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MANUSCRIPT

Generalization: strategies and representations used by sixth to eighth graders in a functional context

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Abstract

We conducted a descriptive exploratory study in which we analyzed 313 sixth to eighth grade students' answers to a word problem, accompanied by diagrams, involving generalization in an algebraic functional context. In this research, we jointly addressed two objectives: (a) to determine the strategies deployed by students to generalize and (b) to identify the types of representation used to express their generalizations. We integrated how regularities are produced, evidenced in structures and represented by students. One of the most prominent findings was that functional strategy was used by almost all the students who generalized. They expressed the generalization using verbal, symbolical, or multiple representations. Ways of expressing regularities that are not restricted to algebraic symbolism are also shown. Although the potential to identify functional relationships was observed in sixth graders, seventh and eighth school students were able to represent more varied and structurally complex relationships. However, no relevant differences in generalization strategies were found between students of different ages with and without previous algebraic training.

Keywords Algebraic thinking · Functional thinking · Generalization · Representations · Strategies

Introduction

The literature suggests that the key areas to address to foster algebraic thinking include generalizing and representing generalizations, identifying algebraic structures, grasping the meaning of variables, and understanding their dynamic interrelationships (Radford, 2018; Warren et al., 2016; Wilkie, 2016). In this study, we focused on that thinking from a functional approach to school algebra where the

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function is the pivotal mathematical concept. Functional thinking is a component of algebraic thinking based on "the construction, description and representation of and reasoning with and about functions and their constituent elements" (Cañadas & Molina, 2016, p. 211).

Our focus is on generalization as an essential element in algebraic thinking (Kaput, 2008; Radford, 2018). In that context, this study contributes to the study of two areas of research: students' generalization strategies and the representations they used to express generalization. Different researchers have focused on the study of the strategies that elementary and high school students, and even adults, use in solving generalization tasks. Some of these studies focus on the variety of strategies used or what elements determine their use, mainly with figural pattern tasks (e.g., Amit & Neria, 2008; El Mouhayar & Jurdak, 2015; Stacey, 1989; Wilkie, 2016; Zapatera Llinares, 2018). Other researchers have described types of reasoning related to strategies (e.g., numerical, figural) (e.g., Becker & Rivera, 2005; El Mouhayar & Jurdak, 2016; Rivera & Becker, 2005). Earlier authors also reported that students were able to identify and establish functional relationships (Akkan, 2013; Amit & Neria, 2008; Stacey, 1989). Such studies also revealed different types of representation used either to express functional relationships or their generalization (e.g., Amit & Neria, 2008; Ureña et al., 2019; Blanton et al., 2019; Pinto et al., 2021; Wilkie, 2016). Our emphasis is to analyze the strategies used by students in the last year of elementary school (sixth graders without previous algebraic training) and students at the beginning of middle school (seventh to eighth graders with algebraic training) when solving a generalization word problem which involves a function and diagrams. They are implicitly required to construct a regularity based on their own productions, unlike some of the mentioned works in which figural patterns are explicitly given (e.g., Amit & Neria, 2008; El Mouhayar & Jurdak, 2015; Stacey, 1989). Specifically, we provide an in-depth description of the students' strategies to generalize, integrating the ways in which the recognized regularity is generated and evidenced through functional relationships structures, and how the students show in turn ways to represent generalizations. At the same time, we are interested to show nuances according to their age and grade. We offer a very comprehensive study with students who had either received no prior algebraic instruction or who had been introduced to algebra as the generalization of arithmetic (conventional approach with contents such as introduction to unknowns, simplification of algebraic expressions using structural properties, or equation solving).

Generalization

Generalization plays an instrumental, even a core, part in algebra (Mason et al., 2005). It is commonly defined to consist of recognizing and representing regularity and generating new particular cases. Pólya (1989) conceived generalization to consist of establishing new cases based on the regularity identified in a suite of elements. For Kaput (1999), to generalize is to extend reasoning beyond the cases at hand by either explaining the similarity present or broadening the reasoning involved to embrace patterns, procedures and structures, and their inter-relationships. According to Radford (2010), algebraic generalization is the identification of regularity in various elements of a sequence, which is then generalized to the rest, and using it to formulate an expression that represents the entire sequence. Stephens et al. (2017) distinguished between generalization as a process and as a product, maintaining that the latter would be obtained by (a) identifying the regularity of a suite of elements, (b) reasoning beyond the cases at issue, and (c) broadening the results beyond particular cases.

In this study, we adapted Kaput's (1999) definition of generalization, cited above, to a functional context. So conceived, generalization entails the extension, to other cases, of the recognized regularity in a task involving the establishment of a relationship between quantities, integrating the external representation of that regularity through a general rule.

Strategies and generalization

The procedures deployed to solve a problem, draw conclusions from a corpus of ideas, and establish relationships, known as strategies (Rico, 1997), provide insight into students' thought processes when solving problems. Interest has recently been voiced by research in exploring the strategies used by students to solve problems involving generalization (Amit & Neria, 2008; El Mouhayar & Jurdak, 2015) more exhaustively, both in the conventional and the functional approaches to algebra (Morales et al., 2018).

Studies focusing on generalization in the latter years of elementary school (ages 11 to 12, generally with no algebraic instruction) or the first two years of middle school (13- to 15-year-olds with some algebraic instruction) showed that students used a variety of strategies (e.g., El Mouhayar & Jurdak, 2015; Wilkie, 2016; Wilkie & Clarke, 2016) and described the difficulties their subjects encountered to apply the ones best suited to the task (Amit & Neria, 2008; Barbosa et al., 2012; Stacey, 1989; Zapatera Llinares, 2018). Even some research results suggest that students found functional relationships hard to identify and justify due in part to their choice of strategies (Moss & Beatty, 2006).

From functional thinking, students can use different approaches when progressing towards the generalization of functions (e.g., Blanton et al., 2015). Smith (2008) proposed three types of relationships: (a) recursive patterns, which focus on the variation of just one of the variables and finding its values from others previously obtained; (b) correspondence, in which the values of pairs of independent/dependent variables are correlated; and (c) covariation, in which the effect of changes in one variable on the other is analyzed. These relationships correspond with the mathematical relationships between variables.

From a research perspective, different studies highlight strategies used by students of different ages to generalize in solving problems involving on patterns or functions. Among these strategies are functional, recursive, or proportionality. Functional ones deal with expressing, analyzing, or using implicitly or explicitly a functional relationship between two variables (such as those described above) (e.g., Amit & Neria, 2008; El Mouhayar & Jurdak, 2015; Lannin et al., 2006; Stacey, 1989;

Zapatera Llinares, 2018). The recursive strategy involves the difference between consecutive terms (i.e., mainly procedures of the form $f(n)=f(n-1)+d$ are followed, with d as the common difference) (e.g., Amit & Neria, 2008; El Mouhayar & Jurdak, 2015; Lannin et al., 2006; Stacey, 1989). Underlying the third strategy is proportionality reasoning in which a value is determined as a product of others, or explicit use is made of the direct proportionality rule (e.g., Lannin et al., 2006; Stacey, 1989; Zapatera Llinares, 2018). However, several of the mentioned authors observed direct proportionality to be misused, primarily in general cases.

The presence of functional strategies was a prominent feature of the studies, given the association with generalization. Zapatera Llinares (2018), for instance, found that moving from additive strategies in near cases to functional strategies (which involve more complex processes) in far generalization ensured that third to sixth graders (ages 8 to 12) would successfully generalize linear patterns. While mathematically talented 11- to 13-year-old students were observed to deploy both functional and recursive strategies (Amit & Neria 2008) when generalizing linear and non-linear patterns, the former prevailed in both scope and efficacy. El Mouhayar and Jurdak (2015) also highlighted these strategies to be used in all tasks with linear and quadratic figural patterns across grades 4 to 11 (9 to 17 years old). The use of functional strategy tended to grow as the demand for generalization towards the general case increased, as revealed in other studies (e.g., Akkan, 2013; Lannin et al., 2006). Although an infrequent use of the functional strategy to generalize was detected, this use increased by grade level among students aged 10 to 15 years.

Other strategies are also recognized in previous studies. For example, counting elements of a figure (Barbosa et al., 2012; Stacey, 1989), the use of arithmetic operations not related to any specific pattern or regularity, and repetition of the problem statement (Merino et al., 2013). Other procedures are the use of acquired knowledge (e.g., knowledge of arithmetic progression), multiple of difference, stating and testing a rule not necessarily applicable to the situation (Akkan, 2013; Güner et al., 2013), or answering the questions posed without explaining how they found the answer (e.g., Zapatera Llinares, 2018). However, these are often less linked to generalizations.

Among the different strategies, results revealed a generalized tendency in elementary education, high school, and university to use more numerical approaches (i.e., based on numerical products) than figural ones (i.e., considering relationships between figures and their elements) (e.g., Becker & Rivera, 2005; El Mouhayar & Jurdak, 2016; Rivera & Becker, 2005).

Identifying and visualizing structures for generalizing are the key elements in the development of algebraic thinking (Hunter & Miller, 2022). The relationships and approaches of students to generalization tasks may be characterized in terms of the structures revealed in their productions, i.e., how an inter-variable regularity is organized and expressed (Pinto & Cañadas, 2017), or how indeterminate and/or numerical values operate when used or represented in the regularity. The structures identified by students when generalizing have been the subject of research. For example, Torres et al. (2019) described the structures recognized by second graders (7- to 8-year-old) when generalizing in a linear function context and even recognizes differences between working with particular and general cases. For instance, in a task involving the function $y=2x$, students reflected in their answers to questions with particular cases structures such as $y=x+x$ (adding the value of the independent variable to itself). Hunter and Miller (2022) recognized sophisticated forms of generalization in second year students (6 years old) that revealed structures of functional relationships in figural patterns. Such structures are means to understand how students interpret and generalize regularities since they show connections and relationships between mathematical concepts and processes (Ramírez et al., 2022).

From a functional context in the framework of early algebra, with sixth to eighth graders at the beginning of middle school (starting their algebraic training), we recognize an opportunity to complement the findings shown above by studying the strategies used to generalize, delving into the regularities and structures evidenced by the students.

Representation and generalization

As noted earlier, representation is closely linked to generalization and algebraic thinking (Kaput, 2008; Radford, 2018; Wilkie, 2016). It may involve progressive symbolization (Kaput, 2008). Specifically, we assume that representation of generalization refers to how the recognized generalization is evidenced and externally expressed (Ureña et al., 2019).

The ways of representing generalization are not restricted to algebraic symbolism. Ureña et al. (2019) describe different representations of the generalization of functional relations: (a) verbal expressions involving the indeterminacy of variables, (b) generic examples suggesting general relations, or (c) algebraic symbols. There is also the case where a regularity recognized in particular cases is expressed numerically by referring to the specific quantities involved.

Coordinate graphs, value charts (Torres et al., 2022; Wilkie, 2016), or pictorial representations are also useful for tasks that fosters functional thinking (Hunter & Miller, 2022), as well as combinations of various representations (Pinto et al., 2021).

Generalization research has distinguished differences in students' representations of generalization. Students benefitting from early algebra instruction proved able to identify inter-variable dependence and their use of (tabular, verbal, symbolic, or similar) representation progresses with their grade (Blanton et al., 2015; Blanton & Kaput, 2004; Carraher et al., 2008; Pinto et al., 2021). In a longitudinal study, Radford (2018) observed that second (7- to 8-year-old) to seventh (11- to 12-year-old) graders used a variety of semiotic (gestures, natural language, symbols) systems to express generalization. He argued that the information furnished by the semiotic systems used to represent generalization varies, for the way they address variables and their interrelationships, as well as the structure of the sequences involved in tasks, differ. Other studies focusing on pattern generalization with elementary school or early middle school students describe that students primarily use numerical representations in elementary education. As they progress to middle school, they generalize using algebraic symbolism (e.g., Akkan, 2013; Amit & Neria, 2008; Wilkie, 2016). But other forms of expressing functional relationships such as verbal expressions, combinations of symbols, and letters are also found (Wilkie, 2016), and pictorial representation proved to be a valuable resource to explore relationships and structures from the visualization in patterns (Hunter & Miller, 2022).

From the research context of functional thinking in elementary school, several researchers have studied the different representations when solving generalization tasks. These showed that second graders (ages 7 to 8) with no prior instruction in functional tasks used numerical or verbal representations to answer questions but did not generalize the functional relationships identified (Torres et al., 2019). There is evidence that, from third grade (8–9 years old), students use mainly verbal representations to express their reasoning, as is also the case with older students (10–11 years old) (e.g., Merino et al., 2013; Pinto & Cañadas, 2017, 2021). Pinto et al. (2021) highlighted that a significant part of third year students expressed correspondence functional relationships mainly through verbal, numerical, and multiple (combination of both) representations. Ureña et al. (2019), in turn, found that fourth graders (ages 9 to 10) represent generalization in functional relationships numerically, verbally, or symbolically and particularly with generic examples.

In line with above, this paper seeks to enrich existing research on algebraic thinking by focusing on the exploration and qualitative description of generalization representations in coordination strategies employed by sixth to eighth graders when solving a functional generalization task.

Research questions

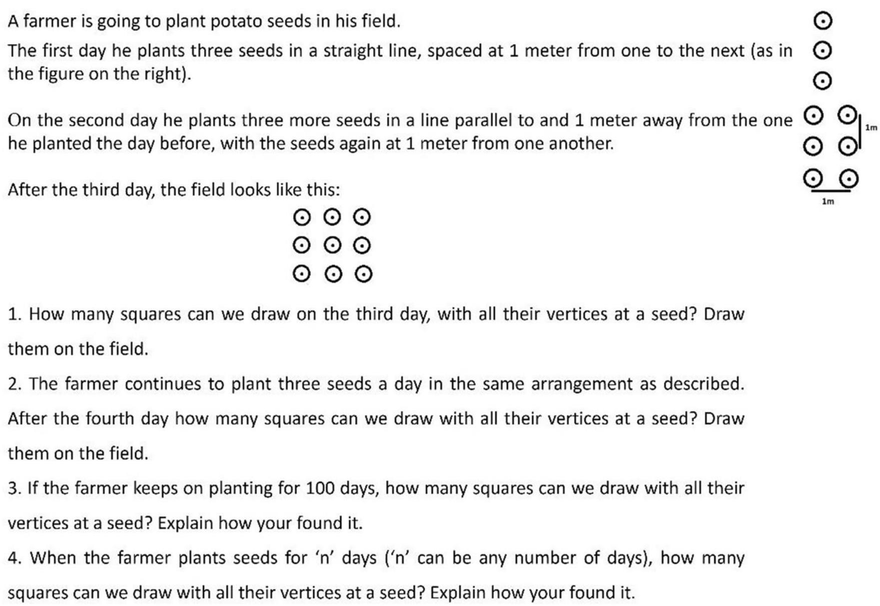
Building on the studies cited above, here, we proposed to provide answers to the following two research questions with sixth graders, without formal algebraic training, and seventh and eighth graders, who are beginning their introduction to school algebra. What strategies are associated with generalization in functional contexts? How do students represent their generalizations? Similar questions have been raised in previous studies. We intend to provide an in-depth description of strategies to generalize, integrating how students build and evidence regularities, the structures they showed in their productions, and the representations to express generalizations. From this description, we expect to qualify the differences found among students, taking into account the stage before and after their introduction to school algebra.

Methodology

This qualitative, descriptive, and exploratory study involved 313 students (ages 11–13) from different locations in Andalucía, Spain: 33 last year elementary (sixth grade) and 167 first and 113 s year middle school students (seventh and eighth grade, respectively). All students participated voluntarily, answering a questionnaire used as a test for admission to a mathematics skills stimulus program, oriented to students from these three grade levels (Ramírez & Cañadas, 2018). They were previously nominated by their teachers as good mathematical problem solvers and for showing an interest in mathematics. Those students with the best grades and performance on the test participated for 2 years in a math stimulation program with curricular enrichment sessions outside of school hours. The participants in this work were all sixth to eighth grade students who voluntarily solved the test, which determined the unequal distribution by ages. All of them were selected intentionally so that we could assume they have a positive attitude toward mathematics, a willingness to answer the questionnaire and, a priori, the aptitudes required to work with specific cases with no difficulties or to generalize. On those grounds, they could clearly be expected to constitute a valid source of information on the strategies used by students to generalize and the types of representation applied to exteriorize their reasoning. None of them previously worked with the proposed tasks.

In keeping with the area of interest addressed in the study, we analyzed students' replies to the only task on the questionnaire (the "potato seed" problem, Fig. 1, in fact, is designed with the purpose of the study in mind) that called for generalizing a functional relationship. After consulting the literature, we adopted the following criteria. The problem wording involved both verbal and pictorial representation. In the first two particular cases, students were asked to represent the situation pictorially. The task had an underlying linear function and an inductive structure. It could be performed using a number of strategies and included a question asking participants to justify their results.

The task was validated by program elementary and middle school teachers after assessing it for suitability to the students' age and mastery of mathematics, progressive complexity, and focus on the generalization of functional relationships.


A farmer is going to plant potato seeds in his field.

The first day he plants three seeds in a straight line, spaced at 1 meter from one to the next (as in the figure on the right).

On the second day he plants three more seeds in a line parallel to and 1 meter away from the one he planted the day before, with the seeds again at 1 meter from one another.

After the third day, the field looks like this:

1. How many squares can we draw on the third day, with all their vertices at a seed? Draw them on the field.
2. The farmer continues to plant three seeds a day in the same arrangement as described. After the fourth day how many squares can we draw with all their vertices at a seed? Draw them on the field.
3. If the farmer keeps on planting for 100 days, how many squares can we draw with all their vertices at a seed? Explain how your found it.
4. When the farmer plants seeds for 'n' days ('n' can be any number of days), how many squares can we draw with all their vertices at a seed? Explain how your found it.

Fig. 1 The potato seed problem

Task

Students' answers to the "potato seed" problem (Fig. 1), which called for generalizing functional relationships, were analyzed (Ramírez & Cañadas, 2018). The problem involved determining and justifying the number of squares (whose vertices were defined as specific points on a lattice) formed after 3, 4, 100, or n days of sowing. The task was meant to prompt students to recognize the underlying regularity and identify and generalize the functional relationships implicit in its formulation (Ramírez & Cañadas, 2018). That approach was expected to help them transition from one type of generalization to another, starting with cases designed to familiarize them with the task, then move on to a tentative informal generalization and from there to expressing it more formally with algebraic symbolism (Amit & Neria, 2008). The function relating the number of days to the number of squares is $y = 4n - 6$ (four new squares appear daily except the first and second days, when zero and two squares are formed, respectively).

Students could solve the task in different ways, for example, drawing the figures and counting the squares of each type, analyzing the corresponding areas and studying the formation of different squares for a given area, or analyzing the recursive construction process of the sequence of squares by determining the increment of squares at each step. Using a functional approach, they could also establish relationships between the number of days and the number of squares, for example, by positing that the number of squares is always 4 times the number of days minus 6 ($y=4n-6$).

Analysis

The unit of analysis adopted was each student's answers to all four cases proposed (3, 4, 100, and n days). Since not all the squares were always recognized or drawn, the correctness of the answers was not considered. We respect whether students recognized all the squares (e.g., Fig. 2a), the squares that rested on the base (e.g., Fig. 2b), or others (e.g., Fig. 2c). We analyzed all strategies used by the students when answering the different questions of the task, to describe in more detail those associated with their generalizations. We also studied how students represented generalization. The categories of analysis emerged from a preliminary review of the data, which were then refined and grounded in previous research. To that end, the

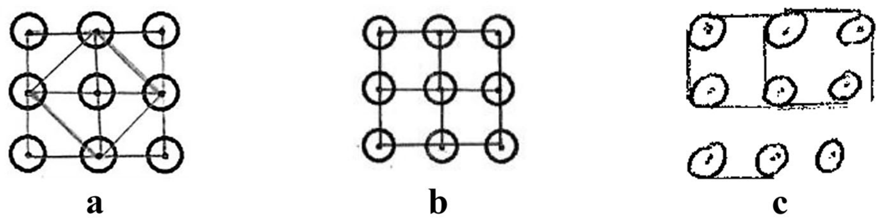


Fig. 2 **a** All squares, **b** squares resting on the base, and **c** other squares

first author formulated two sets of categories based on the representations of generalization defined by Ureña et al. (2019) and earlier studies on problem-solving strategies (e.g., Akkan, 2013; Amit & Neria, 2008; Barbosa et al., 2012; Merino et al., 2013; Stacey, 1989; Zapatera Llinares, 2018).

In the analysis of the data and description of the results, we labelled students with the letter "S" followed by a number from 1 to 313 and a numerical subscript to signify the grade (6, sixth; 7, seventh; 8, eighth). For instance, student $S110_7$ refers to seventh-grade student number 110.

To ensure data analysis validity and reliability, the other authors subsequently validated the coding by analyzing a random sample of students' written answers to the task in keeping with the categories proposed. After all the researchers agreed on how to code those results, they defined the following categories for strategies (Table 1) and representations of generalization (Table 2), evidenced by the students.

In correspondence with the generalization and based on the consistency of the procedures as the students progressed from case to case, three groups of strategies were recognized (Ureña et al., 2022). In the first group, the students answered founded on prior collected information and extended their reasoning to general cases (functional and arithmetic progression). Students in the second group incorporated procedures or reasoning applied to specific or isolated cases (counting, additive, and multiplicative operational and unexplained answer). In the third group, the students used a strategy based exclusively on the data found in the immediately preceding case or used procedures involving prior formulas or knowledge unrelated to the data of the problem (proportionality, recursive patterning, and other). This grouping of strategies made it possible to discriminate the procedures that led to a generalization and how it was evidenced.

When establishing the categories for representations of generalization, we deemed students to have represented generalization when they represented a general rule according to a regularity recognized in their productions and extended the regularity to other cases included the same task. Three types of representations of generalization were distinguished in the students' responses (see Table 2).

We recognize that other representations (e.g., tabular, pictorial, or numerical) were used by students to support their procedures or generalizations. However, they did not evidence explicitly regularity through them. In this line, the representations of generalization are studied in relation to the evidenced structures and the strategy used.

Results

We arranged the results under two headings: (a) generalization strategies and (b) representation of generalization.

Generalization strategies

Students were observed to deploy a range of strategies that varied depending on their grade of schooling and the case (=number of days) involved. We organized in Table 3 the total number of students who used different strategies by grade in each case posed in the task.

Table 1 Strategies used by students to reply to the questions posed

Strategy	Description	Example
Functional	Consists of establishing and making use of the functional (correspondence) relationship between related variables to describe the situation considered	100-day case: $S30_6$ write the expression Solution = No. of days × 3 = 100 × 3 = 300 squares
Arithmetic progression	Consists of applying the arithmetic progression formula $a_n = a_1 + (n - 1)d$ where a_n is the general value of the progression, a_1 is the value of the first term in the progression, and d is the difference between consecutive values	100-day case: $S312_8$ wrote "$a_n = a_1 + (n - 1)d$; $a_n = 2 + (100 - 1)4 = 398$ *squares*"
Counting	Consists of counting the elements comprising the solution	4-day case: $S1_6$ drew the squares and counted them (see Fig. 3)
Additive operationality	Consists of explicitly or implicitly applying separate sums unrelated to the operations performed in prior or subsequent answers	100-day case: $S44_7$ expressed the total number of squares as 300 + 7 = 307 (the first summand had been obtained by "100 *days* × 3 *seeds each day*." The second is the number of squares obtained in the 4-day case)
Multiplicative operationality	Consists of explicitly or implicitly applying separate products (multiplication) or quotients (division) unrelated to the operations performed in prior or subsequent answers	100-day case: $S10_6$ divided the total number of seeds by 4 (number of vertices in a square), obtaining 300 ÷ 4 = 75 squares
Unexplained answer	Consists of solving the problem with no evidence of the procedure followed	*n*-day case: $S119_7$ replied directly "133 squares."
Proportionality	Consists of applying proportional reasoning to obtain one answer as the result of another. It is distinguished from multiplicative operationality in its focus on the reasoning and procedure involved	100-day case: $S3_6$ wrote 3 days – 20 100 days – x 100 · 20 ÷ 3 = 66 I have done it by rule of 3
Recursive patterning	Consists of adding the difference between consecutive solutions to the value of the preceding case	*n*-day case: $S284_8$ answered "to find the number of squares x we have to look at the number of squares from the previous day and multiply by two."
Other	Consists of using procedures that cannot be classified as any of the above and are not applicable to the problem posed or the data given in the task or calculated by the student	*n*-day case: to calculate the number of squares $S47_7$ wrote "12 seeds + n = x seeds there are."

The data in the table show that arithmetic progression and recursive patterning were scantly used and only by eighth graders in the items referring to the 100- or n-day cases. Additive and multiplicative operationality were likewise sparsely used. The latter was nonetheless present in all grades in the third and fourth questions, primarily in the third, involving 100 sowing days. Additive operationality was deployed to a lesser extent and only by seventh and eighth graders.

The strategies most frequently used, for particular case 3 and 4 days, were unexplained answer and counting (Fig. 3). A wider spectrum of strategies was found for cases 100 and n days, with proportionality and especially functional the most common. Functional strategy was used more intensively by sixth graders for the 100-day case (30.00% compared to 21.56% by seventh and 23.00% by eighth graders) and conversely by seventh and eighth graders in the n-day case (18.56% in seventh and 21.24% in eighth grade compared to 15.15% in sixth).

In all three grades, all but one of the students who generalized used functional strategy in the items about the 100- or n-day, whereas the single exception applied arithmetic progression ($S312_8$). Using the approach proved to be conducive to representing the regularity identified. In this sense, there was no difference between the courses in the strategy to generalize.

In keeping with our interest on a deeper description of the strategies invoked by students to generalize, the following is a more detailed analysis of the answers of the participants who applied the functional strategy either generalized or who at least evidenced recognized a regularity with that approach. Its use was observed to result in one of two outcomes.

(a) Partial regularity: students used a regularity based on their analysis of the solutions for two particular cases or after analyzing a single particular case unrelated to any of the other solutions.
(b) Full regularity: students identified a regularity consistent with their analysis of the solutions to the preceding particular cases.

A breakdown of the number of students recognizing partial or full regularity with functional strategy in either the 100- or the n-day case, or both is given in Table 4.

Eleven sixth graders, 38 seventh graders, and 26 eighth graders used functional strategy in at least one of the last two cases posed in the task.

For a more complete description of the functional strategy, Table 5 illustrates functional relationship structures that students used implicitly or explicitly according to the type of squares they identified ($y = 4n - 6$ for all squares or $y = 3n - 4$ for squares that rested on the base), being this information a resource to later describe the generalization

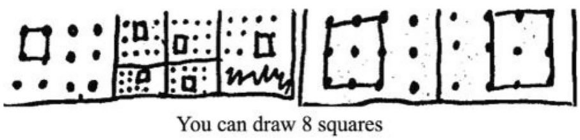


Fig. 3 Counting strategy by $S1_6$

Table 2 Representations of generalization manifested by the students in their answers

Representation	Description	Example
Verbal	The regularity detected is expressed in natural language, citing interrelated indeterminate quantities, and their relationships	100-day case: $S208_8$, expressed "the number of squares is always the number of days times 3 – [minus] 4."
Symbolic	The regularity detected is expressed using algebraic symbols to represent indeterminate quantities and their relationships	n-day case: $S245_8$ wrote $s = (n - 4) \cdot 3 + 8$
Multiple	The regularity detected is expressed using a combination of verbal and symbolic representation	100-day case: $S30_6$ answered $\text{Solution} = \text{No. of days} \times 3$

Table 3 Strategies used by students by case

	Cases 3 and 4			Case 100			Case *n*		
Strategy	6th	7th	8th	6th	7th	8th	6th	7th	8th
Counting	19	74	73	0	0	0	1	3	0
Additive operationality	0	1	0	0	4	1	0	0	1
Multiplicative operationality	0	0	0	3	15	7	3	4	1
Functional	0	0	0	10	36	25	5	31	24
Proportionality	0	0	0	5	30	21	3	10	8
Arithmetic progression	0	0	0	0	0	2	0	0	1
Recursive patterning	0	0	0	0	0	0	0	0	2
Unexplained answer	10	63	20	8	23	19	5	12	7
Other	2	9	9	3	12	6	3	11	7
No answer	2	20	11	4	47	32	13	96	62
Total	33	167	113	33	167	113	33	167	113

representations evidenced). It coordinates, by grade, the use of the functional strategy with the regularity recognized.

In partial regularity, the use of functional strategy translated primarily into the application of functional relationships with a multiplicative structure: $2n$, $3n$, or $4n$ (Table 5). One sixth-grade, seven seventh-grade, and three eighth-grade students invoked those structures, defined further to an analysis of the solutions for the 3- and 4-day cases, where the number grew at a constant rate of 2, 3, or 4 squares per day. Functional relationships based on the analysis of a single particular case, 4 days, were likewise used in connection with this strategy. Some students used the solution to derive a general structure applied to the following cases. $S240_8$, for instance, wrote the structure symbolically as $[(n-1)\cdot 2]+\left(\frac{n}{2}\right)+\left(\frac{n}{2}\right)$ based on the (six large, two medium, and two small) squares identified after 4 days. A similar approach was adopted by other students, although they used other structures. All the regularities recognized here were imprecise and unrelated to the first particular cases. For the n-day case, $S155_7$ symbolically represented the small squares as $n\cdot 2-2$ but then divided by four to determine the number of large squares (Fig. 4). They may have proceeded to divide by 4, an operation unrelated to their previous results, because they associated the solution to the first particular case (3 days), in which four small squares determined a larger square.

Table 4 Use of functional strategy: outcomes by grade for cases 100 and n

	Grade			Total
	6th ($n=33$)	7th ($n=167$)	8th ($n=113$)	
Partial regularity	3	15	8	26
Full regularity	8	23	18	49
Total	11	38	26	75

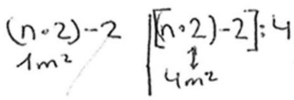
Table 5 Functional relationship structures by regularity

Regularity	Used structures	Examples	Grade		
			6th	7th	8th
Functional relationship $y = 3n - 4$					
Partial regularity	Multiplicative structure	$3n$ $2n$	0	6	2
	Other	$2n - 1$ $(n \div 2) + (n - 1) \cdot 2$ $(n \cdot 2) - 2 + [(n \cdot 2) - 2] \div 4$	0	6	3
Full regularity	Equivalent structure to $y = 3n - 4$	$3(n - 4) + 8$ $(n - 1) \cdot 2 + (n - 2)$	4	12	10
	Other	$(n - 1) \cdot 2$	4	7	3
Functional relationship $y = 4n - 6$					
Partial regularity	Multiplicative structure	$3n$ $4n$	1	1	1
	Other	$[(n - 1) \cdot 2] + \left(\frac{n}{2}\right) + \left(\frac{n}{2}\right)$ $(n + 2) + (n - 2)$	2	2	2
Full regularity	Equivalent structure to $y = 4n - 6$	$(n - 5) \cdot 4 + 14$ $(2 + 2 \cdot (n - 2)) + (n - 2) + (n - 2)$	0	4	4
	Other	$(n - 2) + (n - 2) + 4(n - 3)$	0	0	1
Total			11	38	26

Functional strategy leading to the identification of full regularity was the strategy most widely used (see Table 5). It was mainly recognized in the use of equivalent structures, according to the identified squares, revealing in turn that only eight students responded correctly to the task using the structure $y = 4n - 6$ or equivalents (Table 5). In this category, students used a consistent correspondence function in their solutions to the particular cases, as observed in S102$_7$'s answers. In the 100-day case, they used the functional relationship $(n - 5) \cdot 4 + 14$ to find all the squares formed, and for the *n*-day case, they wrote it out symbolically as $(n - 5) \cdot 4$. The starting point was the 14 squares formed in 5 days (Fig. 5).

Functional strategy linked to full regularities was based mostly on the grounds of the number of squares that rested on the base (Table 5). That is implicit in S1$_6$'s explanation "three more were added daily, so $(3 \cdot 96) + 8$" ($3(n - 4) + 8$) and equivalent expressions, such as proposed by S299$_8$ ($(n - 2) \cdot 3 + 2$) or by other seventh or eighth graders ($3n - 4$). We also recognize nuances in the structures of the functions used. On the one hand, the expressions depended on the size of the squares (e.g., Fig. 6). And on the other middle school, students used other structures to describe an even smaller quantity of squares. For instance, four seventh and one eighth grader used the structure $(n - 1) \cdot 2$ in their recognition of small,

Fig. 4 S155$_7$'s answer to the *n*-day case



$(n \cdot 2) - 2$ little squares can be form of 1m² or $[(n \cdot 2) - 2] + 4$ squares of 4 m²

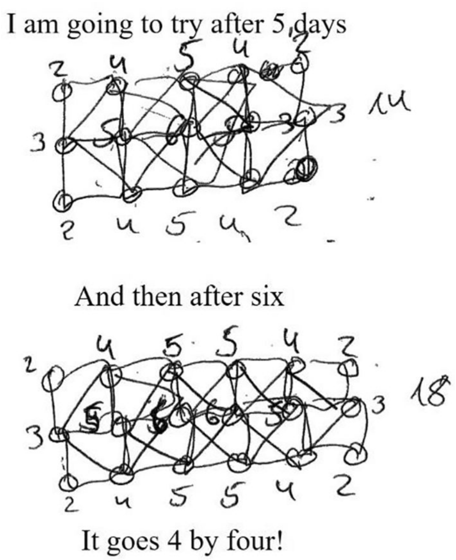


Fig. 5 Other additional cases considered by $E102_7$, 100-day case

interior squares. Student $S204_8$, in turn, represented the small squares forming in the upper row as $n-1$.

Although functional strategy based on full regularity was the most frequent in all grades, we identified differences in the structures of the functional relationships employed. Seventh and eighth school students represented varied and structurally complex relationships, being similar in both grades.

Representations of generalization

Students generalized in their answers to the items on the 100- and n-day cases. They represented the generalization of a regularity verbally, symbolically, or through multiple representation in both cases. Nevertheless, they could use other representations (e.g., pictorial, tabular) on which they supported or based their answers (e.g., see Figs. 7 and 8).

Fig. 6 $S179_7$'s answer to the 100-day case

Because the number of
squares is the addition
of the small ones
$(2 + 2 \cdot (n^{\underline{o}} days - 2)$
the middle ones
$(n^{\underline{o}}\ days - 2)$
and big ones
$(n^{\underline{o}}\ days - 2)$

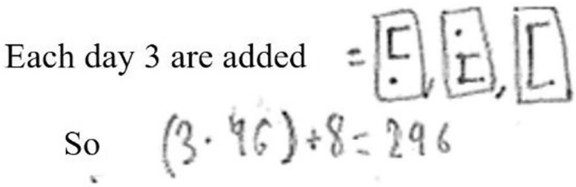


Fig. 7 $S1_6$'s answer to the 100-day case

The data on the categories used to represent generalization by grade of schooling and case are given in Table 6. The table also specifies the number of students who failed to represent the generalization but who gave proof of having identified a regularity, all of them applying functional strategy.

Further to the data in Table 6, all but four of the students who used functional strategy in the 100-day case recognized a regularity consistent with their answers to the particular cases. In the same case, other two sixth, eight seventh, and eighth graders although failed to represent the generalization in a general way, evidenced the identification of a regularity. They expressed relationships between the quantities adopted by the variables used to answer the items on the questionnaire numerically. Student $S85_7$, for instance, answered that $98 + 99 \cdot 2 = 296$ squares would form, using the structure represented symbolically in case n as $\text{No.of squares} = (n-2) + (n-1) \times 2$. In their replies to the n-day case questions, one sixth grader in this group of students used verbal representation of generalization and all the seventh and eighth graders' symbolic representation.

In general, for all grades, it stands out that the verbal representation of generalization was the most common in case 100. However, it contrasts that almost exclusively seventh and eighth grade students represented the generalization symbolically. While the multiple representations were used only by sixth and seventh grade students, the latter also represented the generalization symbolically.

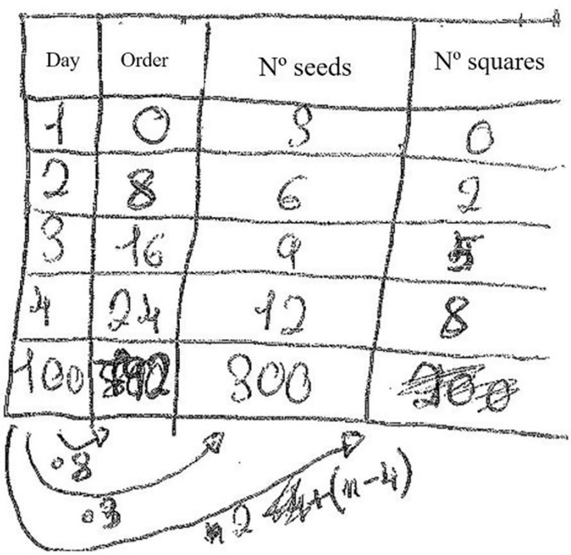


Fig. 8 $S292_8$'s table for the 100-day case

Table 6 Students' representations of generalization using the functional strategy

Case *n*					
Case 100	Absence of representation and identification of a regularity	Absence of representation and presence of identification of a regularity	Verbal rep	Symbolic rep	Multiple rep
Absence of identification and representation of a regularity	(0, 0, 0)	(0, 0, 0)	(1, 0, 0)	(0, 2, 1)	(0, 0, 0)
Absence of representation and presence of identification of a regularity	(1, 0, 1)	(0, 0, 0)	(1, 0, 0)	(0, 8, 7)	(0, 0, 0)
Verbal representation	(4, 7, 1)	(0, 0, 0)	(1, 7, 4)	(1, 10, 10)	(1, 0, 0)
Symbolic representation	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 1, 2)	(0, 0, 0)
Multiple representation	(1, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 3, 0)	(0, 0, 0)

In the parentheses, the forms a_1, a_2, and a_3, respectively, refer to the number of sixth, seventh, and eighth graders evidencing the representations of generalization, as appropriate

The sub-sections below describe the representation of generalization used by students broken down by the categories defined in the methodology.

Verbal representation

Verbal representation was the type that most widely used for the 100-day case by students in all three grades (7 sixth, 24 seventh, and 15 eighth graders). That representation of generalization was used less frequently in the *n*-day case, although the expressions proposed were similar to those observed in the 100-day case. Generalization was represented verbally in both last cases by all the seventh and eighth graders and one sixth grader.

Verbal generalization was often expressed along the lines of "three [squares] are added every day," and it was associated with correspondence functions with different structures that were observed under these representations. By way of example, student $S1_6$ (Fig. 7) used the structure $3(n-4)+8$ in the 100-day case, whilst $S214_8$ found that 285 squares would be formed by multiplying 95·3, i.e., applying the structure $3(n-5)+11$, but without the constant 11. In seventh and eighth graders, these verbal representations also were associated with multiplicative functional relationships such as $3n$, $2n$, or $4n$.

In connection with verbal representation, we found that seventh and eighth graders described the structure of the functional relationship used more accurately than sixth graders. In the 100-day case, $S98_7$ recognized that there were "$x = 2 + 98 \cdot 3 = 296$ because I realized that every day there were three more squares than the day before, so I thought since there were two squares in two days I would need to add three [every day] for 98 days." The functional relationship describing the student's reply was $2 + (n - 2) \cdot 3$. $S208_8$, in turn, contended that "the number of squares is always the number of days times 3 – [minus] 4," referring to the structure $3n - 4$.

Verbal representation was used but less frequently (one student per grade) to describe generalization in other functional relationships in fashions not consistent with the student's own calculations or results.

Symbolic representation

Symbolic representation, involving algebraic symbolism, was observed primarily in the *n*-day case. It was applied by one sixth, 24 seventh, and 20 eighth graders.

The sole sixth grader who represented generalization symbolically ($S1_6$) wrote "there are $(n \cdot 3)$ because every day three more are formed." Although that was consistent with the student's verbal representation for the 100-day case, it was inconsistent with the structure $3(n - 4) + 8$ applied in that case (Fig. 7).

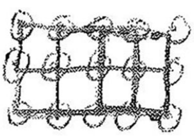


Fig. 9 $S272_8$'s answer to the 100-day case

It can be formed the double of squares of the number of the day in which they are planted. The number of the day + 2 is the number of little squares with area $2m^2$. The number of the day – 2 is the number of little squares with area $1m^2$

Fig. 10 $S14_6$'s answer to the *n*-day case

Seventh and eighth graders used a variety of structures in their symbolic representations. $S292_8$, for instance, described the structure as $2n + (n - 4)$ based on the table they built for the 100-day case (Fig. 8). Other equivalent structures were also represented symbolically for squares resting on the base as $(n - 2) + (n - 1) \cdot 2$ ($E85_7$), $n \cdot 3 - 4$ ($E114_7$), or $(n - 4) \cdot 3 + 8$ ($E245_8$), along with structures representing all the squares that could form, for instance, $(n - 4) \cdot 4 + 10$ ($E301_8$) or $2 + 4(n - 2)$ ($E179_7$).

Generalization was also represented symbolically in the 100-day case, although by seventh and eighth graders only. One seventh and two eighth graders used the same representation as in the *n*-day case (Fig. 9).

In connection with this type of representation, eight students in seventh and seven in eighth grades surprisingly moved from not representing generalization in the 100-day case, although identifying a regularity, to the use of symbolic representation in the *n*-day case (Table 6). A further 10 students in each of the two middle school grades and one in sixth grade moved from verbal representation in the 100-day case to symbolic representation in the *n*-day case.

Multiple representation

Multiple representation was found when students represented variables verbally and used numbers as terms in arithmetic operations to describe intervariable relationships (as in Fig. 6, for instance). Two sixth graders used this type of representation. For the 100-day case, $S30_6$ proposed the expression $\text{Solution} = \text{No. of days} \times 3$. Student $S14_6$, in turn, moved from verbal representation in the 100-day case to multiple representations in the *n*-day case (Fig. 10). Three seventh graders used multiple representation the same way as $S103_7$ with the expression $(\text{number of days} \times 3) - 4 = 296$. In the *n*-day case, all three represented generalization symbolically and consistently with the structure of the functional relationship. In that general case, they replaced verbal representation of the variable with the letter *n*. $S103_7$, for instance, gave the structure of the functional relationship as $n \times 3 - 4$.

Discussion and conclusions

This article describes the generalization strategies and representations used by a wide group of students with either no formal (sixth graders) or some initial (seventh and eighth graders) algebraic instruction. It enriches existing algebraic thinking literature focused on generalization by conducting a comprehensive analysis of ways in which the students, in line with generalization strategies, generated and evidenced the regularity, showed different structures, and represented their generalizations.

That approach also enabled us to compare their performance in an original school algebra generalization task in a functional context not only by type of case involved, but by year of schooling.

Unlike work with given patterns, here, students required to construct a regularity based on their own productions. In this context, functional strategy was almost the only procedure linked with generalization in all three grades. This strategy was evidenced using different structures of functional relationships and varied representations of generalization.

The research also distinguishes regularities on which the generalizations were based. In addition, it describes and contrasts by grade level, three representations of the generalization of a regularity (verbal, symbolic, and multiple), incorporating the nuance of evidence of recognition of a regularity at the numerical level. Implications for instruction are also given. We highlight that although the same generalization strategy is used at all three levels, differences in generalization representations are recognized between elementary and middle grade students. We also recognize similarities in the structures and representations of generalization applied by seventh and eighth graders that may be attributable to the fact that both had been introduced to algebra prior to completing the questionnaire. More varied and complex generalized and represented structures are identified in these grades.

The original functional task proposed was observed to induce the use, recognition, and representation (not necessarily in conventional algebraic notation) of variables and their relationships generating a space for enriching algebraic thinking, while at the same time, introducing students to functions. Those findings validated, complemented, and strengthened previous authors' results (e.g., Amit & Neria, 2008; Blanton et al., 2011; Lannin et al., 2006; Pinto et al., 2021; Radford, 2018; Warren et al., 2016; Wilkie, 2016).

Generalization strategies

With sixth to eight graders, functional strategy seems to be a significant way to generalize, complementing, from a functional context, findings reported by previous research developed mainly with patterns (e.g., Amit & Neria, 2008; El Mouhayar & Jurdak, 2015; Wilkie, 2016).

This strategy was found to be used most intensely in cases prompting generalization (here 100- or *n*-day cases). Its use may be induced by the existence of far cases and the concomitant cognitive demand involved in transitioning from near particular cases (with which students are familiar) to those requiring a different approach and more effective solving and generalizing strategies (Lannin et al., 2006). We support that the use of strategies is influenced by the nature of the cases (e.g., near, far or general) (El Mouhayar & Jurdak, 2015; Lannin et al., 2006). We agree that the use of functional strategy reveals student readiness to progress to other strategies depending on the demands of the case at issue and constitutes a key factor in representing generalization (Amit & Neria, 2008; Zapatera Llinares, 2018). This strategy stands out for being more complex, advanced, and efficient compared to the others by allowing relationships to be established and regularities identified to be generalized (El Mouhayar & Jurdak, 2015), thus being a reference to recognize students' algebraic thinking.

One of the contributions of this study is the distinction between two types of functional strategy depending on the degree of regularity referred to in the structure. Additionally, the identification of a variety of functional relationship structures informed the ways in which students interpreted and posited regularities and established relationships between data. Functional strategy was associated more often with full regularity, although on occasion, the structures were written or otherwise represented incompletely between one case and the next (with numerical coefficients either incorrectly applied or omitted). Other notable finding was the greater frequency of the partial recognition of regularities based on multiplicative structures not applied to the task in middle than in elementary school students. Partial regularities could be characterized from other strategies in the literature as "guess and check," where vaguely predicting a rule whose accuracy or validity was of little or no concern (Akkan, 2013; Güner et al., 2013).

Despite the broad presence of the functional strategy in students' generalizations, it is striking that few students responded correctly to the task. The inter-case differences in the structures derived from full regularities, along with the use of structures resulting from the partial recognition of regularities (i.e., not wholly applicable to the task proposed or consistent with students' own results) might be attributed to arithmetic errors or to a tendency to grant higher priority to finding an answer than to checking its accuracy. We do not discard a scant experience with generalization in functional contexts or the difficulty of the task itself as it involves modeling an unfamiliar situation (Lepak et al., 2018). These results likewise corroborate other authors' observation that students were not in the habit of checking their work (e.g., Akkan, 2013; Stacey, 1989). This information would provide elements to be taken into account for the instruction.

Lastly, we believe that the strategies used furnish information on students' instruction, experience, and practices as elements to be addressed. Their knowledge can be a valuable source for decision-making (El Mouhayar & Jurdak, 2015) not only in research but in classroom. For example, the study of strategies allows to recognize when a procedure is more appropriate, providing inputs for the strengthening or reorientation of different ways of reasoning and solving. The research revealed with elementary and middle school students' information on the difficulty entailed in the use of strategies suitable to generalizing, in an original word problem with diagrams, all of which serves as a complement to earlier findings (e.g., Amit & Neria, 2008; Barbosa et al., 2012; Moss & Beatty, 2006; Stacey, 1989; Wilkie, 2016). Information gleaned from the strategies used might also be deemed an important resource for designing tasks geared to developing essential components of algebraic thinking.

Representations of generalization

Another contribution lies in the description of the representations used by sixth to eighth-grade students to represent generalization in connection with the functional strategy, the regularities raised, and the structures used. The findings visualize the wealth and flexibility of students' use of (not only conventional symbolic) representation to organize and exteriorize their thinking about algebraic notions. This paper proposes a rearrangement of the representations of generalization put forward by Ureña et al. (2019), distinguishing between students who represented generalization and those who, while failing to do so, exhibited signs, specifically through numerical calculations, of having recognized a regularity. That recognition is important insofar as it means that students were able to establish what a set of specific cases had generally in common and be working with functional relationships implicitly, revealing ways of expressing regularities that are not restricted to algebraic symbolism, complementing the results of other studies (e.g., Amit & Neria, 2008; Wilkie, 2016). Each representation of generalization reveals differently the variables, their relationships, and the depth to which they are addressed (as contended by Radford, 2018). Concerning verbal representation or in the productions of the students who did not represent a generalization but evidenced the recognition of a regularity, the variables and the structure of the functional relationship are implicit, unlike the symbolic or multiple representations.

Verbal representation of generalization was more common in the 100-day, and symbolic representation in the *n*-day, item. Multiple representations were observed only in the 100-day case. These findings suggest that when students are asked about particular, even far particular, cases, they prioritize the use of verbal or multiple representation, confining the use of algebraic symbolism to cases that were called for in the task.

Although there was a wide use of the verbal representation, its inconsistency in describing the structures of the functional relationships stood out. The result could be due to the ambiguity of this representation (Molina, 2014), as opposed to multiple or symbolic representation, where the variables and functional relationship structures are obvious. Its prominence in elementary school coincides with the results in previous studies (e.g., Ureña et al., 2019; Merino et al., 2013; Pinto et al., 2021), attributable to students' familiarity with such representation (Merino et al., 2013; Pinto et al., 2021) and few opportunities with varied representations as contrasted by results in early algebra research (e.g., Blanton et al., 2015). Here, we observed middle school (seventh- and eighth-grade) students to also use verbal representation; most replaced it in the *n*-day or general case with symbolic representation.

Symbolic representation (or algebraic symbolism) was found primarily in *n*-day case and mainly among middle school students, clearly as result of these students' familiarity with the use of letters. A notable observation was that seventh and eighth graders used symbolic representation to express diverse and complex structures, showing the regularities through different ways. Results reinforce that the higher the grade of schooling, the greater student ability to flexibly apply a variety of representations, in keeping with the instruction received and students' cognitive development (Akkan, 2013; Blanton & Kaput, 2004; El Mouhayar & Jurdak, 2015; Radford, 2018). The conclusion that may be drawn is that whereas functional strategy was not found to be conditioned by age or level of instruction, the representations of generalization were.

Multiple representations involved words as variables in quasi-algebraic expressions revealing to be a preliminary to the symbolic representation of generalization, or even semi-symbolic representation (Amit & Neria, 2008). Most of the students applying that procedure replaced the verbal representation of the variable with the letter *n* when prompted to do so.

As in other research (e.g., Ramírez et al., 2022; Hunter & Miller, 2022), representations such as pictorial were supportive of some students' generalization. However, drawings were used by very few students (and only seventh and eighth graders) outside of the particular cases where they were instructed to draw, despite their presence in the task. From it, the structure underlying the pattern could have been extracted and generalized, even using a figural term as a generic example as it could be inferred from Kücheman (2010). Those findings are indicative of students' scant familiarity with such generalization tasks, the challenge they constitute for elementary and middle graders, or their learning experiences with varied representations. This could be a cause of the low number of correct answers in which all the squares that could be formed were recognized and generalized. We perceive a bias toward numerical approaches that might derive from instructional models (e.g., Becker & Rivera, 2005; El Mouhayar & Jurdak, 2016; Rivera & Becker, 2005).

As this research shows, there have been differences between students from different grades, before and after algebraic instruction only in the generalization representations exhibited, but not in the strategies to generalize. This information may have educational implications since it could suggest a school algebraic education more focused on the approach of conventional representations, such as the symbolic one, than on the development of diverse solving strategies and their articulation with multiple representations.

In the same line, we identified difficulties and issues in connection with other studies, as implications for teaching (e.g., Ureña et al., 2019; Barbosa et al., 2012; Blanton et al., 2019; Rivera & Becker, 2005; Stacey, 1989; Warren et al., 2016; Wilkie, 2016). Those include the efficient and suitable use of multiple representations, the identification of regularities associated with variability, the switch from one type of representation to another, the correct formulation and representation of the structure expressing the regularity identified, the coordination between different modes of reasoning (e.g., numerical, visual), and the processes of validation and justification of generalizations. As Mason et al. (2005) contend, students have the potential to think algebraically and the capacity to generalize and express generalization, but those aptitudes need to be harnessed and developed. Findings from early algebra research (e.g., Blanton & Kaput, 2004; Blanton et al., 2015, 2019; Carraher et al., 2008) show that students are able to understand and work with variables in the form of letters as indeterminate quantities or to represent and generalize functional relationships once introduced to such notions. This research suggests not only exposing students of different levels to generalization contexts but also considering approaches that integrate multiple components of algebraic thinking and mathematical skills. As Stacey's (1989), instruction plays an essential role in guiding students as they learn to organize their ideas, find suitable problem-solving strategies, and explore resources to express themselves.

Finally, we recognize that the limitations of the study were the small number of sixth graders who participated in the study restricting the variety of responses and making difficult to establish more balanced contrasts between school grades and the analysis of only written answers to a single task. Also, the small number of particular cases proposed may have restricted the recognition of generalizations prior to 100- or n-day cases. At the same time, the task did not allow us to recognize other types of generalization strategies, unlike in other works (e.g., Amit & Neria, 2008; El Mouhayar & Jurdak, 2015; Lannin et al., 2006). The large number of students who did not provide explanations for their reasoning or did not answer, mainly in the last two cases, among other things could suggest that it proposed a difficult task or that it required more mechanisms to obtain complement answers. Also, as the information collected comes from students with particular interests in strengthening their mathematical training, it would be interesting to expand the research with data from with different academic profiles in varied generalization contexts.

Author contribution Jason Ureña: conceptualization, data analysis, original draft preparation, and writing—review and editing. Rafael Ramírez: conceptualization, methodology, task design, data collection, supported and validated the analysis, and writing—review and editing. Marta Molina: methodology, supported and validated the analysis, and writing—review and editing. María C. Cañadas: methodology, task design, data collection, supported and validated the analysis, and writing—review and editing.

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Declarations

Ethics approval All the information published complies with the requirements of confidentiality, anonymity, and the necessary guarantees and permissions for its disclosure.

Competing interests The authors declare no competing interests.

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