation for (X', x') is $h' := h/u_i^{m(x)}$, where

$$h' \equiv \gamma z'^{m(x)} \text{ mod } (u_i, \{u_k, \ k \notin A\}),$$

and

$$E' \subset \text{div}\left(u_i \times \prod_{j \neq i, j \in A} \left(\frac{u_j}{u_i}\right) \prod_{k \notin A} u_k\right),$$

which proves the lemma.

Proof of 4.1 If $\mathrm{Sing}_{m(x)}(X)=\{x\}$, take $E=\mathrm{div}(u_1u_2u_3)$ with coordinates as in (23) above.

If $\dim(\mathrm{Sing}_{m(x)}(X))\geq 1$, then any regular closed set $Y\subset S:=S_0$ is permissible for X. In any case, we have $(m(x_n),-\tau(x_n))\leq(m(x),-\tau(x))$ since centers are permissible.

Let (C,x) be any curve contained in S. Since it is assumed that v has rank one, x_n does not belong to the strict transform C_n of C in X_n for $n\gg 0$ if we take $Y_i=\{x_i\}$, the center of v in X_i for $i\geq 0$. In particular, it can be assumed that the strict transform S_n of S in X_n has pure dimension two. Take $n=0$ in what follows.

We now apply classical embedded resolution theorems for S with $\dim(S)=2$ ([11] for suitable generality). This involves blowing up closed points or regular curves on the successive strict transforms of S. By blowing up finitely closed many points as before, it can be assumed that every blown up curve is equicharacteristic $p=\mathrm{char}k(x)$. We reach the following situation: the strict transform S_n of S at x_n is empty or an irreducible surface with normal crossings with the (equicharacteristic) reduced exceptional divisor E_n of $(Z_n,x_n)\rightarrow(Z,x)$. If S itself is equicharacteristic, enlarge E_n to $E_n\cup S$. Otherwise, we blow up finitely many times irreducible components of $S\cap E_n$ (i.e. equicharacteristic curves) to get $x_n\notin S_n$. This is possible again because v has rank one.

5 Refined directrix, transverseness, encombrement

Assume that the conclusion of Proposition 4.1 (ii) holds. We will perform local blowing ups which are permissible in Hironaka's sense, with center Y_n having normal crossing with E_n. Take $n=0$ in what follows, $E=E_0$, and consider a local blowing up:

$$\pi:(Z',x')\longrightarrow(Z,x)$$

of center $Y\subset X$, permissible for (X,x) and at normal crossing with E. We assume that

$$(m(x'),\tau(x'))=(m(x),\tau(x)),$$

where $x'\in X'$ is the center of v. By Lemma 4.2, we have

$$C_{x'}(E')\perp C_{x'}(X')_{red},$$

where $E':=\pi^{-1}(E)_{red}$, X' the strict transform of X in Z'.

Definition and Notation 5.1 *Let (z,u) be fully prepared with $E\subset div(u_1u_2u_3)\subset Spec(R)$ as above. Let*

$$F:=in_\delta(h)=Z^m+\sum_{2\leq j\leq m}Z^{m-j}F_j(U_1,U_2,U_3)\in gr_\delta(R)=k(x)[Z,U_1,U_2,U_3],$$

where $Z=in_\delta(z)$, $U_i=in_\delta(u_i)$, $1\leq i\leq 3$ (notations of **3.3.1***). Each F_j is zero or homogeneous of degree $j\delta(x)$; we have $F_j=0$ if $j\delta(x)\notin\mathbb{N}$.*

We define the refined tangent ideal *of X at x as the ideal*

$$I_x:=\left(Z,\prod_{div(u_i)\subset E}U_i^{-m!d_i}F_j^{\frac{m!}{j}},\ 1\leq j\leq m\right)\subset k(x)[Z,U_1,U_2,U_3].$$

We define the refined directrix *of X at x as the smallest vector subspace $\mathcal{V}\subseteq\langle U_1,U_2,U_3\rangle$ such that*

$$\{U_i^{-m!d_i}F_j^{\frac{m!}{j}}\mid 1\leq j\leq m,\ div(u_i)\subset E\}\subseteq k(x)[\mathcal{V}].$$

Let $\rho(x):=dim(\mathcal{V})$.

Remark 5.1.1 The following holds:

(i) $U_i^{\lceil jd_i \rceil}$ divides F_j, $2 \leq j \leq m$, for $1 \leq i \leq 3$ such that $\mathrm{div}(u_i) \subset E$.
(ii) I_x and $\mathcal{V}$ do not depend upon choices of z, u satisfying the assumptions.
(iii) the polynomials

$$U_i^{-m!d_i} F_j^{\frac{m!}{j}}, \quad 1 \leq j \leq m, \; 1 \leq i \leq 3$$

are zero or homogeneous of degree $m!\epsilon(x)$.

Statement (i) is a consequence of the definition of d_i. For (ii), suppose (z', u') is fully prepared, where $E \subset \div(u'_1 u'_2 u'_3)$. Let

$$u_j = a_{1,j}u'_1 + a_{2,j}u'_2 + a_{3,j}u'_3 + b_j z, \; a_{i,j}, b_j \in R, \; 1 \leq i, j \leq 3,$$

for some matrix $(a_{i,j}) \in \mathrm{GL}(3, R)$. Since $\tau(x) = 1$, we have

$$\deg U_j = \deg U'_j = \frac{1}{\delta(x)} < \deg Z = 1$$

in gr_δ. Computing w.r.t. the r.s.p. (z, u'), we get

$$\mathrm{in}_\delta(h) = Z^m + \sum_{2 \leq j \leq m} Z^{m-j} F'_j(U'_1, U'_2, U'_3) \in \mathrm{gr}_\delta(R) = k(x)[Z, U'_1, U'_2, U'_3],$$

with $F'_j(U'_1, U'_2, U'_3) = F_j(\overline{\mathcal{M}}.(U'_1, U'_2, U'_3))$, $\overline{\mathcal{M}}$ being the residue of $\mathcal{M}$ in $\mathrm{GL}(3, k(x))$. Since (z', u') is fully prepared, no term in Z^{m-1} occurs in $\mathrm{in}_\delta(h) \in k(x)[Z', U'_1, U'_2, U'_3]$ and this implies that $\langle \mathrm{in}_\delta(Z)\rangle = \langle \mathrm{in}_\delta(Z')\rangle$.

Statement (iii) immediately follows from definition **3.7**.

Definition 5.2 Let E be a fixed normal crossings divisor and (z, u) be fully prepared (always with the condition $E \subset \mathrm{div}(u_1u_2u_3) \subset \mathrm{Spec}(R)$) as above.

We call "transverseness" index of x, denoted by $t(x)$, the maximal dimension of a subspace of $\mathcal{V}$ which is transverse to $\langle U_i, \; \mathrm{div}(u_i) \subset E\rangle$. This is independent of the choice of a fully prepared r.s.p. (z, u) by Remark 5.1.1(ii).

We call "encombrement" of x, denoted by $e(x)$, the minimum number of U_i's among all possible fully prepared (z, u) necessary to write a basis of $\mathcal{V}$.

We define an invariant

$$\iota(x) := (m(x), -\tau(x), \epsilon(x), -\rho(x), -t(x), -e(x)) \in \mathbb{N} \\ \times \{-4, -3, -2, -1\} \times \mathbb{Q}_{\geq 0} \times \{-3, -2, -1, 0\}^3.$$

For convenience, we extend the definition when $\tau(x) \geq 2$ by letting $\iota(x) := (m(x), -\tau(x), 0, 0, 0)$, Theorem 1.3 being already proved in this special case (Sect. 2). Note that $\epsilon(x) \in \frac{1}{m(x)!}\mathbb{N}$, so any decreasing sequence of values (for the lexicographical ordering) taken by ι is finite.

Example 5.2.1 Assume $p \geq 5$.

(i) $h = z^3 + u_1^2(u_1 + u_2 + u_3)^2$, $E = \mathrm{div}(u_1)$. Then $\delta(x) = \frac{4}{3}, d_1(x) = \frac{2}{3}, \mathcal{V} = \langle U_1 + U_2 + U_3\rangle$, $\rho(x) = 1$, $t(x) = 1$, $e(x) = 1$: take the r.s.p. $(z, u_1, u_1 + u_2 + u_3, u_3)$.
(ii) $h = z^3 + u_1^2(u_1 + u_2 + u_3)^2$, $E = \mathrm{div}(u_1u_2u_3)$. Then $\delta(x) = \frac{4}{3}$, $d_1(x) = \frac{2}{3}$, $d_2(x) = d_3(x) = 0$, $\mathcal{V} = \langle U_1 + U_2 + U_3\rangle$, $\rho(x) = 1$, $t(x) = 0$, $e(x) = 3$: the only choice allowed upon (u_1, u_2, u_3) is permuting or multiplying by a unit in R.

(iii) $h = z^3 + u_1^2(u_1^2 + u_2^2)$, $E =$ div(u_1). Then $\delta(x) = \frac{4}{3}$, $d_1(x) = \frac{2}{3}$, $\mathcal{V} = \langle U_1, U_2\rangle$, $\rho(x) = 2$, $t(x) = 1$, $e(x) = 2$.
(iv) $h = z^3 + u_1^2(u_1^2 + u_2^2)$, $E =$ div(u_1u_2). Then $\delta(x) = \frac{4}{3}$, $d_1(x) = \frac{2}{3}$, $d_2(x) = 0$, $\mathcal{V} = \langle U_1, U_2\rangle$, $\rho(x) = 2$, $t(x) = 0$, $e(x) = 2$.

Remark 5.2.2 The French "encombrement" was proposed by J. Giraud twenty years ago (English: "cumbersomeness index" roughly).

6 Permissible blowing ups, behaviour of the invariants

We stick to the assumptions of the previous section and assume furthermore that $\epsilon(x) > 0$.

Definition 6.1 An ϵ-permissible center (permissible center for short) Y at x is one of the following:

(i) either $Y := \{x\} = V(z, u_1, u_2, u_3)$,
(ii) or $Y := V(z, u_1, u_2)$ with (z, u) fully prepared,

$$d_1 + d_2 + \epsilon(x) \geq 1 \tag{24}$$

and

$$l(h, u, z) = m, \tag{25}$$

where L denotes the linear form $L(x_1, x_2, x_3) = \frac{x_1+x_2}{d_1+d_2+\epsilon(x)}$ (Definition 3.2).

Proposition 6.2 *An ϵ-permissible center at x is permissible in Hironaka's sense.*

Proof Indeed, we have just to look at the case of a curve V(z, u_1, u_2). In that latter case, as $d_1 + d_2 + \epsilon(x) \geq 1$, we have

$$\text{ord}_\eta(h) \geq v_{L,h,u,z}(h) = m,$$

where η is the generic point of V(z, u_1, u_2), so

$$\text{ord}_\eta(h) = m,$$

which means exactly that V(z, u_1, u_2) is permissible in Hironaka's sense.
‘

Proposition 6.3 *Let $\pi : Z' \longrightarrow Z$ be the blowing up along an ϵ-permissible center Y at x, X' be the strict transform of X (with transformed equation h' at the center $x' \in X'$ of v). We have:*

(i) *$\iota(x') \leq \iota(x)$ (Definition 5.2). If equality holds (in which case we say that x' is "very near" x), then $E' := \pi^{-1}(E)_{red}$ is transverse to the directrix $\mathcal{T}'$ of X' at x'.*
(ii) *if $Y = \{x\}$ and $(m(x'), -\tau(x'), \epsilon(x')) = (m(x), -\tau(x), \epsilon(x))$, then x' lies on*

$$Proj k(x)[Z, U_1, U_2, U_3]/(Z, \mathcal{V}) \subset Proj(gr_{\mathfrak{M}}(R)) = \mathbb{P}^3_{k(x)}$$

with notations as in **5.1**. *The refined directrix $\mathcal{V}'$ at x' satisfies $\mathcal{V}' \equiv U^{-1}\mathcal{V} \mod\langle U\rangle$ where $U =$ in$_{x'}(u)$, u an equation of the exceptional divisor of π.*

Proof First assume that $Y = \{x\}$. By [18, thm.3, p. 331], if $m(x') = m(x)$, then x' lies on the strict transform of $z = 0$. The variables u_1, u_2, u_3 play symmetric roles; o after reordering, it can be assumed that x' belongs to the affine chart $\mathrm{Spec}R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$. Let

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3/u_1).$$

We can choose a r.s.p. at x' in the following way: if x' is the origin, take (z', u'_1, u'_2, u'_3); f x' belongs to the strict transform of, say $\mathrm{div}(u_2)$, we can take (z', u'_1, u'_2, v_3) with $v_3 = \sum_a \lambda_a {u'_3}^a$, $\lambda_a \in R$ a unit or zero (the sum is finite) whose residue $\sum_a \overline{\lambda_a} {U_3}^a \in k(x)[U_3]$ is an irreducible polynomial; in the general case $u'_2(x')u'_3(x') \neq 0$, we take (z', u'_1, v_2, v_3) where $v_c = \sum_{a,b} \lambda_{a,b,c} {u'_2}^a {u'_3}^b$, $c = 2, 3$ (sums are finite), $\lambda_{a,b,c} \in R$ a unit or zero, and

$$\left\langle \sum_{a,b} \overline{\lambda_{a,b,c}} {U_2}^a {U_3}^b,\ c = 1, 2 \right\rangle \subset k(x)[U_2, U_3]$$

is a maximal ideal. Let

$$\begin{aligned} F &:= Z^m + \sum_{2 \le j \le m} Z^{m-j} U_1^{a(1,j)} U_2^{a(2,j)} U_3^{a(3,j)} G_j(U_1, U_2, U_3) \\ &:= Z^m + \sum_{2 \le j \le m} Z^{m-j} F_j(U_1, U_2, U_3) := \mathrm{in}_\delta(h) \in \mathrm{gr}_\delta(R) = k(x)[Z, U_1, U_2, U_3], \end{aligned} \tag{26}$$

with $a(i, j) \ge j d_i$, $2 \le j \le m$, $1 \le i \le 3$, $G_j \in k(x)[U_1, U_2, U_3]$ homogeneous, $G_j = 0$ or $\deg(G_j) = j\delta(x) - (a(1, j) + a(2, j) + a(3, j))$ and G_j not divisible by U_i, $1 \le i \le 3$.

Let $h' := h/u_1^m$ define the strict transform of h. We define the linear form

$$L'(x'_1, x'_2, x'_3) := \frac{x'_1}{\delta(x) - 1}$$

with associated valuation $v' := v_{L',h',u',z'}$ (Definition 3.2). We have

$$\mathrm{in}_{v'}(h') = Z'^m + \sum_{2 \le j \le m} Z'^{m-j} {U'_1}^{j(\delta(x)-1)} {u'_2}^{a(2,j)} {u'_3}^{a(3,j)} G_j(1, u'_2, u'_3) \in \mathrm{gr}_{v'}(R'), \tag{27}$$

where $\mathrm{gr}_{v'}(R') = R'/(z', u'_1)[Z', U'_1]$. Here, the meaning of $G_j(1, u'_2, u'_3)$ is given by the inclusion

$$k(x) = R/\mathfrak{M} \to R'/(u'_1) \to R'/(z', u'_1).$$

By (27), $x'_1 = \delta(x) - 1$ is the minimum value of the first coordinate of points in $\Delta(h', u'_1, v_2, v_3, z')$. Since z is δ-prepared, no vertex of $\Delta(h', u'_1, v_2, v_3, z')$ with first coordinate equal to $x'_1 = \delta(x) - 1$ is solvable. We get

$$d_1(x') = \delta(x) - 1,$$

and for at least one vertex $(x'_1 = \delta(x) - 1, x'_2, x'_3)$, we have

$$x'_2 + x'_3 \le \min \left\{ \frac{\mathrm{ord}_{x'}({u'_2}^{a(2,j)} {u'_3}^{a(3,j)} G_j(1, u'_2, u'_3))}{j}, 2 \le j \le m \right\}.$$

In case x' belongs to the strict transform of some $\mathrm{div}(u_i), i = 2, 3$, we have $d_i(x') = d_i(x)$ for $u'_i(x') = 0$ by **3.6**(iii). This leads to:

$$\epsilon(x') \leq \min \left\{ \frac{\mathrm{ord}_{x'}(u_2'^{a(2,j)-jd_2} u_3'^{a(3,j)-jd_3} G_j(1, u'_2, u'_3))}{j}, 2 \leq j \leq m \right\} \leq \epsilon(x) \quad (28)$$

with the convention: $\mathrm{ord}_{x'}(u_i'^a) = 0$ when $a \in \mathbb{Q}_+$ and $u'_i(x') \neq 0$, $\mathrm{ord}_{x'}(u_i'^a) = a$ when $u'_i(x') = 0$. This proves $(-\tau(x'), \epsilon(x')) \leq (-\tau(x), \epsilon(x))$.

Assume that $(-\tau(x'), \epsilon(x')) = (-\tau(x), \epsilon(x))$. Then

$$\mathrm{ord}_{x'} \left(\prod_{\mathrm{div}(u_i) \subset E} u_i'^{-m!(a(i,j)-jd_i)} F_j(1, u'_2, u'_3)^{\frac{m!}{j}} \right)$$
$$= \deg \left(\prod_{\mathrm{div}(u_i) \subset E} U_i^{-m!(a(i,j)-jd_i)} F_j^{\frac{m!}{j}} \right)$$

for each j with $F_j \neq 0$. By [18, Theorems 2 and 3], this means that x' lies on

$$\mathrm{Proj} k(x)[Z, U_1, U_2, U_3]/(Z, \mathcal{V}) \subset \mathrm{Proj} k(x)[Z, U_1, U_2, U_3] = \mathbb{P}^3_{k(x)}. \quad (29)$$

This proves the first assertion of (ii) in this case. All other assertions are easy consequences of (28) and of its explicitation (29).

Assume now that $u'_2(x')u'_3(x') \neq 0$. If x' is rational over x, i.e. $u'_2(x') = \lambda \in k(x)$, $u'_3(x') = \mu \in k(x)$, we have

$$I_{x'} \equiv (Z', G_j(1, V_2 - \lambda, V_3 - \mu)^{\frac{m!}{j}}, 2 \leq j \leq m) \bmod (U'_1), \quad (30)$$

where $I_{x'}$ is the refined tangent ideal of x' (cf. **5.1**(ii)). This proves the last assertion of (ii) in this case. Finally, if x' is not rational over x, then $\dim(\mathcal{V}) = 1$. We get

$$\mathcal{V} = \langle aU_1 + bU_2 + cU_3 \rangle,\ a, b, c \in k(x),\ (b, c) \neq (0, 0).$$

If $b \neq 0$, we take $v_2 := a + bu'_2 + cu'_3 \mathrm{mod}(u_1)$ and we get by **5.1.1**

$$I_{x'} = (Z', V_2^{m!\epsilon(x)}) \mod (U'_1), \quad (31)$$

which proves the last assertion of (ii) in this case. All other assertions are easy as in the previous case.

We now consider blowing up along a curve $Y = V(z, u_1, u_2)$.

By [18, thm.3, p. 331], if $m(x') = m(x)$, then x' lies on the strict transform of $z = 0$. The variables u_1, u_2 play symmetric roles; so after reordering, it can be assumed that x' belongs to the affine chart $\mathrm{Spec} R[z/u_1, u_2/u_1, u_3] \subset Z'$. Let

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3).$$

We can choose a r.s.p. at x' in the following way: if x' is the origin, take (z', u'_1, u'_2, u'_3); otherwise take $(z', u'_1, v_2, \underline{u'_3})$ where $v_v = \sum_a \lambda_a u_3'^a$, $\lambda_a \in R$ a unit or zero (the sum is finite) whose residue $\sum_a \overline{\lambda_a} U_3{}^a \in k(x)[U_3]$ is an irreducible polynomial.

With notations (26), since $\mathrm{V}(z, u_1, u_2)$ is ϵ-permissible, we have

$$a(3, j) = jd_3,\ G_j \in k(x)[U_1, U_2].$$

Let $h' := h/u_1^m$ define the strict transform of h. Equation (27) gets replaced by

$$\begin{aligned} \mathrm{in}_{v'}(h') &= \Gamma_0 Z'^m + \sum_{2\leq j\leq m} Z'^{m-j} U_1'^{j(\delta(x)-1-d_3)} u_2'^{a_2} u_3'^{jd_3} \Gamma_j \in \mathrm{gr}_{v'}(R') \\ &= R'/(z', u_1')[Z', U_1'], \end{aligned} \tag{27*}$$

with $\Gamma_j \in R'/(z', u_1')$ whose residue in $R'/(z', u_1', u_3')$ is $G_j(1, u_2')$ for $2 \leq j \leq m$, Γ_0 a unit.

By (27*), $\delta(x) - 1 - d_3$ is the minimum value of the first coordinate of points in $\Delta(h', u_1', v_2, u_3', z')$. As in (26) there is no Z'^{m-1}, each vertex of $\Delta(h', u_1', v_2, u_3', z')$ with first coordinate equal to $\delta(x) - 1 - d_3$ is not solvable. Then

$$d_1(x') = \delta(x) - 1 - d_3,$$

and for at least one vertex $(x_1' = \delta(x) - 1 - d_3, x_2', x_3')$, we have

$$x_2' + x_3' \leq \min\left\{ \frac{\mathrm{ord}_{x'}(u_2'^{a_2} u_3'^{a_3} G_j(1, u_2'))}{j},\ 2 \leq j \leq m \right\}.$$

By Theorem 3.6(iii), we have $d_3(x') = d_3$. If $u_2'(x') \neq 0$, this gives

$$\epsilon(x') \leq \min\left\{ \frac{\mathrm{ord}_{x'}(u_3'^{a_3-d_3} G_j(1, u_2'))}{j},\ 2 \leq j \leq m \right\} \leq \epsilon(x).$$

If $u_2'(x') = 0$, we also have $d_2(x') = d_2$ by **3.6**(iii). This leads to:

$$\epsilon(x') \leq \ \mathrm{inh}\left\{ \frac{\mathrm{ord}_{x'}(u_2'^{a_2-d_2} u_3'^{a_3-d_3} G_j(1, u_2'))}{j}, 2 \leq j \leq m \right\} \leq \epsilon(x)$$

with the convention: $\mathrm{ord}_{x'}(u_i'^{a}) = 0$ when $a \in \mathbb{Q}^+$ and $u_i'(x') \neq 0$, $\mathrm{ord}_{x'}(u_i'^{a}) = a$ when $u_i'(x') = 0$.

If $\epsilon(x') = \epsilon(x)$, then

$$\mathrm{ord}_{x'}\left(\prod_{\mathrm{div}(u_i)\subset E} u_i'^{-m!(a(i,j)-jd_i)} G_j(1, u_2')^{\frac{m!}{j}} \right) = \ \deg\left(\prod_{\mathrm{div}(u_i)\subset E} U_i^{-m!(a(i,j)-jd_i)} G_j^{\frac{m!}{j}} \right)$$

for each j with $G_j \neq 0$. By [18, Theorems 2 and 3], this means that x' lies on

$$\mathrm{Proj} k(x)[Z, U_1, U_2]/(Z, \mathcal{V}) \subset \ \mathrm{Proj} k(x)[Z, U_1, U_2], \tag{29*}$$

where the latter is identified with $\pi^{-1}(x) \subset Z'$. The proof now runs parallel to the case $Y = \{x\}$.

Proposition 6.4 *With assumption as in* **6.3**, *assume $e(x) = 3$ and $Y = \{x\}$. Then x' is not very near x, i.e. $\iota(x') < \iota(x)$ (Definition 5.2).*

Proof If $\rho(x) = 3$, this follows from **6.3**(ii), since $\mathrm{Proj} k(x)[Z, U_1, U_2, U_3]/(Z, \mathcal{V}) = \emptyset$.

When $\rho(x) = 2$, we have $t(x) < 2$ necessarily: otherwise we should have

$$E \subseteq \ \mathrm{div}(u_1),\ \mathcal{V} = \langle U_2, U_3 \rangle \ \mathrm{mod}\ (U_1).$$

By a linear change on the free variables (u_2, u_3), we would get

$$\mathcal{V} = \langle U_2, U_3 \rangle,$$

i.e. $e(x) = 2$, a contradiction.

When $\rho(x) = 2, t(x) = 1$, we can choose parameters such that

$$E = \operatorname{div}(u_1u_2),\ \mathcal{V} = \langle U_3, \alpha U_1 + U_2\rangle,\ \alpha \in k(x)^\times.$$

By Proposition 6.3(ii), we have

$$\mathcal{V}' \equiv \langle U_3', V_2'\rangle \mod (U_1'),$$

with $E' = \operatorname{div}(u_1')$, $V_2' = \operatorname{in}_{x'}(a + u_2/u_1)$, where $a \in R$ is a preimage of α. Then $\rho(x') \geq t(x') \geq 2$.

When $\rho(x) = 2$, $t(x) = 0$, then, up to a permutation on u_1, u_2, u_3, we have $E = \operatorname{div}(u_1u_2u_3)$ and

$$\mathcal{V} = \langle U_1 + \alpha U_2, \beta U_2 + U_3, \rangle,\ \alpha \in k(x)^\times,\ \beta \in k(x).$$

By Proposition 6.3(ii), we can take $\pi^{-1}(x) = \operatorname{div}(u_2)$ locally at x', and r.s.p.

$$(z', v_1', u_2', v_3') := (z/u_2, u_1/u_2 + a, u_2, u_3/u_2 + b),$$

where $a, b \in R$ are preimages of α, β. In particular we get $E \subseteq \operatorname{div}(u_2'u_3')$. On the other hand, we have

$$\mathcal{V}' \equiv \langle V_1', V_3'\rangle \bmod (U_2'),$$

and this proves that $t(x') \geq 1$.

When $\rho(x) = 1$, then $e(x) = 3$ implies $E = \operatorname{div}(u_1u_2u_3)$ (so $t(x) = 0$) and

$$\mathcal{V} = \langle \alpha U_1 + \beta U_2 + U_3, \rangle,\ \alpha, \beta \in k(x)^\times$$

up to renumbering parameters. By Proposition 6.3(ii), we can choose $E' \subseteq \operatorname{div}(u_1'u_2')$, say $\pi^{-1}(x) = \operatorname{div}(u_1)$ locally at x' and r.s.p.

$$(z', u_1', v_2', v_3') := (z/u_1, u_1, v_2', a + bu_2/u_1 + u_3/u_1),$$

at x', where $a, b \in R$ are preimages of α, β. Since $V_3' \in \mathcal{V}' \oplus \langle U_1\rangle$, we get $t(x') \geq 1$.

Proposition 6.5 *Let x satisfy the conclusion of Proposition 4.1*(ii) *and assume $e(x) = 2$. There exists a finite sequence of local blowing ups*

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n), \tag{32}$$

where $x_0 = x$, $x_i \in X_i$ (X_i denoting the strict transform of X), $0 \leq i \leq n$, is the center of v, each blowing up center is $Y_i = \{x_i\}$ such that $\iota(x') < \iota(x)$.

Proof First assume that $\rho(x) = 1$. Then $t(x) = 0$, $\operatorname{div}(u_2u_3) \subseteq E$ and we have $\mathcal{V} = \langle \alpha U_2 + U_3\rangle, \alpha \in k(x)^\times$ after possibly renumbering parameters. If x' does not belong to the strict transform of $\operatorname{div}(u_2)$, we can take $\pi^{-1}(x) = \operatorname{div}(u_2)$ locally at x', and r.s.p.

$$(z', v_1', u_2', v_3') := (z/u_2, v_1', u_2, a + u_3/u_2),$$

where $a \in R$ is a preimage of α. In particular we get $E' \subseteq \operatorname{div}(u_1'u_2')$, with $u_1' = u_1/u_2$. On the other hand, we have

$$\mathcal{V}' \equiv \langle V_3'\rangle \mod (U_2'),$$

whence $t(x') \geq 1$, so $\iota(x') < \iota(x)$. Assume now that x' belongs to the strict transform of $\operatorname{div}(u_2)$. We can take $\pi^{-1}(x) = \operatorname{div}(u_1)$ locally at x', and r.s.p.

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3/u_1).$$

We get $E = \operatorname{div}(u'_1 u'_2 u'_3)$ and

$$\mathcal{V}' \equiv \langle \alpha U'_2 + U'_3 \rangle \mod (U'_1).$$

If $\iota(x') = \iota(x)$, then $\mathcal{V}' = \langle \alpha U'_2 + U'_3 \rangle$ and iterate. Since the valuation v has rank one, say $v(u_2) < nv(u_1)$ for some $n > 0$, the process stops after iterating n times.

Assume that $\rho(x) = 2$. Then $\mathcal{V} = \langle U_2, U_3 \rangle$ after possibly renumbering parameters. We can take $\pi^{-1}(x) = \operatorname{div}(u_1)$ locally at x' and r.s.p.

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3/u_1).$$

We get $\operatorname{div}(u'_1) \subseteq E'$ and

$$\mathcal{V}' \equiv \langle U'_2, U'_3 \rangle \mod (U'_1),$$

hence $\langle U'_2 + \alpha_2 U'_1, U'_3 + \alpha_3 U'_1 \rangle \subseteq \mathcal{V}'$ for some $\alpha_2, \alpha_3 \in k(x)$. If $\iota(x') = \iota(x)$, then equality holds; moreover $\alpha_i = 0$ whenever $\operatorname{div}(u_i) \subseteq E$, $i = 2$ or $i = 3$.

Iterating, there exists a *regular* formal curve $\widehat{Y} \subset X$ passing through all points $x, x_1 := x', \ldots, x_n$, taking $Y_i = \{x_i\}$ for each $i \geq 0$. By standard arguments, $\widehat{Y} \subseteq \operatorname{Sing}_{m(x)}(X)$. Our assumptions (beginning of Sect. 5) force $\widehat{Y} \subset E$, say $\widehat{Y} \subset \operatorname{div}(u_2)$. One concludes as in the case $\rho(x) = 1$.

7 Proof of the main theorem

By Theorem 3.9, a reduction in $m = m(x)$ can be achieved when $\epsilon(x, E) = 0$ for some normal crossings divisor $E \subseteq \operatorname{div}(u_1 u_2 u_3)$. The previous section (Propositions 6.4 and 6.5) reduces Theorem 1.3 to the only case $\epsilon(x) > 0$, $e(x) = 1$. There remains to prove the following:

Theorem 7.1 *Let x satisfy the conclusion of Proposition 4.1*(ii) (*w.r.t.* $E \subseteq$ *div*$(u_1 u_2 u_3)$) *and assume* $\epsilon(x) > 0$, $e(x) = 1$. *There exists a finite sequence of local blowing ups*

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n), \tag{33}$$

where $x_0 = x$, $x_i \in X_i$ (X_i denoting the strict transform of X), $0 \leq i \leq n$, is the center of v, each blowing up center $Y_i \subset X_i$ is permissible in Hironaka's sense, such that

(i) $\iota(x_n) < \iota(x)$, *and*
(ii) x_n *satisfies the conclusion of Proposition 4.1*(ii) (*w.r.t. the strict transform E_n of E in Z_n*) *if* $((m(x_n), \tau(x_n)) = (m(x), \tau(x))$ *and* $\epsilon(x_n) > 0$).

The proof is long, needing new invariants and the control of their behavior under permissible blowing ups. There are two different cases:

(i) $t(x) = 0$, $e(x) = 1$,
(ii) $t(x) = e(x) = 1$.

In both cases, we choose the indices so that $\mathcal{V} = \langle U_3 \rangle$. We assume that
(P1) (z, u) is fully prepared, and
(P2) $E \subseteq \operatorname{div}(u_1 u_2 u_3)$.

7.2 A new invariant B, preparation of the free variable (case (ii)).

Let us remind the convention $d_i(x) = 0$ for $\mathrm{div}(u_i) \not\subset E$, $1 \leq i \leq 3$. In particular, $d_3(x) = 0$ in case (ii). For $B \in \mathbb{Q}_+$, define the monomial valuation v_B by

$$v_B(z) = 1,\ v_B(u_3) := \frac{1}{\epsilon(x) + d_3 + \frac{d_1+d_2}{B}} =: Bv_B(u_1) = Bv_B(u_2).$$

We choose $B \in \mathbb{N} \cup \{+\infty\}$ maximal such that, up to the multiplication by an element of $k(x)^{\times}$, $\mathrm{in}_{v_B}(h)$ takes the following form:

$$\mathrm{in}_{v_B}(h) = Z^m + \sum_{1\leq j\leq m} Z^{m-j}\Phi_j(U_1, U_2, U_3),$$

with

$$\begin{aligned} \deg_{U_3}(\Phi_j) &\leq j(d_3 + \epsilon(x)), \quad 1 \leq j \leq m,\\ \Phi_j(U_1, U_2, U_3) &= U_1^{jd_1} U_2^{jd_2} U_3^{jd_3}(\lambda_j U_3^{j\epsilon(x)} + P_j(U_1, U_2, U_3)) \end{aligned} \tag{34}$$

with $\lambda_j \in k(x)$ and $(jd_1, jd_2, j\epsilon(x)) \in \mathbb{N}^3$ whenever equality holds; furthermore, equality holds for some j, $2 \leq j \leq m$.

Note that we necessarily have $\Phi_j(U_1, U_2, U_3) \neq \lambda_j U_3^{j\epsilon(x)}$ for some j if $B < +\infty$, since B is taken to be maximal. Moreover, since (z, u) is fully prepared and $\mathcal{V} = \langle U_3\rangle$, we necessarily have $B \geq 1$ and $\deg_{U_3}(\Phi_1) < d_3 + \epsilon(x)$.

This construction builds up a face of $\Delta(h, u, z)$ with equation

$$\frac{x_1 + x_2}{B(\epsilon(x) + d_3) + d_1 + d_2} + \frac{x_3}{(\epsilon(x) + d_3 + \frac{d_1+d_2}{B})} = 1,$$

for some B which contains the point $\mathbf{x} := (d_1, d_2, \epsilon(x) + d_3)$ and at least another point.

Let p be the projection

$$p : \mathbb{R}^3 - \{\mathbf{x}\} \longrightarrow \{x_3 = 0\}.$$

For analytic computations, note that if $M = z^{m-j}u_1^{a_1}u_2^{a_2}{u_3}^{a_3}$ is a monomial appearing with nonzero coefficient in some expansion **3.1**(1) of h and $j \geq 1$, then M defines the point $\mathbf{x}_M$:

$$M = z^{m-j}u_1^{a_1}u_2^{a_2}{u_3}^{a_3} \leftrightarrow \mathbf{x}_M = \left(\frac{a_1}{j}, \frac{a_2}{j}, \frac{a_3}{j}\right) \in \Delta(h, u, z),$$

and

$$p(\mathbf{x}_M) = \left(d_1 + \frac{\frac{a_1}{j} - d_1}{d_3 + \epsilon(x) - \frac{a_3}{j}}, d_2 + \frac{\frac{a_2}{j} - d_2}{d_3 + \epsilon(x) - \frac{a_3}{j}}\right). \tag{35}$$

Then $B + d_1 + d_2$ is the minimum value $x_1 + x_2$ for points in $p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\})$.

We define $\Delta_2(h; u_1, u_2; u_3) \subseteq (\mathbb{R}_+)^2$ by the formula

$$(d_1, d_2) + \Delta_2(h; u_1, u_2; u_3) := p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\}).$$

The main idea is that $\Delta_2(h; u_1, u_2; u_3)$ acts as the characteristic polyhedron of a surface singularity and in the following, we mimic [2,11], all these following Hironaka.

In case (ii) ($\mathrm{div}(u_3) \not\subset E$), we will require two extra conditions (to be achieved in **7.3** below by possibly changing u_3):

(P3) there is no homogeneous $P \in k(x)[U_1, U_2]$, $P \neq 0$, such that

$$\mathrm{in}_{v_B}(h) = Z^m + \sum_{2 \leq j \leq m} Z^{m-j} \lambda_j U_1^{jd_1} U_2^{jd_2} (U_3 + P(U_1, U_2))^{j\epsilon(x)}, \tag{36}$$

with the convention $\lambda_j = 0$ when $(jd_1, jd_2, j\epsilon(x)) \notin \mathbb{N}^3$;

(P4) if $B < +\infty$, let $\mathbf{x}_2 = (d_1 + A(1), \beta + d_2)$ be the vertex of $\Delta_2(h; u_1, u_2; u_3)$ with minimal first coordinate. Then $\mathbf{x}_2$ does not vanish by changing u_3 to $u_3 + \gamma u_1^\alpha u_2^\beta$, $\gamma \in R$, γ invertible.

Proposition 7.3 *With assumptions as above, there exist* (z, u_1, u_2, u_3), $z, u_3 \in \widehat{R}$ *such that (P1)(P2) and (P3)(P4) (in case (ii) with* $B < \infty$*) are satisfied.*

Proof The conditions (P1) (P2) can be achieved easily. If (P3) or (P4) is not achieved, we make a translation on u_3: we replace u_3 by $u_3 + \sum_{a_1,a_2} \gamma_{a_1,a_2} u_1^{a_1} u_2^{a_2}$, $\gamma_{a_1,a_2} \in R$, $(a_1, a_2) \in p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\})$. To achieve (P3), we take

$$\sum_{a_1,a_2} \overline{\gamma_{a_1,a_2}} U_1^{a_1} U_2^{a_2} := P(U_1, U_2),$$

$P(U_1, U_2)$ as in (36), which makes B increase if (P3) is not achieved. To achieve (P4) we change u_3 to $u_3 + \gamma u_1^\alpha u_2^\beta$ as in (P4)), which makes $(A(1), \beta)$ strictly increase for the lexicographical ordering.

In both cases, this translation makes $\Delta_2(h; u_1, u_2; u_3)$ smaller. These translations may spoil (P1), so each must be followed by a translation on z to get again (P1). This translation makes $\Delta_2(h; u_1, u_2; u_3)$ not bigger. The process may be infinite, but since $\Delta_2(h; u_1, u_2; u_3)$ gets smaller at each step, this converges to some $z, u_3 \in \widehat{R}$.

Definition 7.3.1 With assumptions as above, a r.s.p. (z, u_1, u_2, u_3), $z, u_3 \in \widehat{R}$ such that (P1)(P2) and (P3)(P4) (in case (ii) with $B < \infty$) are satisfied is said to be well prepared. For such (z, u_1, u_2, u_3), the number B defined above is denoted by $B(z, u_1, u_2, u_3)$ or $B(x)$ for short, even if it may depend on the choice of (z, u_1, u_2, u_3).

7.4. We begin the proof of Theorem VI.1 by the special case $B(x) = \infty$.

When $B(z, u_1, u_2, u_3) = \infty$, $\Delta(h, u, z)$ has only one vertex with coordinates $(d_1, d_2, \epsilon(x) + d_3)$. Since $\epsilon(x) > 0$, we have $\mathrm{div}(u_3) \not\subset E$, hence $t(x) = 1$ (case (ii)) and $E \subseteq \mathrm{div}(u_1 u_2)$, $d_3 = 0$. The proof is a variation of that of Theorem 3.9, checking carefully the algebraicity of the blowing up centers.

It has been assumed from Sect. 5 on that $\mathrm{Sing}_m(X) \subseteq E$, so $\epsilon(x, E) < 1$ necessarily since $V(z, u_3) \subseteq \mathrm{Sing}_m(X)$ otherwise.

By blowing up the surfaces $V(z, u_i)$, $\mathrm{div}(u_i) \subseteq E$, it can be assumed w.l.o.g. that $d_i < 1$. Similarly, it can be assumed that $d_1 + d_2 < 1$ by blowing up $V(z, u_1, u_2)$.

Assume that $V(z, u_i, u_3) \subseteq \mathrm{Sing}_m(X) \subseteq E$, i.e. $d_i + \epsilon(x) \geq 1$, $i = 1$ or $i = 2$. Then $C_i := V(z, u_i, u_3)$ is a *formal* irreducible component of $\mathrm{Sing}_m(X)$. By excellence, its Zariski closure $\overline{C_i}$ is a curve on X. On the other hand, $\overline{C_i}$ is contained in $V(z, u_i)$, so C_i itself is a curve on X. By blowing up C_i, we may assume that $d_i + \epsilon(x) < 1$, $i = 1, 2$.

At this point, we have reached the situation of Theorem 3.9(17) and the proof therein extends without changes: we eventually get reduction in $(m(x), \tau(x))$ by blowing up closed points. We observe that Theorem 7.1 can also be phrased as follows in this case: E_n can be enlarged to a new normal crossings divisor F_n such that $\epsilon(x_n, F_n) = 0$.

From now on, we assume that

$$e(x) = 1, \; B(x) < \infty. \tag{Hyp}$$

Definition 7.4.1 *(New invariants)* We define A_1, β by: $(d_1 + A_1, \beta + d_2)$ is the vertex of minimal first coordinate of

$$p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\}).$$

We define A_2 by: $d_2 + A_2$ is the minimal second coordinate of the points of

$$p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\}).$$

We define $C(u, z)$ (or $C(x)$ for short) by;

$$C(u, z) = B(u, z) - A_1 - A_2.$$

Finally, we define $\gamma(u, z)$ (or $\gamma(x)$ for short) as follows:

(i) $\gamma(u, z) := \lceil \beta(u, z) \rceil \geq 0$ if ($E \subseteq$ div(u_1) and $t(x) = 1$);
(ii) $\gamma(u, z) := \lceil \beta(u, z) \rceil \geq 0$ if ($E \subseteq$ div$(u_1 u_3)$ and $t(x) = 0$);
(iii) $\gamma(u, z) := 1 + \lfloor C(u, z) \rfloor \geq 1$ otherwise, i.e if ($E =$ div$(u_1 u_2)$ and $t(x) = 1$) or if ($E =$ div$(u_1 u_2 u_3)$ and $t(x) = 0$).

Proposition 7.4.2 (Behaviour of the new invariants under blowing up along an ϵ-permissible curve). *Assume that (Hyp) is true,* (z, u) *is well prepared and let*

$$\mathcal{C}_i := V(z, u_i, u_3), \quad i = 1 \ \text{or}\ i = 2.$$

Assume that $\mathcal{C}_i$ *is* ϵ*-permissible in* $\widehat{X} = Spec(\widehat{R}/(h))$*, for some* i*,* $i = 1, 2$*, then:*

(i) $\epsilon(x) + d_3 + d_i \geq 1$,
(ii) $\mathcal{C}_i$ *is algebraic, i.e., if in achieving (P3)(P4), we get* $z, u_3 \in \widehat{R}$*, then there exists a curve in* $Spec R$ *whose formal completion is* $V(z, u_i, u_3)$.
(iii) *let* $\pi_i : (Z', x') \to (Z, x)$ *be the blowing up along* $\mathcal{C}_i$*,* $X' \subset Z'$ *the strict transform of* X *and* $x' \in X'$ *the center of* v*, with* $\iota(x') = \iota(x)$*. Then:*
(iv) *if* $i = 1$ *and* x' *is the point of* Z' *with parameters*

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2, u_3/u_1),$$

these are well-prepared parameters and

$$\beta(x') = \beta(x),\ A_1(x') = A_1(x) - 1,\ A_2(x') = A_2(x),$$
$$d_1(x') = d_1(x) + \epsilon(x) + d_3(x) - 1,\ d_2(x') = d_2(x),\ d_3(x') = d_3(x);$$

(v) *if* $i = 2$ *and* x' *is the point of* Z' *with parameters*

$$(z', u'_1, u'_2, u'_3) := (z/u_2, u_1, u_2, u_3/u_2),$$

these are well-prepared parameters and

$$\beta(x') = \beta(x) - 1,\ A_1(x') = A_1(x),\ A_2(x') = A_2(x) - 1,$$
$$d_2(x') = d_1(x) + \epsilon(x) + d_3(x) - 1,\ d_1(x') = d_1(x).$$

Proof of (i). Condition (i) is equivalent to $h \in (z, u_3, u_i)^m$.

Proof of (ii)(iii). Let us note that (ii) is clear when div$(u_3) \subset E$, because in that case, we do not make (P3)(P4), $z, u_1, u_3 \in R$. When div$(u_3) \not\subset E$, we will prove that

(ii)' $\mathcal{C}_i$ is the only analytic branch in div$(u_i) \cap \mathrm{Sing}_m(X) \cap \{y \in X :\ \epsilon(y) > 0\}$ not contained in div(u_j), $j = 1, 2$, $j \neq i$.

By **3.10**, **3.11**, this will prove (ii). We compute $\pi_i : X' \subset Z' \longrightarrow X' \subset Z'$. By symmetry, we suppose $i = 1$. Let us expand:

$$h = \sum_{A,m-j,0\leq j\leq m} C_{A,m-j} z^{m-j} u_1^{a_1} u_2^{a_2} u_3{}^{a_3},$$

$C_{A,m-j} \in R$, $C_{A,m-j}$ invertible or zero, $C_{A,m-j} \neq 0 \Rightarrow a_1 + a_2 + a_3 \geq j\delta(x)$, $a_i \geq jd_i$, $i = 1, 2, 3$, $C_{0,0}$ invertible.

Since $h \equiv \delta z^m \mod(u_1, u_2, u_3)$, $\delta \in R$ a unit, $X' \cap \operatorname{Spec} R[u_1/z, u_3/z] \subset Z'$ does not contain the point $(z, u_1/z, u_2, u_3/z)$. Assume now that x' belongs to the affine chart $\operatorname{Spec} R[z/u_3, u_1/u_3] \subset Z'$. Let

$$(z', u_1', u_2', u_3') := (z/u_3, u_1/u_3, u_2, u_3).$$

We have

$$h' := u_3^{-m} h = \sum C_{A,m-j} z'^{m-j} u_1'^{a_1} u_2^{a_2} u_3'{}^{a_1+a_3-j}, \tag{37}$$

$$h' = C_{0,m} z'^m + \sum_{2\leq j\leq m} \gamma_j u_1'{}^{jd_1} u_2^{jd_2} u_3{}^{j(d_1+d_3+\epsilon(x)-1)} \text{ modulo } I^+(z', u_1', u_2, u_3), \tag{38}$$

where, $\gamma_j \in R$, γ_j invertible or zero, $\gamma_j = 0$ when one exponent is not integer,

$$\gamma_j = C_{jd_1, jd_2, j(d_3+\epsilon(x)), m-j} \text{ modulo } \mathfrak{M}$$

when γ_j is invertible and $I^+(z', u_1', u_2, u_3)$ is generated by

$$z'^{m+1}, z'^{m-j} u_1'{}^a u_2^b u_3^c,$$

with $1 \leq j \leq m$, $a \geq jd_1$, $b \geq jd_2$, $c \geq j(d_1 + d_3 + \epsilon(x) - 1)$, $a + b + c > j(d_1 + d_2 + d_1 + d_3 + \epsilon(x) - 1)$.

Note that (38) implies $d_3(x') \geq d_1 + d_3 + \epsilon(x) - 1$, in fact there is equality. Otherwise, by [17], there would exist $t = z' + \gamma u_3^e$, $e \geq d_1 + d_3 + \epsilon(x) - 1$, $\gamma \in R$, $\operatorname{ord}_{u_i} \gamma \geq d_i$, $i = 1, 2$, with $h' = C_{0,0} t^m$ modulo $I^+(t, u_1', u_2, u_3)$. As $I^+(z', u_1', u_2, u_3) = I^+(t, u_1', u_2, u_3)$ and, in (38), there is no term in z'^{m-1}, this is impossible. As $d_1(x') = d_1(x)$ and $d_2(x') = d_2(x)$, by (38), we get $\epsilon(x') = 0$: there is no x' very near x in this chart. This gives the first statement in (iv). This gives also (ii)', because if there was a curve in $\operatorname{div}(u_2) \cap \operatorname{Sing}_m(X) \cap \{y \in X : \epsilon(y) > 0\}$, the strict transform of this curve would have a non empty intersection with our affine chart and there would exist in this chart some x' with $\epsilon(x') \geq 1$.

Proof of (iv). Now $x' \in \operatorname{Spec} R[z/u_1, u_3/u_1] \subset Z'$ is the point with parameters $(z', u_1', u_2', u_3') := (z/u_1, u_1, u_2, u_3/u_1)$. Then, using the notations of (37),

$$h' := u_1^{-m} h = \sum C_{A,m-j} z'^{m-j} u_1^{a_1+a_3-j} u_2^{a_2} u_3'{}^{a_3},$$

$\Delta(h', u', z')$ is obtained as follows: take the convex hull of the set $\{(a+c-1, b, c) | (a, b, c) \in \Delta(h, u, z)\}$ and add $\mathbb{R}^{+3}$, then

$$\partial(\Delta(h', u', z')) \subset \{(a + c - 1, b, c) | (a, b, c) \in \partial(\Delta(h, u, z))\},$$
$$\operatorname{in}(h', \Delta')_{u',z'} = \overline{C_{0,m}} Z'^m + \sum_{2\leq j\leq m, A/j\in\partial(\Delta(h',u',z'))} \lambda_{j,A} Z'^{m-j} U_1'^{a_1+a_3-j} U_2'^{a_2} U_3'^{a_3},$$

where the $\lambda_{j,A} \in k(x)$ are defined by:

$$\operatorname{in}(h, \Delta)_{u,z} = \overline{C_{0,m}} Z^m + \sum_{2\leq j\leq m, \frac{A}{j}\in\partial(\Delta(h,u,z))} \lambda_{j,A} Z^{m-j} U_1{}^{a_1+a_3-j} U_2{}^{a_2} U_3{}^{a_3}.$$

Let $\mathcal{M}'$ be the set of monomials $M' = z'^{m-j} u_1^{a_1+a_3-j} u_2^{a_2} u_3'^{a_3}$ which appear with a non zero coefficient in the expansion of h', let $\mathcal{M}$ be the set of monomials $M = z^{m-j} u_1^{a_1} u_2^{a_2} u_3^{a_3}$ which appear with a non zero coefficient in the expansion of h:

$$d_1(x') = \mathrm{inh}_{M' \in \mathcal{M}'} \left(\frac{a_1 + a_3 - j}{j} \right) = d_1 + d_3 + \epsilon(x) - 1,$$
$$d_i(x') = \mathrm{inh}_{M' \in \mathcal{M}'} \left(\frac{a_i}{j} \right) = d_i, \ i = 2, 3.$$

As x' is very near x, $\epsilon(x) = \epsilon(x')$, $\delta(x') = d_1(x') + d_2(x') + d_3(x') + \epsilon(x)$. The only point on the first side of $\Delta(h', u', z')$ is

$$(d_1(x'), d_2(x), d_3(x) + \epsilon(x))$$

let p' be the projection on $x_3 = 0$ from this vertex. A monomial M' defines a point $(\frac{a_1+a_3}{j} - 1, \frac{a_2}{j}, \frac{a_3}{j})$ that we call also M', when $a_3 < d_3(x) + \epsilon(x)$,

$$p'(M') = \left(d_1(x') + \frac{\frac{a_1+a_3}{j} - 1 - d_1(x')}{d_3(x) + \epsilon(x) - \frac{a_3}{j}}, d_2(x) + \frac{\frac{a_2}{j} - d_2(x)}{d_3(x) + \epsilon(x) - \frac{a_3}{j}} \right),$$

as $\frac{\frac{a_1+a_3}{j} - 1 - d_1(x')}{d_3(x)+\epsilon(x)-\frac{a_3}{j}} = \frac{\frac{a_1}{j} - d_1(x)}{d_3(x)+\epsilon(x)-\frac{a_3}{j}} - 1$ and by **7.2** (35)

$$p(M) = \left(d_1(x) + \frac{\frac{a_1}{j} - d_1(x)}{d_3(x) + \epsilon(x) - \frac{a_3}{j}}, d_2(x) + \frac{\frac{a_2}{j} - d_2(x)}{d_3(x) + \epsilon(x) - \frac{a_3}{j}} \right).$$

We get

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x')),$$

i.e. the polyhedron $p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\})$ translated by the vector $-(d_1(x'), d_2(x'))$, from

$$p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))$$

by making an horizontal translation of -1. This gives the other assertions of (iv). Mutatis mutandis, we get (v).

Proposition 7.4.3 (Behaviour of the new invariants under blowing up a closed point) *Assume that (Hyp) is true and (z, u) is well prepared. Let $\pi_i : (Z', x') \to (Z, x)$ be the blowing up along x, $X' \subset Z'$ the strict transform of X and $x' \in X'$ the center of v, with $\iota(x') = \iota(x)$. Then*

(i) *x' belongs to the strict transform of $V(z, u_3)$,*
(ii) *if $x' \in \ SpecR[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$ is the point*

$$(z', u_1', u_2', u_3') := (z/u_1, u_1, u_2/u_1, u_3/u_1),$$

these parameters are well-prepared and

$$\beta(x') \leq \beta(x),\ A_1(x') = B(x) - 1,\ A_2(x') = A_2(x),\ C(u', z') \leq C(u, z),$$
$$d_1(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1,\ d_2(x') = d_2(x),\ d_3(x') = d_3(x);$$

(iii) *if* $x' \in \mathit{SpecR}[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$ *and* $x' \neq (z', u'_1, u'_2, u'_3)$, *then*

$$\beta(x') \leq 1 + \left\lfloor \frac{C(u,z)}{2} \right\rfloor,\ A_1(x') = d(x) - 1,$$
$$d_1(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1,\ d_2(x') = 0,\ d_3(x') = d_3(x),$$

where $\lfloor . \rfloor$ *denotes lower integral part. If moreover (*$E \subseteq \mathrm{div}(u_1u_3)$ *and* $0 < \beta(x)$*), then* $\beta(x') \leq \beta(x)$.
We have

$$\gamma(x') \leq \gamma(x).$$

More precisely: if (x' *is not rational over* x *and* $\gamma(x) \geq 3$*), then* $\gamma(x') < \gamma(x)$*; if (*$\gamma(x') = \gamma(x) = 2$ *and* x' *is not rational over* x*), then* $\beta(x) = 2$, $div(u_2) \not\subset E$ *and* $\beta(x') < \beta(x) = 2$*;*

(iv) *if* $x' \in \mathit{SpecR}[z/u_2, u_1/u_2, u_3/u_2] \subset Z'$ *is the point with parameters*

$$(z', u'_1, u'_2, u'_3) := (z/u_2, u_1/u_2, u_2, u_3/u_2),$$

these are well prepared parameters and

$$\beta(x') = \beta(x) + A_1(x) - 1,\ A_1(x') = A_1(x),\ A_2(x') = B(x) - 1,$$
$$d_2(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1,\ d_1(x') = d_1(x),$$
$$\gamma(x') \leq \gamma(x),\ C(u', z') \leq \frac{\beta(x)}{2}.$$

Proof (i) is a consequence of **6.3**(ii) and **6.3** (28).
Proof of (ii). Write

$$h = \sum C_{A,m-b} z^{m-b} u_1^{a_1} u_2^{a_2} u_3^{a_3},\ C_{A,m-b} \in R^\times \text{ or } C_{A,m-b} = 0,$$

where the sum runs along $b \leq m$, $A = 0$ when $b = 0$, and $A = (a_1, a_2, a_3) \in b\Delta(h, u, z)$. Then

$$h' := u_1^{-m} h = \sum C_{A,m-j} z'^{m-j} u_1^{a_1+a_2+a_3-j} {u'_2}^{a_2} {u'_3}^{a_3},$$

and $\Delta(h', u', z')$ is obtained as follows: take the convex hull of the set

$$\{(a + b + c - 1, b, c) + {\mathbb{R}_+}^3 | (a, b, c) \in \Delta(h, u, z)\}.$$

Let $\mathcal{M}'$ be the set of monomials $M' = z'^{m-j} u_1^{a_1+a_2+a_3-j} {u'_2}^{a_2} {u'_3}^{a_3}$ which appear with a non zero coefficient in the expansion of h', let $\mathcal{M}$ be the set of monomials $M = z^{m-j} u_1^{a_1} u_2^{a_2} {u_3}^{a_3}$ which appear with a non zero coefficient in the expansion of h:

$$d_1(x') = \inf_{M' \in \mathcal{M}'} \left(\frac{a_1 + a_2 + a_3 - j}{j} \right) = d_1 + d_2 + d_3 + \epsilon(x) - 1,$$
$$d_i(x') = \inf_{M' \in \mathcal{M}'} \left(\frac{a_i}{j} \right) = d_i(x),\ i = 2, 3.$$

As x' is very near to x, $\epsilon(x) = \epsilon(x')$, $\delta(x') = d_1(x') + d_2(x') + d_3(x') + \epsilon(x)$. The only point on the first side of $\Delta(h', u', z')$ is

$$(d_1(x'), d_2(x), d_3(x) + \epsilon(x)).$$

Let p' be the projection on $x_3 = 0$ from this vertex. A monomial M' corresponds to a point $\mathbf{x}_{M'}$ $(\frac{a_1+a_2+a_3}{j} - 1, \frac{a_2}{j}, \frac{a_3}{j})$. When $a_3 < d_3(x) + \epsilon(x)$,

$$p'(M') = \left(d_1(x') + \frac{\frac{a_1+a_2+a_3}{j} - 1 - d_1(x')}{d_3(x) + \epsilon(x) - \frac{a_3}{j}}, d_2(x) + \frac{\frac{a_2}{j} - d_2(x)}{d_3(x) + \epsilon(x) - \frac{a_3}{j}} \right),$$

as $\frac{\frac{a_1+a_3}{j} - 1 - d_1(x')}{d_3(x)+\epsilon(x)-\frac{a_3}{j}} = \frac{\frac{a_1}{j} - d_1(x)}{d_3(x)+\epsilon(x)-\frac{a_3}{j}} - 1$ and by **7.2** (35)

$$p(M) = \left(d_1(x) + \frac{\frac{a_1}{j} - d_1(x)}{d_3(x) + \epsilon(x) - \frac{a_3}{j}}, d_2(x) + \frac{\frac{a_2}{j} - d_2(x)}{d_3(x) + \epsilon(x) - \frac{a_3}{j}} \right).$$

So we get

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x'))$$

from

$$p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))$$

as follows: take the convex hull of the set

$$\{(a + b - 1, b) + {\mathbb{R}_+}^2 | (a, b) \in p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))\}.$$

These are the usual transformation laws of the characteristic polyhedra of surfaces see the appendix of H. Hironaka in [3]. To get the other assertions of (ii), the proof runs along the same lines as **7.4.2** (37).

Proof of (iv). Mutatis mutandis, we get all assertions of (iv), except the last line that we prove now. In fact, we get

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x'))$$

from

$$p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))$$

as follows: take the convex hull of the set $\{(a, a+b-1) + {\mathbb{R}_+}^2 | (a, b) \in p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))\}$. We get $A_1(x') = A_1(x)$, $\beta(x') = \beta(x) + A_1(x) - 1$ and $A_2(x) = d(x) - 1$.

Let us denote by (α_2, β_2) and (α_3, β_3) with $\alpha_2 \le \alpha_3$, the coordinates of the (maybe equal) vertices of the first side of $p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))$.

Then $(\alpha_2, \alpha_2 + \beta_2 - 1) = (\alpha_2, B(x) - 1)$ is the vertex of smaller second coordinate of

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x')).$$

Note that $(A_1(x), A_1(x) + \beta(x) - 1)$ is the vertex of smaller first coordinate of

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x')).$$

All this leads to:

$$A_1(x') = A_1(x), \ A_2(x') = B(x) - 1,$$
$$C(u', z') \le \beta(x') - A_2(x') = A_1(x) + \beta(x) - 1 - (B(x) - 1) = \beta(x) - (B(x) - A_1(x)),$$
$$C(u', z') \le \alpha_2 - A_1(x') = \alpha_2 - A_1(x) \le \alpha_2 + \beta_2 - A_1(x) = B(x) - A_1(x).$$

Then either $B(x) - A_1(x) \leq \frac{\beta(x)}{2}$, then $C(u', z') \leq \frac{\beta(x)}{2}$ by the last inequality; or $B(x) - A_1(x) > \frac{\beta(x)}{2}$, then $C(u', z') < \frac{\beta(x)}{2}$ by the first of the two inequalities just above. The inequality $\gamma(x') \leq \gamma(x)$ is left to the reader.

Proof of (iii). Recall the notations and assumptions of **7.2** (34). We write

$$\Phi_j(U_1, U_2, U_3) = U_1^{jd_1} U_2^{jd_2} U_3^{jd_3} (\lambda_j U_3^{j\epsilon(x)} + \sum_{i \in \mathbb{Q}^+} U_1^{a(i,j)} U_2^{b(i,j)} U_3^{j\epsilon(x)-i} Q_{i,j}(U_1, U_2)) \tag{39}$$

with $\lambda_j \in k(x)$, $\lambda_j = 0$ if $(jd_1, jd_2, j\epsilon(x)) \notin \mathbb{N}^3$. In this expansion, we take:

$$Q_{i,j} \in k(x)[U_1, U_2],\ Q_{i,j} = 0 \text{ or } (U_1 \nmid Q_{i,j} \text{ and } U_2 \nmid Q_{i,j}),$$
$$Q_{i,j} = 0 \text{ when } (jd_1 + a(i,j), jd_2 + b(i,j), jd_3 + j\epsilon(x) - i) \notin \mathbb{N}^3.$$

Note that at least one $Q_{i,j}$, $2 \leq j \leq m$ is nonzero and at least one $\lambda_{j'}$, $2 \leq j' \leq m$ is nonzero.

By definition of $C(u, z)$, when $Q_{i,j} \neq 0$, $\deg(Q_{i,j}) \leq iC(u, z)$, where deg is the usual homogeneous degree. When $Q_{i,j} \neq 0$, let us denote $d(i, j) = \deg(Q_{i,j})$. Then we have, with natural notations, the relation:

$$\begin{aligned} v_B(u_3^{j\epsilon(x)}) &= v_B(U_1^{a(i,j)} U_2^{b(i,j)} U_3^{j\epsilon(x)-i} Q_{i,j}(U_1, U_2)) \\ j\epsilon(x) v_B(u_3) &= (j\epsilon(x) - i) v_B(u_3) + (a(i,j) + b(i,j) + d(i,j)) v_B(u_1) \\ j\epsilon(x) v_B(u_3) &= (j\epsilon(x) - i) v_B(u_3) + (a(i,j) + b(i,j) + d(i,j)) \frac{v_B(u_3)}{B}, \end{aligned}$$

which leads to:

$$a(i,j) + b(i,j) + d(i,j) - j(d_1 + d_2) = iB. \tag{40}$$

Then, in the expansion of $U_3^{j\epsilon(x)+jd_3-i} U_1^{a(i,j)} U_2^{b(i,j)} Q_{i,j}(U_1, U_2)$, the monomial with non zero coefficient and minimal exponent in U_1 is

$$U_1^{a(i,j)} U_2^{iB-a(i,j)} U_3^{j\epsilon(x)+jd_3-i}$$

which gives the point (cf. **7.2** (35))

$$\left(d_1 + \frac{\frac{a(i,j)}{j} - d_1}{d_3 + \epsilon(x) - \frac{jd_3 + j\epsilon(x) - i}{j}}, d_2 + \frac{\frac{iB - a(i,j)}{j} - d_1}{d_3 + \epsilon(x) - \frac{jd_3 + j\epsilon(x) - i}{j}} \right)$$

in $p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\})$. As

$$d_1 + \frac{\frac{a(i,j)}{j} - d_1}{d_3 + \epsilon(x) - \frac{jd_3 + j\epsilon(x) - i}{j}} = d_1 + \frac{a(i,j) - jd_1}{i},$$

we deduce that

$$\begin{aligned} A_1(u, z) = \inf \Bigg\{ \frac{a(i,j) - jd_1}{i} \mid 2 \leq j \leq m(x), 0 < i \leq j\epsilon(x), j\epsilon(x) \\ + jd_3 - i \in \mathbb{N}, i \in \mathbb{Q}, Q_{i,j} \neq 0 \Bigg\}. \end{aligned} \tag{41}$$

Similarly,

$$A_2(u,z) = \inf\left\{\frac{b(i,j)-jd_1}{i} \mid 2 \le j \le m(x), 0 < i \le j\epsilon(x), j\epsilon(x) \right. \\ \left. +jd_3 - i \in \mathbb{N}, i \in \mathbb{Q}, Q_{i,j} \ne 0\right\} \tag{42}$$

and, finally, by (40), when $Q_{i,j} \ne 0$,

$$d(i,j) = i\left(B - \frac{a(i,j)-jd_1}{i} - \frac{b(i,j)-jd_1}{i}\right) \le iC(u,z). \tag{43}$$

Since $x' \in \operatorname{Spec} R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$, x' is not the origin

$$(z', u'_1, u'_2, u'_3) = (z/u_1, u_1, u_2/u_1, u_3/u_1),$$

and x' belongs to the strict transform of $\mathrm{V}(z, u_3)$ then $z'(x') = u'_1(x') = u'_3(x') = 0$. We complete (z', u'_1, u'_3) to a r.s.p. (z', u'_1, v', u'_3) at x' where

$$v' = u_2'^{\,n} + \sum_{0 \le a \le n-1} \mu_a u_2'^{\,n-a}, \quad \mu_a = 0 \text{ or } \mu_a \in R^\times,$$

for some irreducible polynomial

$$P := U_2^{\,n} + \sum_{0 \le a \le n-1} \overline{\mu_a} U_2^{\,n-a} \in k(x)[U_2].$$

The following lemma will end the proof of **7.4.3**(iii).

Lemma 7.4.4 *With hypotheses and notations as in* **7.4.3***(iii), let* $d := [k(x') : k(x)]$. *We have:*

(i) $A_1(x') = B(u,z) - 1$*;*
(ii) *if* $div(u_3) \subset E$, *then*

$$\beta(x') \le \frac{C(u,z)}{d} \le \frac{\beta(x)}{d}$$

(iii) *in general,*

$$\beta(x') < 1 + \left\lfloor \frac{C(u,z)}{d} \right\rfloor, \tag{44}$$

(iv) *if (*$E \subseteq div(u_1)$, $0 < \beta(x)$ *and* x' *is rational over* x*), then*

$$\beta(x') \le \beta(x).$$

Proof As x' is very near to x, we have $\epsilon(x) = \epsilon(x')$, $\delta(x') = d_1(x') + d_2(x') + d_3(x') + \epsilon(x)$. As x' is on the strict transform of $\mathrm{div}(u_3)$ and not on the strict transform of $\mathrm{div}(u_2)$, we get:

$$d_2(x') = 0, \; d_3(x') = d_3(x).$$

With notations as in the proof of **7.4.3**(ii):

$$h = \sum C_{A,m-b} z^{m-b} u_1^{a_1} u_2^{a_2} u_3^{a_3}, \; C_{A,m-b} \in R^\times \text{ or } C_{A,m-b} = 0,$$

where the sum runs along $b \leq m$, $A = 0$ when $b = 0$, $A = (a_1, a_2, a_3) \in b\Delta(h, u, z)$,

$$h' := u_1^{-m} h = \sum C_{A,m-j} z'^{m-j} u_1'^{a_1+a_2+a_3-j} u_2'^{a_2} u_3'^{a_3}.$$

Up to multiplying h by an unit, we may assume $\overline{C_{0,m}} = 1 \in k(x)$. Then, with the notations of **3.2**, we have

$$\delta(x) = d_1 + d_2 + d_3 + \epsilon(x),$$
$$\mathrm{in}_{\delta,u,z}(h) = Z^m + \sum_{2 \leq j \leq m} \mu_j Z^{m-j} U_1^{jd_1} U_2^{jd_2} U_3^{jd_3 + j\epsilon(x)}, \ \mu_j \in k(x), \tag{45}$$

$\mu_j = 0$ whenever $(jd_1, jd_2, jd_3 + j\epsilon(x)) \notin \mathbb{N}^3$, $\mu_j = \overline{C_{jd_1, jd_2, jd_3 + j\epsilon(x), m-j}} \in k(x)$ otherwise. This leads to

$$h' = C_{0,m} z'^m + \sum_{2 \leq j \leq m} z'^{m-j} C_{jd_1, jd_2, jd_3 + j\epsilon(x), m-j} u_1'^{j(d_1+d_2+d_3+\epsilon(x)-1)} u_2'^{jd_2} u_3'^{jd_3+j\epsilon(x)} + h_1' \tag{46}$$

where $h_1' \in \{z'^{m-j} u_1'^{a(j)}, j \in \mathbb{N}, a(j) > j(d_1 + d_2 + d_3 + \epsilon(x) - 1) = j(\delta(x) - 1)\}$. As a consequence,

$$(\delta(x) - 1, 0, d_3(x) + \epsilon(x))$$

is the vertex of smallest first coordinate of $\Delta(h', u_1', v', u_3', t)$ and is not solvable. In the preparation, we may replace z' by $t = z' + \lambda u_1'^a$ with $a \geq \delta(h) - 1$, but, this cannot erase the vertex $(\delta(x) - 1, 0, d_3(x) + \epsilon(x))$. We get

$$d_1(x') = \delta(x) - 1.$$

Let us study the projection of $\Delta(h', u_1', v', u_3', t) \cap \{x_3' < d_3 + \epsilon(x')\}$ on $x_3' = 0$, in particular we are interested in the vertex of smallest first coordinate of this projection. Let w be the monomial valuation on $R' := \mathcal{O}_{X',x'}$ defined by

$$w(z') = 1, \ w(u_3') = \frac{1}{\epsilon(x) + d_3 + \frac{d_1+d_2}{B(u,z)-1}},$$
$$w(u_1') = \frac{1}{(B(u,z) - 1)(\epsilon(x) + d_3) + d_1 + d_2} = \frac{1}{B(u,z) - 1} w(u_3').$$

There is an expansion

$$\mathrm{in}_w(h') = Z'^m + \sum_{2 \leq j \leq m} Z'^{m-j} \Phi_j'(U_1', U_3') \in \ \mathrm{gr}_w(R') = R'/(z', u_1', u_3')[Z', U_1', U_3'],$$

where

$$\Phi_j'(U_1', U_3') = \lambda_j U_1'^{j(\delta(x)-1)} \overline{u_2'}^{jd_2} U_3'^{j\epsilon(x)+jd_3}$$
$$+ \sum_{0<i \leq j\epsilon(x), j\epsilon(x)+jd_3-i \in \mathbb{N}, i \in \mathbb{Q}} U_3'^{j\epsilon(x)+jd_3-i} U_1'^{a(i,j)+b(i,j)+d(i,j)+j\epsilon(x)+jd_3-i-j} \overline{u_2'}^{b(i,j)}$$
$$Q_{i,j}(1, \overline{u_2'}),$$

where $d(i,j) = \deg(Q_{i,j})\overline{u'_2}$ is the image of u'_2 in $\frac{R'}{(z',u_1,u'_3)} = k(x)[\overline{u'_2}]_{(\overline{v'})}$, $\overline{v'}$ being the image of v' in $\frac{R'}{(z',u_1,u'_3)}$. Let us recall that

$$\mathrm{in}_w(h') \in \mathrm{gr}_w(R') := \bigoplus_{r\in\mathbb{Q}_{\geq 0}} \frac{I_r}{I_r^+},$$

with $I_r = \{a \in R' | w(a) \geq r\}$, $I_r^+ = \{a \in R' | w(a) > r\}$. By **7.4.3** (40), $\mathrm{in}_w(h') =$

$$Z'^m + \sum_{2\leq j\leq m} Z'^{m-j} U_1'^{j(\delta(x)-1)}[\lambda_j U_3'^{j\epsilon(x)+jd_3} + U_3'^{j\epsilon(x)+jd_3-i} U_1'^{i(B(u,z)-1)}\overline{u'_2}^{b(i,j)}$$
$$Q_{i,j}(1,\overline{u'_2})]. \tag{47}$$

This means that

$$\frac{1}{(B(u,z)-1)(\epsilon(x)+d_3)+d_1+d_2}x'_1 + \frac{1}{\epsilon(x)+d_3+\frac{d_1+d_2}{B(u,z)-1}}x'_3 = 1$$

is the defining equation of a face of $\Delta(h', u'_1, v', u'_3)$.

7.4.4.1 When $\mathrm{div}(u_3) \subset E$, then $\mathrm{div}(u'_3) \subset E'$, we have just to make (P2) in the preparation, we may replace z' by $t = z' + r$, $r \in R'$ and, as Z'^{m-1} does not appear in (2), $w(r) > 1$, $w(t) = w(z') = 1$. This means that

$$\frac{1}{(B(u,z)-1)(\epsilon(x)+d_3)+d_1+d_2}x'_1 + \frac{1}{\epsilon(x)+d_3+\frac{d_1+d_2}{B(u,z)-1}}x'_3 = 1$$

is the defining equation of a face of $\Delta(h', u'_1, v', u'_3, t)$. By **7.2** (34),

$$A_1(u'_1, v', u'_3, t) = B(u,z) - 1 \quad \text{and} \quad \beta(x') = \inf\{\mathrm{ord}_{x'}(Q_{i,j}(1,\overline{u'_2}))/id\}.$$

By **7.4.3**(5), $\beta(x') \leq C(u,z)/d$ and this gives **7.4.4** in the case $\mathrm{div}(u_3) \subset E$.

7.4.4.2 From now on, $\mathrm{div}(u_3) \not\subset E$, in particular $d_3(x) = 0$. Then, to get (P1),...,(P4), we may replace z' by $t = z' + r$, $r \in R'$ and, as $Z'^{m(x)-1}$ does not appear in (46), $w(r) > 1$, $w(t) = w(z') = 1$. We possibly have to make the projection of $\Delta(h', u'_1, v', u'_3, t) \cap \{x'_3 < d_3 + \epsilon(x')\}$ on $x'_3 = 0$ smaller by changing u'_3 to $v_3 = u'_3 + \lambda u_1'^a$ with $a \geq B(u,z) - 1$ and $\lambda \in R'$, λ not divisible by u'_1.

Assume that $a > B(u,z) - 1$ (this is always the case when $B(u,z) \notin \mathbb{N}$). Then $\mathrm{in}_w(v_3) = \mathrm{in}_w(u'_3)$, we get $A_1(u'_1, v', u'_3, t) = B(u,z) - 1$ and

$$\beta(x') = \inf\{\mathrm{ord}_{x'}(Q_{i,j}(1,\overline{u'_2}))/id\}.$$

By **7.4.3** (43),

$$\beta(x') \leq C(u,z)/d$$

which gives **7.4.4** in this case.

7.4.4.3 From now on,

$$B(u,z) \in \mathbb{N},\ a = B(u,z) - 1.$$

If there exists a couple (i, j_0) such that in (47) above

$$\lambda_{j_0} = 0 \quad \text{and} \quad Q_{i,j_0} \neq 0,$$

then the translations $t = z' + r$ and $v_3 = u'_3 + \lambda {u'_1}^a$ will not modify the term

$$ {U'_3}^{j_0\epsilon(x)-i_0} {U'_1}^{i_0(B(u,z)-1)} \overline{u'_2}^{b(i_0,j_0)} Q_{i_0,j_0}(1,\overline{u'_2}) $$

with $i_0 := \min\{i : Q_{i,j_0} \neq 0\}$. More precisely, in the expansion

$$ \mathrm{in}_w(h') = T^m + \sum_{2\leq j\leq m} T^{m-j} {U'_1}^{j(\delta(x)-1)} [\lambda_j {V_3}^{j\epsilon(x)} + \mu_{i,j} {U'_3}^{j\epsilon(x)-i} {U'_1}^{i(B(u,z)-1)} \times {V'}^{e(i,j)}], $$

$\mu_{i,j} \in \frac{R'}{(t,u'_1,u'_3)}$, $e(i,j) \in \mathbb{N}$, we will have

$$ \overline{u'_2}^{b(i_0,j_0)} Q_{i_0,j_0}(1,\overline{u'_2}) = \mu_{i_0,j_0} \times {V'}^{e(i_0,j_0)}. $$

Then

$$ \beta(x') \leq \mathrm{ord}_{x'}(Q_{i_0,j_0}(1,\overline{u'_2}))/id, $$

which, by **7.4.3** (43), gives

$$ A_1(x') = B(u,z) - 1,\ \beta(x') \leq C(u,z)/d $$

and implies **7.4.4** in this case.

7.4.4.4 From now on, we assume the implication:

$$ Q_{i,j} \neq 0 \Rightarrow \lambda_j \neq 0. $$

In particular, we have $j\epsilon(x) \in \mathbb{N}$, $j\delta(x) \in \mathbb{N}$ and all the indices i in (46) (47) are integers. Let us define

$$ F_j \in \mathrm{gr}_{v_B}(R) = k(x)[U_1, U_2, U_3, Z] $$

by

$$ F_j = \lambda_j {U_3}^{j\epsilon(x)} + \sum_{1\leq i\leq j\epsilon(x)-1} {U_3}^{j\epsilon(x)-i} {U_1}^{a(i,j)} {U_2}^{b(i,j)} Q_{i,j}(U_1,U_2). $$

$$ F'_j \in \mathrm{gr}_w(R') = \frac{R'}{(u'_1,u'_3,z')}[U'_1, U'_3, Z'] = \frac{R'}{(u'_1,u'_3,z')}[U'_1, U'_3, T] $$

by

$$ F'_j = \lambda_j {U'_3}^{j\epsilon(x)} + \sum_{1\leq i\leq j\epsilon(x)-1} {U'_3}^{j\epsilon(x)-i} {U'_1}^{i(B(u,z)-1)} \overline{u'_2}^{b(i,j)} Q_{i,j}(1,\overline{u'_2}), $$

so (47) can be rewritten:

$$ \mathrm{in}_w(h') = T^{m(x)} + \sum_{2\leq j\leq m(x)} T^{m(x)-j} {U'_1}^{j(\delta(x)-1)} F'_j. \tag{47'} $$

The preceding remarks rewrite $j\epsilon(x) \notin \mathbb{N} \Rightarrow F_j = 0$, $F'_j = 0$. Let

$$ G_j = F_j^{\frac{m!\epsilon(x)}{j\epsilon(x)}},\ G'_j = {F'_j}^{\frac{m!\epsilon(x)}{j\epsilon(x)}}\ 2 \leq j \leq m,\ j\epsilon(x) \in \mathbb{N}, $$

$\deg_{U_3}(G_j) = m!\epsilon(x)$ or $G_j = 0$, and $\deg_{U'_3}(G'_j) = m!\epsilon(x)$ or $G'_j = 0$.

Let $\mu_1, \mu_2 \in k(x)$, $j_1, j_2,\ 2 \leq j_1, j_2 \leq m$, let

$$G = \mu_1 G_{j_1} + \mu_2 G_{j_2} = \mu_{m!\epsilon(x)} U_3{}^{m!\epsilon(x)} + \sum_{1\leq i\leq m!\epsilon(x)-1} U_3{}^{m!\epsilon(x)-i} U_1{}^{a(i)} U_2{}^{b(i)} Q_i(U_1, U_2),$$

where $Q_i = 0$ or Q_i neither divisible by U_1 nor by U_2. Let us denote $d(i) := \deg(Q_i)$. Assume that for some i, $Q_i \neq 0$, then, by **7.4.3** (43),

$$d(i) = i\left(B - \frac{a(i)}{i} - \frac{b(i)}{i}\right) \leq iC(u, z). \tag{48}$$

Assume that not all G_j's are collinear in the $k(x)$-vector space $\mathrm{gr}_{v_B}(R)$. Then there is some $G \neq 0$ as above with $\lambda = 0$. Let

$$G = \mu_1 G_{j_1} + \mu_2 G_{j_2} = \sum_{1\leq i\leq m!\epsilon(x)-1} U_3{}^{m!\epsilon(x)-i} U_1{}^{a(i)} U_2{}^{b(i)} Q_i(U_1, U_2),$$

with some $Q_i \neq 0$. Let $i_0 :=\ \min\{i : Q_{i_0} \neq 0\}$. Let

$$G' = \lambda G'_{j_1} + \mu G'_{j_2} = \sum_{1\leq i\leq m!\epsilon(x)-1} U_3'^{\,m!\epsilon(x)-i} U_1'^{\,i(B(u,z)-1} \overline{u_2'}^{\,b(i)} Q_i(1, u_2').$$

Replacing U_3' by V_3, we get

$$\begin{aligned} G' = \mu_1 G'_{j_1} + \mu_2 G'_{j_2} = {} & V_3{}^{m!\epsilon(x)-i_0} U_1'^{\,i_0(B(u,z)-1} \overline{u_2'}^{\,b(i_0)} Q_{i_0}(1, u_2') \\ & + H',\ deg_{V_3} H' < m!\epsilon(x) - i_0. \end{aligned}$$

Then

$$A_1(x') = B(u, z) - 1,\ \beta(x') \leq C(u, z)/d$$

which implies **7.4.4** in this case.

7.4.4.5 From now on, we assume that all G_j's are collinear in the $k(x)$-vector space $\mathrm{gr}_{v_B}(R)$.

By (P3) for (z, u), any $G_j \neq 0$ is not collinear to a $(m!\epsilon(x))^{\mathrm{th}}$-power, any $F_j \neq 0$ is not collinear to a $(j\epsilon(x))^{\mathrm{th}}$-power. Take some $F_j \neq 0$, and let

$$j\epsilon(x) = p^e q,\ (p, q) = 1. \tag{49}$$

Let $v_3 = u_3' + \lambda u_1'{}^a$, with $a \geq B(u, z) - 1$ and $\lambda \in R'$, λ not divisible by u_1'. Let

$$\begin{aligned} \overline{\lambda} \in \frac{R'}{(z', u_1, u_3')} = {} & k(x)[\overline{u_2'}]_{\overline{v'}},\ b :=\ \mathrm{ord}_{v'}(\overline{\lambda}),\ \beta_0 :=\ \min_i\left(\frac{\mathrm{ord}_{v'}(Q_{i,j}(1, u_2'))}{i}\right) \\ & \leq C(u, z)/d. \end{aligned}$$

When $b < \beta_0$, we have

$$A_1(x') = B(u, z) - 1,\ \beta(x') = b < C(u, z)/d.$$

When $b > \beta_0$, we have

$$A_1(x') = B(u, z) - 1,\ \beta(x') = \beta_0 \leq C(u, z)/d.$$

When $b=\beta_0$ and there exists $i<p^e$ such that $Q_{i,j}\neq 0$, say i_0 is the smallest such i, we get

$$F'_j=\lambda_j {V_3}^{j\epsilon(x)}+{V_3}^{j\epsilon(x)-i_0}{U'_1}^{i_0(B(u,z)-1)}\overline{u'_2}^{b(i_0,j)}Q_{i_0,j}(1,\overline{u'_2})$$
$$+H'_j,\ \deg_{V_3}H'_j<j\epsilon(x)-i_0,$$
$$A_1(x')=B(u,z)-1,\ \beta(x')\leq\frac{\mathrm{ord}_{v'}(Q_{i_0,j}(1,u'_2))}{i_0})\leq C(u,z)/d.$$

When $b=\beta_0$ and for $i\leq p^e$ $Q_{i,j}=0$, then

$$F'_j=\lambda_j {V_3}^{j\epsilon(x)}+{V_3}^{j\epsilon(x)-p^e}{U'_1}^{p^e(B(u,z)-1)}\lambda^q+H'_j,\ \deg_{V_3}H'_j<j\epsilon(x)-p^e,$$
$$A_1(x')=B(u,z)-1,\ \beta(x')\leq b=\beta_0\leq C(u,z)/d.$$

When $b=\beta_0$ and for $i<p^e\,Q_{i,j}=0$ and $\lambda_j^{-1}U_1^{a(p^e,j)}U_2^{b(p^e,j)}Q_{p^e,j}$ is a(p^e)th-power, then

$${\lambda_j}^{-1}U_1^{a(p^e,j)}U_2^{b(p^e,j)}Q_{p^e,j}=U_1^{p^ea}U_2^{p^eb}Q_0^{p^e},$$

with $a\geq A_1(x), b\geq A_2(x)andB_0:=\deg(Q_0)\leq C(u,z)$. Let $W_3:=U_3+U_1^aU_2^bQ_0$, we get

$$F_j=\lambda_j {W_3}^{j\epsilon(x)}+\sum_{1+p^e\leq i\leq j\epsilon(x)-1}{W_3}^{j\epsilon(x)-i}{U_1}^{a_0(i,j)}{U_2}^{b_0(i,j)}Q_{0,i,j}(U_1,U_2),$$

$a_0(i,j)\geq iA_1(x),\ b_0(i,j)\geq iA_2(x),\ d_0(i,j):=\deg(Q_{0,i,j})\leq iC(u,z),$ or $Q_{0,i,j}=0.$

Let $w_3\in R$ such that $\mathrm{in}_{v_B}(w_3)=W_3$, then, with $w'_3=w_3/u_1$, $W'_3=$in$_w(w'_3)$:

$$F'_j=\lambda_j {W'_3}^{j\epsilon(x)}+\deg_{W'_3}<j\epsilon(x)-p^e.$$

Let $v_3=w'_3+\lambda'{u'_1}^{a'}$, with $a'\geq B(u,z)-1$ and $\lambda'\in R'$, λ' not divisible by u'_1. Then we conclude as above:

$$A_1(x')=B(u,z)-1,\ \beta(x')\leq b=\beta_0\leq C(u,z)/d.$$

7.4.4.6 From now on, we assume **7.4.4.2**, **7.4.4.3**, **7.4.4.4**, **7.4.4.5** and for $i<p^e$ $Q_{i,j}=0$ and ${\lambda_j}^{-1}U_1^{a(p^e,j)}U_2^{b(p^e,j)}Q_{p^e,j}$ is NOT a $(p^e)^{\mathrm{th}}$-power (c.f. (49)). In particular $e\geq 1$. Let us recall the following elementary lemma [13, II.5.3.2].

Lemma 7.4.4.7 *Let $F(U_1,U_2)\in k(x)[U_1,U_2]$ be a homogeneous polynomial of degree $d_0\geq 0$, and $a,b\in\mathbb{N}$ be such that $U_1^aU_2^bF(U_1,U_2)\notin(k(x)[U_1,U_2])^p$.*

Let $x'\in$ Speck$(x)[\frac{U_2}{U_1}]$ be a closed point with ideal $(v:=P(1,\frac{U_2}{U_1}))$, $P\in k(x)[U_1,U_2]$ a nonzero homogeneous irreducible polynomial of degree $d:=[k(x'):k(x)]$, unitary in U_2.

Let $A\in T':=k(x)[U_1,\frac{U_2}{U_1}]_{(U_1,v)}$ be such that $U_1^{a+b+d_0}$ (resp. $U_1^{a+b+d_0}v^b$) divides A^p in T' if $P\neq U_2$ (resp. $P=U_2$). There exists an integer $c\geq 0$ such that

$$U_1^{a+b+d_0}\left(\frac{U_2}{U_1}\right)^bF\left(1,\frac{U_2}{U_1}\right)+A^p\equiv U_1^{a+b+d_0}\left(\frac{U_2}{U_1}\right)^b\gamma v^c\ \ mod(U_1^{a+b+d_0+1}T'),$$

with γ invertible in T'. We have the following estimates for c:

(i) *if $P\neq U_2$ (resp. $P=U_2$), then $c\leq 1+\frac{d_0}{d}$ (resp. $c\leq d_0$);*
(ii) *if $P\neq U_2$, then $c<p(1+\lfloor\frac{d_0}{pd}\rfloor)$ (equivalently: for every $N\in\mathbb{N}$ such that $\frac{d_0}{pd}<N$, we have $c<Np$);*

(iii) *if* $d_0 \geq 1$ *and* $b = 0$, *then* $c \leq i$.

Let $f < e$ be the integer defined by:

$$\lambda_j{}^{-1} U_1^{a(p^e,j)} U_2^{b(p^e,j)} Q_{p^e,j} = Q_0^{p^h}, \; Q_0 \text{ is not a } p\text{th}-\text{power}.$$

Let

$$Q_0 =: (U_1^a U_2^b F(U_1, U_2)),$$

with $p^h a = a(p^e, j)$, $p^h b = b(p^e, j)$, $p^h d_0 = d(p^e, j)$, where $d_0 :=\deg(Q_0)$. In particular,

$$d_0 \leq p^{e-h} C(u, z).$$

Then,

$$F'_j = \lambda_j (V_3{}^{j\epsilon(x)} + V_3{}^{j\epsilon(x)-p^e} \lambda_j^{-1} (U_1'{}^{p^e(B(u,z)-1)} \overline{u_2'}^{b(i_0,j)} Q_{i_0,j}(1, \overline{u_2'}) + \overline{\lambda}^{p^e})$$
$$+\deg_{V_3} < j\epsilon(x) - p^e).$$

By Lemma 7.4.4.7, $(\overline{u_2'}^{b(i_0,j)} Q_{i_0,j}(1, \overline{u_2'}) + \overline{\lambda}^{p^e}) = (\gamma v'^c)^{(p^h)} \neq (0)$, so

$$A_1(x') = B(u, z) - 1, \; p^e \beta(x') \leq p^h c.$$

Furthermore, by (i) above, $c \leq 1 + \frac{d_0}{d}$, so:

$$p^e \beta(x') \leq p^h (1 + \tfrac{d_0}{d}) \leq p^h + p^h \tfrac{p^{e-h} C(u,z)}{d} = p^h + \tfrac{p^e C(u,z)}{d},$$
$$\beta(x') \leq \tfrac{1}{p} + \tfrac{C(u,z)}{d}.$$

By (ii),

$$p^e \beta(x') < p^h p \left(1 + \left\lfloor \frac{d_0}{pd} \right\rfloor\right) \leq p^{f+1} \left(1 + \left\lfloor \frac{p^{e-h} C(u, z)}{pd} \right\rfloor\right), \; \beta(x') < 1 + \left\lfloor \frac{C(u, z)}{d} \right\rfloor,$$

which is **7.4.4**(iii). Now **7.4.4**(iv) is a consequence of (iii) above.

7.4.5 Proof of Theorem VI.1: some cases with $\gamma(u, z) = 1$.

The strategy to make the proof is to make a list of different subcases covering this case, from the easiest to the most difficult and to prove them up to the former ones.

All cases ($\beta(u, z) < 1$ and $\mathrm{div}(u_1 u_2) \not\subseteq E$) are covered by **7.4.5.3** below. All cases with ($\gamma(u, z) < 1$ and $\mathrm{div}(u_1 u_2) \subseteq E$) are dealt with in **7.4.5.6**. This includes in particular all remaining cases with $\beta(u, z) < 1$ since $C(u, z) \leq \beta(u, z)$ for $\mathrm{div}(u_1 u_2) \subseteq E$ (see Definition 7.4.1).

Lemma 7.4.5.1 *With assumptions as in* **7.4.3**, *assume furthermore that*

$$A_1(u, z) < 1, \; \beta(u, z) < 1. \tag{50}$$

There exist well prepared parameters (z', u') *at* x' *such that*

$$(A_1(u', z'), \beta(u', z')) <_{\text{lex}} (A_1(u, z), \beta(u, z)), \; \textit{and } \beta(u', z') < 1. \tag{51}$$

Proof This is a direct consequence of Lemma 7.4.4.

Lemma 7.4.5.2 *With assumptions as in* **7.4.3***, assume furthermore that*

$$\beta(u,z) < 1,\ A_1(u,z) \geq 1,\ (d_1(x) + d_3(x) + \epsilon(x) \geq 1 \ \text{or}\ E \subseteq \ \text{div}(u_1u_3)). \tag{52}$$

Then $\mathcal{C}_1 := V(z, u_1, u_3)$ *is an* ϵ*-permissible algebraic curve on* X.

Let $\pi : (Z', x') \to (Z, x)$ *be the blowing up along* $\mathcal{C}_1$*,* $X' \subset Z'$ *the strict transform of* X *and* $x' \in X'$ *the center of* v *and assume* $\iota(x') = \iota(x)$*. Then* $(z', u') = (z/u_1, u_1, u_2, u_3/u_1)$ *are well prepared parameters at* x' *and we have*

$$(A_1(u', z'), \beta(u', z')) = (A_1(u, z) - 1, \beta(u, z)).$$

Proof This follows from Proposition 7.4.2.

Remark 7.4.5.3 Lemmas 7.4.5.1 **7.4.5.2** prove Theorem 7.1 when (Hyp) is true and

$$\beta(u,z) < 1,\ A_1(u,z) \geq 1,\ (d_3(x) + \epsilon(x) \geq 1 \text{ or } E \subset \ \text{div}(u_1u_3)). \tag{53}$$

Indeed, $d_3(x)+\epsilon(x) = d_3(x')+\epsilon(x')$ if $\iota(x') = \iota(x)$ after blowing up. If $E \subseteq \text{div}(u_1u_3)$, then $E' \subseteq \text{div}(u'_1u'_3)$, so condition (3) remains stable after blowing up. A descending induction on $A_1(u, z)$ ends the proof.

7.4.5.4 Proof of Theorem 7.1 in the case $C(u, z) = 0$.

In that special case, we have $\beta(u, z) = A_2(u, z)$. When $A_2(u, z) < 1$ and $A_1(u, z) < 1$, **7.4.5.1** gives the result. Let us see the other cases:

$$A_1(u,z) \geq 1 \quad \text{or} \quad A_2(u,z) \geq 1. \tag{54}$$

Case 1 $d_3(x) + \epsilon(x) \geq 1$. We may assume $A_2(u, z) \geq 1$ by symmetry on u_1, u_2. Then $\mathcal{C} := (z, u_2, u_3)$ is an ϵ-permissible algebraic curve by Proposition 7.4.2 and we get

$$A_1(x') = A_1(x),\ A_2(x') = A_2(x) - 1,\ C(u', z') = 0$$

after blowing up along $\mathcal{C}$ if $\iota(x') = \iota(x)$. A descending induction on $A_2(x)$ and **7.4.5.2** give the result. From now on, we assume

$$d_3(x) + \epsilon(x) < 1.$$

Case 2 $d_3(x) + d_i(x) + \epsilon(x) < 1$, $i = 1$ and $i = 2$. We blow up $\{x\}$ in this case. By Proposition 7.4.3(ii) or (iv), we have

$$\begin{aligned} d_3(x') = d_3(x),\ d_2(x') = d_2(x),\ d_1(x') &= d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1 \\ &< d_1(x),\ C(u', z') = 0, \\ \delta(x') = d_1(x') + d_2(x') + d_3(x') + \epsilon(x') &< \delta(x) \end{aligned}$$

if $\iota(x') = \iota(x)$ and x' is the origin of a chart. Otherwise, Lemma 7.4.4(ii)(iii) gives **7.4.5.2** (52) at x' for some well prepared r.s.p. (z', u').

Case 3 $d_3(x) + \epsilon(x) < 1$, $d_3(x) + d_i(x) + \epsilon(x) \geq 1$ for some $i = 1$ or $i = 2$, $A_j(x) \geq 1$, $j = 1, 2$. We choose an ϵ-permissible blowing up center Y as follows:

if $\text{V}(z, u_i, u_3)$, for $i = 1, 2$ are ϵ-permissible, then $Y := (z, u_i, u_3)$ with

$$(A_i(x), d_i(x)) \geq (A_{i'}(x), d_{i'}(x)),\ \{i, i'\} = \{1, 2\};$$

if $\text{V}(z, u_i, u_3)$ is ϵ-permissible for a unique $i \in \{1, 2\}$, then $Y := (z, u_i, u_3)$;

if $\text{V}(z, u_i, u_3)$ is not ϵ-permissible for $i \in \{1, 2\}$, then $Y := \{x\}$.

Let $n(x) := 2$ if $(A_1(x), d_1(x)) = (A_2(x), d_2(x))$, $n(x) := 1$ otherwise. If $\iota(x') = \iota(x)$, we claim that x' satisfies **7.4.5.2** (52) or falls into cases 1,2 above, or there is a well prepared r.s.p. (z', u') at x' with $C(u', z') = 0$ and

$$(\max_{i=1,2}\{A_i(x)\}, \max_{i=1,2}\{d_i(x)\}, n(x)) <_{\text{lex}} (\max_{i=1,2}\{A_i(x')\}, \max_{i=1,2}\{d_i(x')\}, n(x')). \tag{55}$$

Note that this ends the proof of the case $C(u, z) = 0$, since (3) can repeat but finitely many times. To prove the claim, first assume that $Y = (z, u_1, u_3)$. By Proposition 7.4.2, we have

$$A_1(x') = A_1(x) - 1,\ A_2(x') = A_2(x),\ d_1(x') = d_1(x) + d_3(x) + \epsilon(x) - 1 < d_1(x),\ d_2(x') = d_2(x)$$

and the result is clear. The case $Y = (z, u_2, u_3)$ is similar.

Assume now that $Y = \{x\}$. By symmetry on u_1, u_2, we assume $A_2(x) \geq 1$. If $x' \in \operatorname{Spec} R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$ and x' is the point with parameters $(z/u_1, u_1, u_2/u_1, u_3/u_1)$ (origin of the first chart), we get

$$A_1(x') = A_1(x) + A_2(x) - 1,\ A_2(x') = A_2(x),\ d_1(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1,\ d_2(x') = d_2(x).$$

Since $A_2(x) \geq 1$ and $V(z, u_2, u_3)$ is not ϵ-permissible, we have

$$d_2(x) + d_3(x) + \epsilon(x) < 1,\ d_1(x) + d_3(x) + \epsilon(x) \geq 1 \text{ and } A_1(x) < 1.$$

We get $d_1(x) > d_2(x)$ and $d_1(x') = d_1(x) + d_2(x) + d_3(x)\epsilon(x) - 1 < d_1(x)$, $A_1(x') < A_1(x) \leq A_2(x') = A_2(x)$ which proves the claim.

If $x' \in \operatorname{Spec} R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$ and x' is not the above point, we have **7.4.5.2** (52) at x' for some well prepared r.s.p. (z', u') at x' by lemma 7.4.4(ii)(iii).

If $x' \in \operatorname{Spec} R[z/u_2, u_1/u_2, u_3/u_2] \subset Z'$ and x' is the point with parameters $(z/u_2, u_1/u_2, u_2, u_3/u_2)$ (origin of the second chart), we get

$$A_2(x') = A_1(x) + A_2(x) - 1,\ A_1(x') = A_1(x),\ d_2(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1,\ d_1(x') = d_1(x).$$

We have $A_1(x) < 1$: otherwise, as $V(z, u_1, u_3)$ is not ϵ-permissible, this would imply $d_1(x)+d_3(x)+\epsilon(x) < 1, d_2(x)+d_3(x)+\epsilon(x) \geq 1$, hence $V(z, u_2, u_3)$ ϵ-permissible since $A_2 \geq 1$: a contradiction. We now get $A_1(x) < 1 \leq A_2(x)$ and $A_2(x') = A_1(x)+A_2(x)-1 < A_2(x)$ which completes the proof of the claim.

7.4.5.5 Proof of Theorem 7.1 in the case $C(u, z) < 1,\ \operatorname{div}(u_1u_2) \subseteq E$.

We perform the sequence of local blowing ups

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n) \leftarrow \cdots,$$

where $x_0 = x$, $x_i \in X_i$ (X_i denoting the strict transform of X), $0 \leq i \leq n$, is the center of v, each blowing up center is $Y_i = \{x_i\}$.

If $\iota(x_1) = \iota(x)$, and is not the origin of a chart (*viz.* case 3 in **7.4.5.4**), then x_1 verifies the assumptions of **7.4.5.1** by Lemma 7.4.4(iii).

Assume now that $\iota(x_i) = \iota(x)$ and x_i is the origin of a chart for all $i \geq 0$. By **7.4.3**(ii)(iv), x_i verifies the assumptions of **7.4.5.5** and $C(x_{i+1}) \leq C(x_i)$ for all $i \geq 0$. It is then a very well known fact that $C(x_i) = 0$ for $i >> 0$, i.e. the assumptions of **7.4.5.4** are satisfied.

7.4.5.6 End of the proof of Theorem 7.1. As our invariants $C(u,z)$, $\beta(u,z)$ are discrete, the next lemma shows that we will reach one of the cases (ii) $\beta(u,z) < 1$ or (iii) $C(u,z) < 1$. This ends the proof of Theorem 7.1 (see comments right after **7.4.5.**).

Lemma 7.4.5.7 *With assumptions as in* **7.4.3**, *consider the sequence of local blowing ups*

$$(Z,x) =: (Z_0,x_0) \longleftarrow (Z_1,x_1) \longleftarrow \ \cdots \ \longleftarrow (Z_n,x_n) \leftarrow \cdots,$$

where $x_0 = x$, $x_i \in X_i$ *(*X_i *denoting the strict transform of* X*),* $0 \leq i \leq n$*, is the center of* v*, each blowing up center is* $Y_i = \{x_i\}$*.*

Assume that $\iota(x_i) = \iota(x)$ *for all* $i \geq 0$*. There exists some* $i \geq 0$ *and a well prepared r.s.p.* $(z_i, u_{1,i}, u_{2,i}, u_{3,i})$ *at* x_i *(w.r.t. the reduced inverse image of* E *in* Z_i*) such that one of the following holds:*

(i) $\gamma(u_i, z_i) < \gamma(u,z)$;
(ii) $\beta(u_i, z_i) < 1$;
(iii) $C(u_i, z_i) < 1$;

Proof This breaks up in three cases:
Case 1 for all $i \geq 0$, the point x_i is the origin of one of the two charts of **7.4.3**, i.e. we are always in one of the cases **7.4.3**(ii)(iv). Then $C(u_i, z_i) = 0$ for $i >> 0$ (see **7.4.5.5** above).
Case 2 for all $i \geq 0$, (x_i is rational over x and belongs to the first chart), i.e. x_i is a rational point not on the strict transform of $\mathrm{div}(u_1)$. By **7.4.3**(ii)(iii), x_1 has a r.s.p. of the form $(z/u_1, u_1, u_2/u_1 + \mu_1, u_3/u_1)$ for some $\mu_1 \in R$. A well prepared r.s.p. is of the form

$$z/u_1 + \lambda_1 u_1, u_1, u_2/u_1 + \mu_1, u_3/u_1 + \mu_2 u_1, \ \lambda_1, \mu_1, \mu_2 \in R,$$

with $\mu_2 = 0$ if $\mathrm{div}(u_3) \subseteq E$. Then there exists a regular *formal* curve $\mathcal{C}$ of the form $\mathcal{C} = V(\widehat{z}, \widehat{u_2}, \widehat{u_3})$ on $\mathrm{Spec}(\widehat{R}/(h)$, transverse to E_i for all $i \geq 0$, $\widehat{u_3} = u_3$ if $\mathrm{div}(u_3) \subseteq E$, whose strict transform goes through all points x_i, $i \geq 0$. Necessarily $\mathcal{C} \subseteq \mathrm{Sing}_m(X)$, so we may assume that $\mathcal{C} \subset \mathrm{div}(u_j) \subseteq E$ for $j = 2$ or $j = 3$. In particular, we may take $\widehat{u_j} = u_j$ for $j = 2$ or $j = 3$. This implies that $v(u_j) > v(u_1^n) = nv(u_1)$ for all $n \geq 1$: a contradiction, since our given valuation v has rank one.
Case 3. $E \subseteq \mathrm{div}(u_1 u_3)$ and we are not in case 2, i.e. there exists $i_0 \geq 0$ such that either x_{i_0+1} is not rational over x_{i_0} or E_{i_0+1} has one more component than E_{i_0}), i_0 minimal. Suppose $\beta(x_{i_0}) \geq 1$.

If x_{i_0+1} is not rational over x_{i_0}, we get

$$\beta(x) \geq \beta(x_{i_0}) > \beta(x_{i_0+1})$$

by **7.4.3**(ii)(iii) and **7.4.4**(iii): note that $C(u,z) \leq \beta(u,z)$ since $\mathrm{div}(u_2) \not\subseteq E$ and

$$1 + \left\lceil \frac{x}{2} \right\rceil \leq x \quad \text{for every } x \geq 1.$$

If E_{i_0+1} has one more component than E_{i_0}, we have

$$C(u_{i_0+1}, z_{i_0+1}) \leq \frac{\beta(x_{i_0})}{2} \leq \frac{\beta(x)}{2}$$

by **7.4.3**(ii), (iii), (iv). This gives **7.4.5.7**(iii) if $1 \leq \beta(x) < 2$.

Now, $\gamma(x_{i_0+1}) = 1 + \lfloor C(u_{i_0+1}, z_{i_0+1}) \rfloor$, $\gamma(x) = \lceil \beta(x) \rceil$, so we get **7.4.5.7**(i) if $\beta(x) > 2$. Assume that

$$\beta(x_{i_0}) = \beta(x) = 2. \tag{56}$$

Since $\gamma(x) = 2$, we get **7.4.5.7**(i) unless $\gamma(x_i) = 2$ for $i \geq 0$ by Proposition 7.4.3.

Let $i_1 > i_0$ be the largest index such that E_i has as many components as E_{i_0} for $i_0 \leq i \leq i_1$. We may assume $i_1 < +\infty$ by case 1 and we have

$$\gamma(x_{i_1}) = 2 = 1 + \lfloor C(u_{i_1}, z_{i_1}) \rfloor.$$

By **7.4.4**(iii), we get $\beta(x_{i_1+1}) < 2$. Now the point x_{i_1+1} falls into case 2 above or into case 3 with (56) not satisfied. This concludes the proof in case 3.

The end of the proof of **7.4.5.7** is just a logical game: we reach the assumption $E \subseteq \mathrm{div}(u_1 u_3)$ for some point x_i, $i \geq 0$ provided we are not in case 1.

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