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ORIGINAL PAPER

Complete convergence for arrays of rowwise negatively orthant dependent random variables

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Abstract Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise negatively orthant dependent random variables. Some sufficient conditions for complete convergence for arrays of rowwise negatively orthant dependent random variables are presented without assumptions of identical distribution. As an application, the Marcinkiewicz–Zygmund type strong law of large numbers for weighted sums of negatively orthant dependent random variables is obtained.

Keywords Arrays of rowwise negatively orthant dependent random variables · Sequences of negatively orthant dependent random variables · Marcinkiewicz–Zygmund type strong law of large numbers · Complete convergence

Mathematics Subject Classification (2000) 60F15

1 Introduction

The concept of complete convergence was introduced by Hsu and Robbins [9] as follows. A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant C if $\sum_{n=1}^{\infty} P(|U_n - C| > \varepsilon) < \infty$ for all $\varepsilon > 0$. In view of the Borel–Cantelli lemma, this implies that $U_n \to C$ almost surely (a.s.). The converse is true if the $\{U_n, n \geq 1\}$ are independent.Hsu and Robbins [9] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [7] proved the converse. The result of Hsu–Robbins–Erdös is a fundamental theorem in probability theory and has been generalized

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and extended in several directions by many authors. One of the most important generalizations is the Baum–Katz–Spitzer type result. For more details about the Baum–Katz–Spitzer type results, one can refer to Spitzer [15], Baum and Katz [5], Gut [8], and so forth. The main purpose of the present investigation is to provide the Baum–Katz–Spitzer type results for weighted sums of negatively orthant dependent random variables and arrays of rowwise negatively orthant dependent random variables.

Let us recall the definitions of negatively associated random variables and negatively orthant dependent random variables.

Definition 1.1 A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be negatively associated (NA) if for every pair of disjoint subsets A_1, A_2 of $\{1, 2, \ldots, n\}$,

$$Cov\{f(X_i : i \in A_1), g(X_j : j \in A_2)\} \leq 0, \tag{1.1}$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite sequence $\{X_n, n \geq 1\}$ is NA if every finite subcollection is negatively associated.

An array of random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is called rowwise NA random variables if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is a sequence of NA random variables.

Definition 1.2 A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be negatively orthant dependent (NOD) if

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$. An infinite sequence $\{X_n, n \geq 1\}$ is said to be NOD if every finite subcollection is NOD.

An array of random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is called rowwise NOD random variables if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is a sequence of NOD random variables.

The concepts of NA and NOD sequences were introduced by Joag-Dev and Proschan [10]. Obviously, independent random variables are NOD. Joag-Dev and Proschan [10] pointed out that NA random variables are NOD. They also presented an example in which $X = (X_1, X_2, X_3, X_4)$ possesses NOD, but does not possess NA. So we can see that NOD is weaker than NA. A number of limit theorems for NOD random variables have been established by many authors. We refer to Volodin [17] for the Kolmogorov exponential inequality, Asadian et al. [4] for the Rosenthal's type inequality, Kim [11] for Hájek–Rényi type inequality, Amini et al. [2,3], Ko and Kim [13], and Klesov et al. [12] for almost sure convergence, Amini and Bozorgnia [1], Kuczmaszewska [14], Taylor et al. [16], Zareo and Jabbari [20] and Wu [18,19] for complete convergence, and so on.

Our goal in this paper is to further study the complete convergence for arrays of rowwise NOD random variables under some moment conditions. We will provide the Baum–Katz–Spitzer type results for weighted sums of NOD random variables and arrays of rowwise NOD random variables. As an application, the Marcinkiewicz–Zygmund type strong law of large numbers for weighted sums of NOD random variables is obtained. We will give some sufficient conditions for complete convergence for an array of rowwise NOD random variables without assumption of identical distribution. The results presented in this paper are obtained by using the truncated method and the Rosenthal's type inequality of NOD random variables.

Definition 1.3 An array of random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_{ni}| > x) \leq CP(|X| > x) \tag{1.2}$$

for all $x \geq 0$, $i \geq 1$ and $n \geq 1$.

The following lemmas are useful for the proof of the main results.

Lemma 1.4 (cf. [6]). *Let random variables $X_1, X_2, \ldots, X_n$ be NOD, $f_1, f_2, \ldots, f_n$ be all nondecreasing (or all nonincreasing) functions, then random variables $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are NOD.*

Lemma 1.5 (cf. [4,19]). *Let $p \geq 2$ and $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for every $n \geq 1$. Then there exists a positive constant C depending only on p such that for every $n \geq 1$,*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq C\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}, \tag{1.3}$$

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leq C\log^p 2n\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}. \tag{1.4}$$

Lemma 1.6 *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$E|X_n|^\alpha I(|X_n| \leq b) \leq C_1\left[E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)\right], \tag{1.5}$$

$$E|X_n|^\alpha I(|X_n| > b) \leq C_2 E|X|^\alpha I(|X| > b), \tag{1.6}$$

where C_1 and C_2 are positive constants.

2 Main results

Throughout the paper, let $I(A)$ be the indicator function of the set A. C denotes a positive constant which may be different in various places and $a_n = O(b_n)$ stands for $a_n \leq Cb_n$.

Our main results are as follows.

Theorem 2.1 *Let $\{X_{ni} : i \geq 1, n \geq 1\}$ be an array of rowwise NOD random variables which is stochastically dominated by a random variable X and $\{a_{ni} : i \geq 1, n \geq 1\}$ be an array of real numbers. Assume that there exist some δ with $0 < \delta < 1$ and some α with $0 < \alpha < 2$ such that $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n^\delta)$ and assume further that $EX_{ni} = 0$ if $1 < \alpha < 2$. If for some $h > 0$ and $\gamma > 0$ such that*

$$E\exp\left(h|X|^\gamma\right) < \infty, \tag{2.1}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty, \tag{2.2}$$

where $p \geq 1/\alpha$ and $b_n \doteq n^{1/\alpha}\log^{1/\gamma} n$.

Proof For fixed $n \geq 1$,define

$$X_i^{(n)} = -b_n I(X_{ni} < -b_n) + X_{ni} I(|X_{ni}| \leq b_n) + b_n I(X_{ni} > b_n),\ i \geq 1,$$
$$T_j^{(n)} = \sum_{i=1}^{j} a_{ni} \left(X_i^{(n)} - EX_i^{(n)}\right),\ j = 1, 2, \ldots, n.$$

It is easy to check that for any $\varepsilon > 0$,

$$\left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni} X_{ni}\right| > \varepsilon b_n\right) \subset \left(\max_{1\leq i\leq n} |X_{ni}| > b_n\right) \bigcup \left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni} X_i^{(n)}\right| > \varepsilon b_n\right),$$

which implies that

$$\begin{aligned} & P\left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni} X_{ni}\right| > \varepsilon b_n\right) \\ & \leq P\left(\max_{1\leq i\leq n} |X_{ni}| > b_n\right) + P\left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni} X_i^{(n)}\right| > \varepsilon b_n\right) \\ & \leq \sum_{i=1}^{n} P\left(|X_{ni}| > b_n\right) + P\left(\max_{1\leq j\leq n} \left|T_j^{(n)}\right| > \varepsilon b_n - \max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni} EX_i^{(n)}\right|\right). \end{aligned} \tag{2.3}$$

Firstly, we will show that

$$b_n^{-1} \max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni} EX_i^{(n)}\right| \to 0, \quad \text{as } n \to \infty. \tag{2.4}$$

By $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n^\delta)$ and Hölder's inequality, we have for $1 \leq k < \alpha$ that

$$\sum_{i=1}^{n} |a_{ni}|^k \leq \left(\sum_{i=1}^{n} \left(|a_{ni}|^k\right)^{\frac{\alpha}{k}}\right)^{\frac{k}{\alpha}} \left(\sum_{i=1}^{n} 1\right)^{\frac{\alpha-k}{\alpha}} \leq Cn. \tag{2.5}$$

Hence, when $1 < \alpha < 2$, we have by $EX_{ni} = 0$, (1.6) of Lemma 1.6, (2.5)(Taking $k = 1$), Markov's inequality and (2.1) that

$$
\begin{aligned}
b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}EX_i^{(n)}\right| &\le \sum_{i=1}^{n}|a_{ni}|P(|X_{ni}|>b_n)\\
&\quad +b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}EX_{ni}I(|X_{ni}|>b_n)\right|\\
&\le C\sum_{i=1}^{n}|a_{ni}|P(|X|>b_n)+b_n^{-1}\sum_{i=1}^{n}|a_{ni}|E|X_{ni}|I(|X_{ni}|>b_n)\\
&\le Cn\frac{E\exp(h|X|^{\gamma})}{\exp(hb_n^{\gamma})}+Cb_n^{-1}\sum_{i=1}^{n}|a_{ni}|E|X|I(|X|>b_n)\\
&\le \frac{Cn}{n^{hn^{\gamma/\alpha}}}+Cb_n^{-1}nE|X|I(|X|>b_n)\\
&= \frac{Cn}{n^{hn^{\gamma/\alpha}}}+Cb_n^{-1}n\sum_{k=n}^{\infty}E|X|I(b_k<|X|\le b_{k+1}) \qquad (2.6)\\
&\le \frac{Cn}{n^{hn^{\gamma/\alpha}}}+Cb_n^{-1}n\sum_{k=n}^{\infty}b_{k+1}P(|X|>b_k)\\
&\le \frac{Cn}{n^{hn^{\gamma/\alpha}}}+Cb_n^{-1}n\sum_{k=n}^{\infty}b_{k+1}\frac{E\exp(h|X|^{\gamma})}{\exp(hb_k^{\gamma})}\\
&\le \frac{Cn}{n^{hn^{\gamma/\alpha}}}+Cb_n^{-1}n\sum_{k=n}^{\infty}(k+1)^{1/\alpha}(\log(k+1))^{1/\gamma}k^{-hk^{\gamma/\alpha}}\\
&\le \frac{Cn}{n^{hn^{\gamma/\alpha}}}+Cb_n^{-1}\sum_{k=n}^{\infty}(k+1)^{1/\alpha+1}(\log(k+1))^{1/\gamma}k^{-hk^{\gamma/\alpha}}\\
&\le \frac{Cn}{n^{hn^{\gamma/\alpha}}}+Cn^{-1/\alpha}(\log n)^{-1/\gamma}\to 0,\quad \text{as } n\to\infty.
\end{aligned}
$$

Elementary Jensen's inequality implies that for any $0<s<t$,

$$
\left(\sum_{i=1}^{n}|a_{ni}|^{t}\right)^{1/t}\le\left(\sum_{i=1}^{n}|a_{ni}|^{s}\right)^{1/s}. \qquad (2.7)
$$

Therefore, when $0<\alpha\le 1$, we have by (1.5) of Lemmas 1.6, (2.7), Markov's inequality and (2.1) that

$$
\begin{aligned}
b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}EX_i^{(n)}\right| &\le \sum_{i=1}^{n}|a_{ni}|P(|X_{ni}|>b_n)+b_n^{-1}\sum_{i=1}^{n}|a_{ni}|E|X_{ni}|I(|X_{ni}|\le b_n)\\
&\le C\sum_{i=1}^{n}|a_{ni}|P(|X|>b_n)\\
&\quad +Cb_n^{-1}\sum_{i=1}^{n}|a_{ni}|\left(E|X|I(|X|\le b_n)+b_nP(|X|>b_n)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le Cb_n^{-1}n^{\delta/\alpha}E|X|I(|X|\le b_n)+Cn^{\delta/\alpha}P(|X|>b_n)\\
&\le Cb_n^{-1}n^{\delta/\alpha}\sum_{k=2}^{n}E|X|I(b_{k-1}<|X|\le b_k)+\frac{Cn^{\delta/\alpha}E\exp(h|X|^{\gamma})}{\exp(hb_n^{\gamma})}\\
&\le Cb_n^{-1}n^{\delta/\alpha}\sum_{k=2}^{n}b_kP(|X|>b_{k-1})+\frac{Cn^{\delta/\alpha}}{n^{hn^{\gamma/\alpha}}} \qquad (2.8)\\
&\le Cb_n^{-1}n^{\delta/\alpha}\sum_{k=2}^{n}b_k\frac{E\exp(h|X|^{\gamma})}{\exp(hb_{k-1}^{\gamma})}+\frac{Cn^{\delta/\alpha}}{n^{hn^{\gamma/\alpha}}}\\
&\le Cb_n^{-1}n^{\delta/\alpha}\sum_{k=2}^{n}k^{1/\alpha}(\log k)^{1/\gamma}(k-1)^{-h(k-1)^{\gamma/\alpha}}+\frac{Cn^{\delta/\alpha}}{n^{hn^{\gamma/\alpha}}}\\
&\le Cn^{-1/\alpha}(\log n)^{-1/\gamma}n^{\delta/\alpha}+\frac{Cn^{\delta/\alpha}}{n^{hn^{\gamma/\alpha}}}\\
&= C(\log n)^{-1/\gamma}n^{\delta/\alpha-1/\alpha}+\frac{Cn^{\delta/\alpha}}{n^{hn^{\gamma/\alpha}}}\to 0,\quad \text{as } n\to\infty.
\end{aligned}
$$

By (2.6) and (2.8), we can get (2.4) immediately. Hence, for n large enough,

$$
P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_{ni}\right|>\varepsilon b_n\right)\le\sum_{i=1}^{n}P(|X_{ni}|>b_n)+P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right|>\frac{\varepsilon}{2}b_n\right).
$$

To prove (2.2), we only need to show that

$$
I\doteq\sum_{n=1}^{\infty}n^{p\alpha-2}\sum_{i=1}^{n}P(|X_{ni}|>b_n)<\infty \tag{2.9}
$$

and

$$
J\doteq\sum_{n=1}^{\infty}n^{p\alpha-2}P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right|>\frac{\varepsilon}{2}b_n\right)<\infty. \tag{2.10}
$$

By Definition 1.3, Markov's inequality and (2.1), we can see that

$$
\begin{aligned}
I&\doteq\sum_{n=1}^{\infty}n^{p\alpha-2}\sum_{i=1}^{n}P(|X_{ni}|>b_n)\\
&\le C\sum_{n=1}^{\infty}n^{p\alpha-2}\sum_{i=1}^{n}P(|X|>b_n)\\
&\le C\sum_{n=1}^{\infty}n^{p\alpha-1}\frac{E\exp(h|X|^{\gamma})}{\exp(hb_n^{\gamma})} \qquad (2.11)\\
&\le C\sum_{n=1}^{\infty}\frac{n^{p\alpha-1}}{n^{hn^{\gamma/\alpha}}}<\infty.
\end{aligned}
$$

For fixed $n \geq 1$, it is easily seen that $\{X_i^{(n)}, 1 \leq i \leq n\}$ are still NOD by Lemma 1.4. For $q > 2$, it follows from (1.4) of Lemma 1.5, C_r's inequality and Jensen's inequality that

$$\begin{aligned} J &\doteq \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \frac{\varepsilon}{2} b_n\right) \\ &\leq C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} E\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right|^q\right) \qquad (2.12)\\ &\leq C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \left[\sum_{i=1}^{n} |a_{ni}|^q E\left|X_i^{(n)}\right|^q + \left(\sum_{i=1}^{n} |a_{ni}|^2 E\left|X_i^{(n)}\right|^2\right)^{q/2}\right] \\ &\doteq J_1 + J_2. \end{aligned}$$

Taking $q > \max\{2, \alpha(p\alpha-1)/(1-\delta)\}$, which implies that $p\alpha - 2 + q\delta/\alpha - q/\alpha < -1$ and $q > \alpha$. It follows from C_r's inequality, (1.5) of Lemmas 1.6, (2.7), Markov's inequality and (2.1) that

$$\begin{aligned} J_1 &\doteq C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \sum_{i=1}^{n} |a_{ni}|^q E\left|X_i^{(n)}\right|^q \\ &\leq C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \sum_{i=1}^{n} |a_{ni}|^q \left[E\,|X_{ni}|^q\, I(|X_{ni}| \leq b_n) + b_n^q P(|X_{ni}| > b_n)\right] \\ &\leq C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \sum_{i=1}^{n} |a_{ni}|^q \left[E|X|^q I(|X| \leq b_n) + b_n^q P(|X| > b_n)\right] \\ &\leq C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q} (\log n)^q E|X|^q I(|X| \leq b_n) \\ &\quad + C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} (\log n)^q P(|X| > b_n) \qquad (2.13)\\ &\leq C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q} (\log n)^q \sum_{k=2}^{n} E|X|^q I(b_{k-1} < |X| \leq b_k) \\ &\quad + C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} (\log n)^q \frac{E\exp(h|X|^\gamma)}{\exp(hb_n^\gamma)} \\ &\leq C\sum_{k=2}^{\infty}\sum_{n=k}^{\infty} n^{p\alpha-2+q\delta/\alpha-q/\alpha} (\log n)^{q-q/\gamma} b_k^q P(|X| > b_{k-1}) \\ &\quad + C\sum_{n=2}^{\infty} \frac{n^{p\alpha-2+q\delta/\alpha}(\log n)^q}{n^{hn^{\gamma/\alpha}}} \\ &\leq C\sum_{k=2}^{\infty} b_k^q \frac{E\exp(h|X|^\gamma)}{\exp(hb_{k-1}^\gamma)} + C\sum_{n=2}^{\infty} \frac{n^{p\alpha-2+q\delta/\alpha}(\log n)^q}{n^{hn^{\gamma/\alpha}}} \\ &\leq C\sum_{k=2}^{\infty} \frac{k^{q/\alpha}(\log k)^{q/\gamma}}{(k-1)^{h(k-1)^{\gamma/\alpha}}} + C\sum_{n=2}^{\infty} \frac{n^{p\alpha-2+q\delta/\alpha}(\log n)^q}{n^{hn^{\gamma/\alpha}}} < \infty. \end{aligned}$$

By C_r's inequality, (1.5) of Lemma 1.6, (2.7) and Jensen's inequality, we can get that

$$
\begin{aligned}
J_2 &\doteq C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \left(\sum_{i=1}^{n} |a_{ni}|^2 E\left|X_i^{(n)}\right|^2\right)^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \left(\sum_{i=1}^{n} |a_{ni}|^2 \left[E|X_{ni}|^2 I(|X_{ni}|\le b_n) + b_n^2 P(|X_{ni}|>b_n)\right]\right)^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} (\log n)^q \left(\sum_{i=1}^{n} |a_{ni}|^2 \left[EX^2 I(|X|\le b_n) + b_n^2 P(|X|>b_n)\right]\right)^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q} (\log n)^q \left[EX^2 I(|X|\le b_n) + b_n^2 P(|X|>b_n)\right]^{q/2} \qquad (2.14) \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q} (\log n)^q \left[EX^2 I(|X|\le b_n)\right]^{q/2} \\
&\quad + C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} (\log n)^q \left[P(|X|>b_n)\right]^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q} (\log n)^q E|X|^q I(|X|\le b_n) \\
&\quad + C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} (\log n)^q P(|X|>b_n) < \infty.
\end{aligned}
\tag{2.15}
$$

Therefore, the desired result (2.2) follows from (2.11)–(2.14) immediately. This completes the proof of the theorem. □

Similar to the proof of Theorem 2.1, we can get the following result for sequences of NOD random variables.

Theorem 2.2 *Let $\{X_n, n \ge 1\}$ be a sequence of NOD random variables which is stochastically dominated by a random variable X and $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers. Assume that there exist some δ with $0 < \delta < 1$ and some α with $0 < \alpha < 2$ such that $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n^\delta)$ and assume further that $EX_n = 0$ if $1 < \alpha < 2$. If (2.1) holds true for some $h > 0$ and $\gamma > 0$, then for any $\varepsilon > 0$,*

$$
\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon b_n\right) < \infty, \tag{2.16}
$$

where $p \ge 1/\alpha$ and $b_n \doteq n^{1/\alpha}\log^{1/\gamma} n$.

The following result provides the Marcinkiewicz–Zygmund type strong law of large numbers for weighted sums $\sum_{i=1}^{n} a_i X_i$ of a sequence of NOD random variables.

Theorem 2.3 *Let $\{X_n, n \ge 1\}$ be a sequence of NOD random variables which is stochastically dominated by a random variable X and $\{a_n, n \ge 1\}$ be a sequence of real numbers. Assume that there exist some δ with $0 < \delta < 1$ and some α with $0 < \alpha < 2$ such that*

$\sum_{i=1}^{n} |a_i|^\alpha = O(n^\delta)$ *and assume further that* $EX_n = 0$ *if* $1 < \alpha < 2$. *If* (2.1) *holds true for some* $h > 0$ *and* $\gamma > 0$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n} \left|S_j\right| > \varepsilon b_n\right) < \infty \tag{2.17}$$

and

$$\lim_{n\to\infty} \frac{|S_n|}{b_n} = 0 \ \ a.s., \tag{2.18}$$

where $p \ge 1/\alpha$, $b_n \doteq n^{1/\alpha} \log^{1/\gamma} n$ *and* $S_n = \sum_{i=1}^{n} a_i X_i$ *for* $n \ge 1$.

Proof Similar to the proof of Theorem 2.1, we can get (2.17) immediately, which yields that

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n} \left|S_j\right| > \varepsilon b_n\right) < \infty. \tag{2.19}$$

Therefore,

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n} \left|S_j\right| > \varepsilon b_n\right) \\
&= \sum_{i=0}^{\infty} \sum_{n=2^i}^{2^{i+1}-1} n^{-1} P\left(\max_{1\le j\le n} |S_j| > \varepsilon n^{\frac{1}{\alpha}} (\log n)^{\frac{1}{\gamma}}\right) \\
&\ge \frac{1}{2} \sum_{i=1}^{\infty} P\left(\max_{1\le j\le 2^i} |S_j| > \varepsilon 2^{\frac{i+1}{\alpha}} (\log 2^{i+1})^{\frac{1}{\gamma}}\right).
\end{aligned}$$

By Borel–Cantelli Lemma, we obtain that

$$\lim_{i\to\infty} \frac{\max_{1\le j\le 2^i} |S_j|}{2^{\frac{i+1}{\alpha}} (\log 2^{i+1})^{\frac{1}{\gamma}}} = 0 \ \ a.s.. \tag{2.20}$$

For all positive integers n, there exists a positive integer i_0 such that $2^{i_0-1} \le n < 2^{i_0}$. We have by (2.20) that

$$\frac{|S_n|}{b_n} \le \max_{2^{i_0-1}\le n<2^{i_0}} \frac{|S_n|}{b_n} \le \frac{2^{\frac{2}{\alpha}} \max_{1\le j\le 2^i} |S_j|}{2^{\frac{i_0+1}{\alpha}} (\log 2^{i_0+1})^{\frac{1}{\gamma}}} \left(\frac{i_0+1}{i_0-1}\right)^{\frac{1}{\gamma}} \to 0 \ \ a.s., \quad \text{as } i_0 \to \infty,$$

which implies (2.18). This completes the proof of the theorem. □

Remark 2.1 In Theorems 2.1–2.3, the condition "there exist some δ with $0 < \delta < 1$ and some α with $0 < \alpha < 2$ such that $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n^\delta)$ (or $\sum_{i=1}^{n} |a_i|^\alpha = O(n^\delta)$)" is needed. If we consider the weaker condition "there exists some α with $0 < \alpha < 2$ such that $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ (or $\sum_{i=1}^{n} |a_i|^\alpha = O(n)$)", we can get the following Theorems 2.4–2.6. Their proofs are similar to that of Theorem 2.1, so the details are omitted.

Theorem 2.4 *Let* $\{X_{ni} : i \ge 1, n \ge 1\}$ *be an array of rowwise NOD random variables which is stochastically dominated by a random variable* X *and* $\{a_{ni} : i \ge 1, n \ge 1\}$ *be an array of real numbers. Assume that there exists some* α *with* $0 < \alpha < 2$ *such that* $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$

and assume further that $EX_{ni} = 0$ *if* $1 < \alpha < 2$. *If* (2.1) *holds true for some* $h > 0$ *and* $\gamma > 0$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty, \tag{2.21}$$

where $b_n \doteq n^{1/\alpha}\log^{1/\gamma} n$.

Theorem 2.5 *Let* $\{X_n, n \ge 1\}$ *be a sequence of NOD random variables which is stochastically dominated by a random variable* X *and* $\{a_{ni}, i \ge 1, n \ge 1\}$ *be an array of real numbers. Assume that there exists some* α *with* $0 < \alpha < 2$ *such that* $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ *and assume further that* $EX_n = 0$ *if* $1 < \alpha < 2$. *If* (2.1) *holds true for some* $h > 0$ *and* $\gamma > 0$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_{i}\right| > \varepsilon b_n\right) < \infty, \tag{2.22}$$

where $b_n \doteq n^{1/\alpha}\log^{1/\gamma} n$.

Theorem 2.6 *Let* $\{X_n, n \ge 1\}$ *be a sequence of NOD random variables which is stochastically dominated by a random variable* X *and* $\{a_n, n \ge 1\}$ *be a sequence of real numbers. Assume that there exists some* α *with* $0 < \alpha < 2$ *such that* $\sum_{i=1}^{n} |a_i|^{\alpha} = O(n)$ *and assume further that* $EX_n = 0$ *if* $1 < \alpha < 2$. *If (2.1) holds true for some* $h > 0$ *and* $\gamma > 0$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n}\left|S_j\right| > \varepsilon b_n\right) < \infty \tag{2.23}$$

and

$$\lim_{n\to\infty} \frac{|S_n|}{b_n} = 0 \ \ a.s., \tag{2.24}$$

where $b_n \doteq n^{1/\alpha}\log^{1/\gamma} n$ *and* $S_n = \sum_{i=1}^{n} a_i X_i$ *for* $n \ge 1$.

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