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ORIGINAL PAPER

An iterative method for semistable solutions

Steeve Burnet · Célia Jean-Alexis · Alain Pietrus

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Abstract We consider a semistability property for a solution of variational inclusion of the form $0 \in \varphi(z) + F(z)$ where φ is a single-valued function admitting a second order Fréchet derivative and F is a set-valued map. We show that this property ensures interesting results for the order of convergence for a Hummel-Seebeck type method.

Keywords Set-valued mapping · Semistability property · Hölder-type condition · Superlinear convergence · Superquadratic convergence · Cubic convergence

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1 Introduction

Variational inclusions are an abstract model of a wide variety of variational problems including linear and non-linear complementarity problems, systems of non-linear equations, variational inequalities,…

In the last decade, several iterative methods have been presented for solving variational inclusions of the form

$$0 \in \varphi(z) + F(z) \tag{1}$$

S. Burnet · C. Jean-Alexis · A. Pietrus (✉)
Laboratoire LAMIA - EA 4540, Département de Mathématiques,
Université des Antilles et de la Guyane, 97159 Pointe–à–Pitre, France
e-mail: apietrus@univ-ag.fr

S. Burnet
e-mail: steeve.burnet@univ-ag.fr

C. Jean-Alexis
e-mail: celia.jean-alexis@univ-ag.fr

where φ is a single-valued function and F is a set-valued map. These methods consist to generate an iterative sequence (z_n) obtained by subsequently solving implicit subproblems of the form $0 \in A(z_n, z_{n+1}) + F(z_{n+1})$ where A denotes some approximation of the mapping φ.

When the Fréchet derivative φ' of the function φ is locally Lipschitz, Dontchev [6,7] associates to (1) a Newton-type method based on a partial linearization which is locally quadratically convergent. Using a second-degree Taylor polynomial expansion of φ at z_k, Geoffroy, Hilout and Piétrus [11] considered the relation

$$0 \in \varphi(z_k) + \varphi'(z_k)(z_{k+1} - z_k) + \frac{1}{2}\varphi''(z_k)(z_{k+1} - z_k)^2 + F(z_{k+1}) \tag{2}$$

where $\varphi'(z)$ and $\varphi''(z)$ denote respectively the first and the second Fréchet derivative of φ at z. They prove the cubic convergence of this method when φ' and φ'' are Lipschitz.

Nevertheless, it is well-known that the computation of the second derivative of a function in application is very expansive. To overcome this difficulty, in [13], the author associated to (1) the iteration

$$0 \in \varphi(z_k) + \frac{1}{2}\left(\varphi'(z_k) + \varphi'(z_{k+1})\right)(z_{k+1} - z_k) + F(z_{k+1}) \tag{3}$$

and proved the cubic convergence of this method inspired by the work of Hummel and Seebeck [12].

Let us remark that all these methods have been studied when a pseudo-Lipschitz property is satisfied for the set-valued map $(\varphi + F)^{-1}$ or one of its approximation. For more details on this property, the reader could refer to [2,3,8–10,15].

In the present paper, we study the inclusion (1) where the function φ is defined from $\mathbb{R}^q$ into $\mathbb{R}^q$ and F is a set-valued map from $\mathbb{R}^q$ to the closed subsets of $\mathbb{R}^q$. Instead of continuity properties for set-valued map, we use another assumption which is directly connected to a solution: the semistability concept. This concept has been introduced by Bonnans [4] for variational inequalities. One says that a solution $\overline{z}$ of a variational inclusion is semistable if, given a small perturbation on the left-hand side, a solution z of the perturbed variational inclusion that is sufficiently close of $\overline{z}$, is such that the distance of z to $\overline{z}$ is of the order of the magnitude of the perturbation.

In a recent paper [5], the authors studied the local behavior of an iterative method for solving (1) under some semistability property of the solution of (1), they obtained superquadratic or cubic convergence for the sequence defined by the following method

$$0 \in \varphi(z_k) + \varphi'(z_k)(z_{k+1} - z_k) + M_k(z_{k+1} - z_k)^2 + F(z_{k+1})$$

where M_k is a $q \times q$ matrix.

Following this work, we consider the relation

$$0 \in \varphi(z_k) + \frac{1}{2}\left(M_k + M_{k+1}\right)(z_{k+1} - z_k) + F(z_{k+1}) \tag{4}$$

where M_k and M_{k+1} are $q \times q$ matrices.

Let us remark that when $M_k = \varphi'(z_k)$, we find the method (3).

Our purpose is to show the superquadratic or the cubic convergence of (4) when the solution of (1) is semistable and when the second Fréchet derivative φ'' satisfies first of all, a Lipschitz condition then Hölder-type conditions.

This paper is organized as follows: in Sect. 2, we recall a few preliminary results and in Sect. 3, we study the behavior of the method (4).

In the sequel, all the norms will be denoted by $\|.\|$ and a set-valued map F from $\mathbb{R}^q$ to $\mathbb{R}^q$ is indicated by $F : \mathbb{R}^q \to 2^{\mathbb{R}^q}$.

2 Preliminaries

In this section, we collect some definitions that will need to prove our results.

First of all, we define the notion of semistability which will play a key role in our study.

Definition 1 A solution $\bar{z}$ of (1) is said to be semistable if $c_1 > 0$ and $c_2 > 0$ exist such that, for all $(z, \delta) \in \mathbb{R}^q \times \mathbb{R}^q$, solution of

$$\delta \in \varphi(z) + F(z)$$

and $\|z - \bar{z}\| \leq c_1$ then $\|z - \bar{z}\| \leq c_2\|\delta\|$.

Note that a sufficient condition for semistability is the strong regularity of Robinson [14]. For more details on this topic, the reader can refer to [4].

In the sequel, we will need the following Hölder-type properties.

Definition 2 Let $\varphi : \mathbb{R}^q \to \mathbb{R}^q$ be a function. One says that φ satisfies a Hölder-type condition on a neighborhood Ω of $\overline{x}$ if

$$\exists\ K > 0, \alpha \in (0, 1], \quad such\ that \quad \|\varphi(x) - \varphi(y)\| \leq K\|x - y\|^{\alpha}, \quad \forall x, y \in \Omega.$$

Note that when $\alpha = 1$, we have the Lipschitz condition for φ.

Definition 3 Let $\varphi : \mathbb{R}^q \to \mathbb{R}^q$ be a function. One says that φ satisfies a center-Hölder-type condition on a neighborhood Ω of $\overline{x}$ if

$$\exists\ K' > 0, \alpha' \in (0, 1], \quad such\ \ that \quad \|\varphi(x) - \varphi(\overline{x})\| \leq K'\|x - \overline{x}\|^{\alpha'}, \quad \forall x \in \Omega.$$

The inspiration for considering such a condition comes from [1]. Let us remark that, in some cases, the center-Hölder condition holds whereas the Hölder condition does not.

3 Main result

In this section, we propose an iterative scheme for approximating $\bar{z}$ which is a solution of inclusion (1).

For this, we consider the following approximation ψ of φ:

$$\psi(u, v) = \varphi(u) + \frac{1}{2}(\varphi'(u) + \varphi'(v))(u - v)$$

and we introduce the following algorithm:

- given any starting point z_0 in some neighborhood of $\bar{z}$,
- for $k = 0, 1, ..,$ while z_k does not satisfy (1), choose $\Xi_k(.)$ an approximation of $\psi(z_k, .)$ defined by:

$$\Xi_k(z) = \varphi(z_k) + \frac{1}{2}(M_k + M_{k+1})(z - z_k), \tag{5}$$

- compute the z_{k+1} solution of

$$0 \in \Xi_k(z_{k+1}) + F(z_{k+1}).$$

We can establish the rate of convergence of (z_k) throughout the following result:

Theorem 1 *Let $\bar{z}$ be a semistable solution of (1) and let (z_k) be a sequence computed by (4) which converges towards $\bar{z}$. If φ'' is locally Lipschitz then*

(i) *if $\psi(z_k, z_{k+1}) - \Xi_k(z_{k+1}) = o\left(\|z_{k+1} - z_k\|^2\right)$ then (z_k) converges superquadratically*
(ii) *if $\psi(z_k, z_{k+1}) - \Xi_k(z_{k+1}) = O\left(\|z_{k+1} - z_k\|^3\right)$ then (z_k) converges cubically.*

To prove Theorem 1, we need the following lemma:

Lemma 1 *Let $\varphi : \mathbb{R}^q \to \mathbb{R}^q$ be a function admitting a second order Fréchet derivative and let ψ the approximation of φ previously defined. If φ'' is locally L-Lipschitz, one has, for all $u, v \in \mathbb{R}^q$, the following inequality*

$$\|\varphi(v) - \psi(u, v)\| \leq \frac{5L}{12}\|v - u\|^3. \tag{6}$$

Proof of Lemma 1: Since φ'' is locally L-Lipschitz, we get

$$\begin{aligned}
\|\varphi(v) - \psi(u, v)\| &= \|\varphi(v) - \varphi(u) - \frac{1}{2}(\varphi'(u) + \varphi'(v))(v - u)\| \\
&= \|\varphi(v) - \varphi(u) - \varphi'(u)(v - u) - \frac{1}{2}\varphi''(u)(v - u)^2 \\
&\quad -\frac{1}{2}(\varphi'(v) - \varphi'(u) - \varphi''(u)(v - u))(v - u)\| \\
&\leq \|\varphi(v) - \varphi(u) - \varphi'(u)(v - u) - \frac{1}{2}\varphi''(u)(v - u)^2\| \\
&\quad +\frac{1}{2}\|(\varphi'(v) - \varphi'(u) - \varphi''(u)(v - u))(v - u)\| \\
&\leq \|\int_0^1 (1 - t)(\varphi''(u + t(v - u)) - \varphi''(u))(v - u)^2 dt\| \\
&\quad +\frac{1}{2}\|v - u\|^2 \int_0^1 \|\varphi''(tv + (1 - t)u) - \varphi''(u)\| dt \\
&\leq L\|v - u\|^3 \int_0^1 t(1 - t)dt + \frac{L}{2}\|v - u\|^3 \int_0^1 t dt \\
&\leq \frac{L}{6}\|v - u\|^3 + \frac{L}{4}\|v - u\|^3 \\
&\leq \frac{5L}{12}\|v - u\|^3.
\end{aligned}$$

Proof of Theorem 1: We write (4) as:

$$\psi(z_k, z_{k+1}) - \Xi_k(z_{k+1}) \in \psi(z_k, z_{k+1}) + F(z_{k+1}) \tag{7}$$

and using (6), we get

$$\psi(z_k, z_{k+1}) - \Xi_k(z_{k+1}) + o(\|z_{k+1} - z_k\|^2) \in \varphi(z_{k+1}) + F(z_{k+1}).$$

Since $\bar{z}$ is a semistable solution of (1), we get:

$$\|z_{k+1}-\bar{z}\| = O\left(\|\psi(z_k, z_{k+1}) - \Xi_k(z_{k+1}) + o(\|z_{k+1}-z_k\|^2)\|\right) = o(\|z_{k+1}-z_k\|^2)$$

then

$$\|z_{k+1}-\bar{z}\| = o(\|z_k-\bar{z}\|^2)$$

which proves (i).

The proof of (ii) is similar, using (6), we deduce that

$$\psi(z_k, z_{k+1}) - \Xi_k(z_{k+1}) + O(\|z_{k+1}-z_k\|^3) \in \varphi(z_{k+1}) + F(z_{k+1}).$$

Thanks to the semistability of $\bar{z}$, we obtain

$$\|z_{k+1}-\bar{z}\| = O(\|z_{k+1}-z_k\|^3)$$

which implies

$$\|z_{k+1}-\bar{z}\| = O(\|z_k-\bar{z}\|^3)$$

and the cubic convergence follows.

Let us extend this study to the function φ whose second Fréchet derivative φ'' satisfies a Hölder-type condition. Thus, we get the following results.

Lemma 2 *Let $\varphi : \mathbb{R}^q \to \mathbb{R}^q$ be a function admitting a second order Fréchet derivative and let ψ be an approximation of φ previously defined. If φ'' satisfies a Hölder-type condition with positive constants α and K, one has:*

$$\|\varphi(v) - \psi(u, v)\| \leq \frac{K(\alpha+4)}{2(\alpha+1)(\alpha+2)}\|v-u\|^{\alpha+2}. \quad (8)$$

Proof of Lemma 2: Since φ'' satisfied a Hölder-type condition then we have

$$\begin{aligned}
\|\varphi(v) - \psi(u, v)\| &= \|\varphi(v) - \varphi(u) - \frac{1}{2}(\varphi'(u) + \varphi'(v))(v-u)\| \\
&\leq \|\varphi(v) - \varphi(u) - \varphi'(u)(v-u) - \frac{1}{2}\varphi''(u)(v-u)^2\| \\
&\quad + \frac{1}{2}\|(\varphi'(v) - \varphi'(u) - \varphi''(u)(v-u))(v-u)\| \\
&\leq K\|v-u\|^2 \int_0^1 (1-t)\|u + t(v-u) - u\|^\alpha dt \\
&\quad + \frac{K}{2}\|v-u\|^2 \int_0^1 \|tv + (1-t)u - u\|^\alpha dt \\
&\leq \frac{K}{(\alpha+1)(\alpha+2)}\|v-u\|^{\alpha+2} + \frac{K}{2(\alpha+1)}\|v-u\|^{\alpha+2} \\
&\leq \frac{K(\alpha+4)}{2(\alpha+1)(\alpha+2)}\|v-u\|^{\alpha+2}.
\end{aligned}$$

Thanks to this lemma and to the semistability, we obtain a similar result than Theorem 1.

Theorem 2 *Let $\bar{z}$ be a semistable solution of (1) and let (z_k) be a sequence computed by (4) which converges towards $\bar{z}$. If φ'' satisfies a Hölder-type condition then*

(i) *if* $\psi(z_k, z_{k+1}) - \Xi_k(z_{k+1}) = o\left(\|z_{k+1} - z_k\|^{\alpha+1}\right)$ *then* (z_k) *converges superlinearly*
(ii) *if* $\psi(z_k, z_{k+1}) - \Xi_k(z_{k+1}) = O\left(\|z_{k+1} - z_k\|^{\alpha+2}\right)$ *then* (z_k) *converges superquadratically.*

Proof of Theorem 2: The proof is the same as that of Theorem 1 in writing (4) as (7) and using (8), we get

$$\psi(z_k, z_{k+1}) - \Xi_k(z_{k+1}) + o(\|z_{k+1} - z_k\|^{\alpha+1}) \in \varphi(z_{k+1}) + F(z_{k+1}).$$

In case (i), it follows from semistability that

$$\|z_{k+1} - \bar{z}\| = o(\|z_k - \bar{z}\|^{\alpha+1})$$

hence (z_k) converges superlinearly.

In case (ii), we similarly obtain

$$\|z_{k+1} - \bar{z}\| = O(\|z_{k+1} - z_k\|^{\alpha+2})$$

which implies the superquadratic convergence.

Remark 1 If φ'' satisfies a weaker condition called center-Hölder condition with positive constants K_0 and α_0, the approximation ψ of φ verifies the inequality

$$\|\varphi(v) - \psi(u, v)\| \leq \frac{K_0(2\alpha_0^2 + 9\alpha_0 + 8)}{2(\alpha_0 + 1)(\alpha_0 + 2)} \|v - u\|^{\alpha_0+2}. \tag{9}$$

Using similar assumptions given in Theorem 2, one also obtains superlinear and superquadratic convergence for the sequence (z_k) defined by the method (4).

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