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ORIGINAL ARTICLE

# Interpreting young children's multiplicative strategies through their drawn representations

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## Abstract

The exploration of children's drawings as mathematical representations is a current focus in early years mathematics education research. This paper presents a qualitative analysis of 72 kindergarten to Grade 3 (5 to 8 years old) children's drawings produced during problem-solving tasks centred on multiplicative strategies. Existing frameworks for the developmental sequence of mathematical drawings and the progression of children's strategies for multiplicative situations were an interpretive lens through which to analyse the drawings. Children used pictographic and iconic drawing types to represent the "story" in the problem and the multiplicative strategies employed to solve the tasks. Exploration of the children's drawings suggested that as children's drawings become more structural, schematic in nature, it may be easier for children to show their understanding of the structural elements of multiplicative relationships. Results revealed that structural elements of multiplicative relationships were more easily seen in iconic representations; however, both pictographic and iconic drawings were useful to observe counting, additive, and multiplicative strategies *when* mathematical elements of the problem were visible. Additional representations attached to the drawings (e.g. numerical) were needed to confirm children's strategies when their drawings lacked structure. These findings have implications for how young children's drawings are interpreted by classroom teachers. The interpretation of these drawings suggested that some children may not yet realise how their drawings in mathematics need to shift from illustrations of the problem's story context to representing mathematical ideas and processes — which requires intentional teaching of the purpose of drawings for mathematical contexts.

**Keywords** Primary mathematics · Children's drawings · Representations · Problem solving · Multiplicative strategies

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## Introduction

In mathematics classrooms, teachers' interpretation of young children's understanding of mathematics happens via a rich variety of representations including gesture, drawings, numerals, symbols, and oral and written language. In mathematics education research, representation is viewed as a key element in the teaching and learning of mathematics to access mathematical ideas, solve mathematical tasks (Bakar et al., 2016), develop relational understanding in mathematics (Thomas et al., 2002), and to assess children's understanding and thought processes (Pape & Tchoshanov, 2001). In a longitudinal study of young children's algebraic ideas and representations, Carraher et al. (2006) found that over time, and given the opportunities, children used "representations as a natural means of describing the events of problem" (p. 108). Their research places value on children developing schematic and general representations, incorporating them into their "repertoire of expressive tools" (p. 108).

Goldin and Kaput (1996) unravel some of the complexities of the construct of representation. They simplify the descriptions of internal vs external representations as *mental* and *physical* respectively. Mathematics is an act of sense-making (Schoenfeld, 1989). A child's ability to translate their internal schemas to external representations provides tangible evidence of how they make sense of mathematics. The external representation physically embodied in children's drawings is the focus of the current paper. Previous studies explored how children's internal and external representations align (Goldin & Shteingold, 2001; MacDonald, 2013), how representations are utilised by children as act or artefact (Thom & McGarvey, 2015), the role representations play in learning and doing mathematics (Goldin & Kaput, 1996), and how representations are a lens through which to notice children's mathematical thinking (Brizuela et al., 2000; Cheeseman et al., 2020; Way, 2018) and problem solving (Bakar et al., 2016; Dahl, 2019; Diezmann & English, 2001; Edens & Potter, 2007). In studying children's external representations, Mulligan (2018) cautions that researchers can only ever make inferences about children's external representations of their internal schemas, where the researcher's interpretations are tentative in nature (Thomas et al., 2002).

This study analysed children's drawings from a mathematical perspective during problem-solving tasks related to multiplicative situations. The study searched for evidence in drawings of multiplicative strategies—not mathematical development—since "drawings in themselves do not provide a coherent picture of children's developing mathematics" (Mulligan, 2018, p. 109). This current paper includes a critical appraisal of using drawings as artefacts to notice children's multiplicative strategies, discussing what issues and affordances are present. The findings are intended to add to existing research into children's mathematical drawings and build teacher awareness of "the need to explicitly support the development of mathematical drawing" (Mulligan, 2018, p. 106).

The paper addresses the following research questions:

***RQ1*** *What types of drawings do children create to represent multiplicative situations?*

***RQ2*** *What strategies for multiplicative situations can be observed in children's drawings when problem solving?*
***RQ3*** *What issues and affordances are present when interpreting children's drawings from a mathematical perspective?*

## Research perspectives

### Representational forms

According to Cai and Lester (2005), representations hold a dual function, to help children make sense of mathematical problems and allow communication of thinking to others. Pape and Tchoshanov (2001) depict representations "as tools for cognitive activity rather than products or the end result of a task" (p. 124). In the present paper's case, representations examined are external in nature and are viewed as a means by which children may represent their mathematical thinking regarding the task itself and their ongoing processes towards solutions for the tasks presented.

### Drawings as representations

Drawings are an external expression of children's understanding and ideas in graphical form (Hopperstad, 2008) and can be considered as "multimodal artefacts" (Deguara & Nutbrown, 2018, p. 5) which children use to represent mental images and convey meaning. Drawing representations allow teachers to access, assess, and attend to children's understanding (Thom & McGarvey, 2015). Bakar et al. (2016) reported that teachers and children alike use representations to explain thinking or scaffold understanding. Crespo and Kyriakides (2007) highlight the value in analysing drawings since children are more likely to favour drawings as a reasoning and communication tool than the use of symbolic representation that take time to develop in sophistication. Research during the past two decades has highlighted children's drawings in relation to spatial reasoning or geometric reasoning (Mulligan et al., 2020; Thom, 2018; Thom & McGarvey, 2015); measurement concepts (MacDonald, 2011; MacDonald & Lowrie, 2011; MacDonald & Murphy, 2018; Way, 2019); multiplicative thinking (Cheeseman et al., 2020; Dahl, 2019); subtraction processes (Way, 2018); and mathematical patterning and number structure (Carraher et al., 2006; Mulligan, 2018; Mulligan & Mitchelmore, 2009; Mulligan et al., 2005; Papic, 2015; Thomas et al., 2002). MacDonald (2013) notes several benefits recognised by researchers while investigating the value of drawing in children's mathematics. One benefit noted is that drawings capture children's meaning-making processes when employed "as a means of investigating children's strategies for solving open-ended mathematical problems" (Woleck, 2001 as cited in MacDonald, 2013, p. 68). Saundry and Nicol (2006) note that children use drawings in different ways when representing a process (strategy) or product (solution) during problem-solving.

### The development of children's drawings — pictographic and iconic

Children's drawings take on many forms and "sometimes more than one type of component is present in a drawing" (Thomas et al., 2002, p. 121). Existing research generally classifies children's mark-making (Carruthers & Worthington, 2006) into two main categories: pictorial and iconic. Pictorial, pictogram (Haylock & Cockburn, 2003), or pictographic (Bakar et al., 2016) drawings are defined as pictures of objects (Thomas et al., 2002), or realistic depictions of objects (Bakar et al., 2016) that children may use to represent features of a problem's context. Pictographic drawings usually reflect the narrative or story element of the problem presented such as animals or people. Alternately, iconic drawings "embody the intended objects" (Bakar et al., 2016, p. 89) and may include lines, shapes, tally marks, or dots (Thomas et al., 2002). Iconic drawings potentially reflect children's internal singular, or cluster of schemas (Athey, 1990) coordinated in drawings. Deguara and Nutbrown (2018) suggest schemas, for example, "enveloping and containing", "circular", and "back and forth", are often features of young children's drawings and are signifiers of action. Carruthers and Worthington (2006) suggest that these schemas identified by Athey (1990) are almost all linked to mathematics. Mulligan (2018) and Thomas et al. (2002) also include *notational* as a drawing category where "recordings are distinguished by the predominant use of numerals" (Thomas et al., 2002, p. 121). For Mulligan and Mitchelmore (2009), numerical notations when integrated with other structural features became evidence of a higher stage of structural development. Numerical representations are considered an integral element in children's drawings as tools for numerical reasoning (Kamii et al., 2001).

### Drawing as a way of communicating mathematics

MacDonald (2013) acknowledges a shift in how drawings are interpreted currently in research, particularly mathematical education research, where researchers are using drawings "as a means of investigating what children know" (p. 67). Mulligan (2018) concurs:

> Recent shifts in theoretical approaches based on 'embodied action' have re-directed attention to the role of drawings as more than artifacts that are used to assess what children have learned, "*re*presentations that reveal their cognitive schema— what they 'know' … (Thom, 2018) (p. 106).

Research orientated towards investigating children's drawings in mathematical situations proposes that drawings are not just the artefact as a product of children's representation of the problem's information, but drawings are an *action* that expands mathematical awareness (Thom & McGarvey, 2015). MacDonald (2013) proports that the *process* of drawing provides a useful method for researching with young children due to the familiar and non-threatening nature of the activity itself. It provides time for children to explain ideas in detail and opens a space for them to change or add to their drawing as needed. Child-generated drawings, although static in nature, may reveal mathematical thinking associated with how problems are solved. Analyses of drawings as *product* reflect children's re-imagining of the problem story, and their mathematical "'translation' from images in the mind to pictures on the paper" (Bakar et al., 2016, p. 91). Pape and Tchoshanov (2001) description of the purpose of representations highlights the translatory role of representations, connecting the thinking and doing of mathematics. Representations "serve to organize the individual's further work on a problem" (p. 125) which then may be used to facilitate an argument and to support conclusions.

### Eliciting children's multiplicative strategies through problem-solving tasks

Thom and McGarvey (2015) suggest that "drawing tasks … are a common means to assess geometric understanding" (p. 467). Studies have also been undertaken to assess numerical understanding using drawing tasks such as those by Mulligan and colleagues (Mulligan & Mitchelmore, 2009; Mulligan et al., 2005; Thomas et al., 2002). Mulligan and Mitchelmore (1997) and Cheeseman et al. (2020) studies specifically observed children's multiplicative representations in problem-solving tasks, for example, "there are 2 tables in the classroom and 4 children are seated at each table. How many children are there altogether?" (Mulligan & Mitchelmore, 1997 p. 314) and "can you make 12 little ducks in equal groups? Can you do it a different way? Draw or write what you did" (Cheeseman et al., 2020, p. 5). Both studies selected worded problems for their potential to produce a range of solutions and/or solution strategies and "require them [children] to make decisions about processes" (Cheeseman et al., 2020, p. 3) to be implemented. Downton (2010) study utilised challenging multiplicative problems with Grade 3 students. Although not focusing on drawings, Downton discovered that when challenged, students are capable of using sophisticated strategies. Utilising problem-solving tasks to elicit children's mathematical thinking through drawing representations aligns to the choice to use problem-solving tasks within the current study. Thus, providing scope to potentially address Cheeseman et al. (2020) concluding statement that "further research might investigate other effective problems that elicit young children's thinking about key ideas underpinning multiplicative thinking" (p. 14), and Calabrese et al. (2020) recommendation that "there is a need to explore further how students understand and represent problems requiring the operation of multiplication" (p. 1).

## Theoretical framing

The study is underpinned by the theoretical approach that mathematical knowledge is actively constructed by the learner (Goldin & Kaput, 1996) and that representations reveal what children know (Thom & McGarvey, 2015). Active construction of knowledge by the learner aligns with the theory of social constructivism where the learning process occurs while children are participating in, and contributing to (Cobb & Yackel, 1995), practices within the classroom. Social constructivism influenced the present study. Individual participation *in the act* of problem-solving and joint contribution *to the act* of solving problems are reflected in the choice to investigate children working individually and in small group situations. The creation of a range of learning environments and mathematical situations, such as problem-solving tasks to investigate children's multiplicative strategies, is supported by Schoenfeld (1992) belief that "mathematics is inherently a social activity" (p. 3). Two main theoretical frameworks for (1) mathematical drawing development and (2) multiplicative strategy development (discussed below) guided the analysis of drawings for this paper. Utilising these two frameworks for analysing children's drawings allowed for meaning to be drawn in relation to both, drawing as a semiotic process for children to communicate and represent multiplicative situations, layered with how children's pictographic and iconic representations depict multiplicative strategies.

## Mathematical drawing development

The concept of representation "is essential to understanding constructive processes in the learning and doing of mathematics" (Goldin & Kaput, 1996 p. 398). Young children's drawings provide a window into their internal representations and visualisation of a broad range of mathematical ideas (Crespo & Kyriakides, 2007; Ferguson et al., 2018). Children's drawings as an external representation of ideas can be observed through an interpretive lens. To interpret and classify children's drawing representations within this paper, Way (2018) *types of mathematical drawings* categories are used as an analytic framework. The sequence of six broad drawing categories for the development of mathematical drawings, based on Way's (2018) categories of drawing development adapted by Cartwright et al. (2021), is presented in Table 1.

**Table 1** Developmental sequence of mathematical drawings (Cartwright et al. 2021) adapted from Way (2018)

| Category | Category description |
|---|---|
| 1. Scribble | Incoherent, no representation of the mathematical story |
| 2. Picture | Shows pictures from the story problem (i.e. animal, farm) but no numerical labels or symbolic representations attached |
| 3. Emergent Story — incorrect process/solution | Shows pictures or iconic representations of the story and includes numerical values. No correct mathematical process or solution is visible |
| 4. Partial Story — errors with process or solution | Uses pictures or iconic representations and numerical values to show process of solving the problem. Correct process but incorrect/incomplete solution. Or correct solution with incomplete/incorrect process |
| 5. Partition and Solution | Uses pictures or iconic representations and numerical values during the process. Shows a correct solution |
| 6. Advanced Partition and Solution | Uses pictures or iconic representations and numerical values during the process. May include multiple solutions or patterns to find solutions |

## Multiplicative strategies development

Research-developed progressions of children's strategies when working in multiplicative situations were considered to analyse the strategies evident in children's drawings during problem-solving activities. A range of studies have previously created developmental frameworks or proposed aspects of multiplicative strategies to address (Anghileri, 1989; Cheeseman et al., 2020; Downton, 2010; Jacob & Willis, 2003; Mulligan & Mitchelmore, 1997; Mulligan & Watson, 1998; Steffe, 1994; Thomas et al., 2002). A synthesis of the research (see Table 2) led to four broad categories established for the current study's data analysis process: (1) counting strategies; (2) additive strategies; (3) pre-multiplying strategies; (4) multiplicative strategies. Cross-analysis with children's drawing development sought to discover evidence of both the categories of mathematical drawings and of multiplicative strategies, aiming to shed light on the use of drawings as a complimentary data source to children's other written representations such as numerical and/or symbolic.

**Table 2** Common strategies children employ in multiplicative situations

| Strategy | Description and related research |
|---|---|
| Counting strategies | Numbers represent a count or total |
| | One to one counting (Jacob & Willis, 2003) |
| Additive strategies | Makes equal groups |
| | Skip counting (Mulligan & Watson, 1998) or repeated adding 2, 4, 6, 8, 10 (Transitional counting and building up, Downton, 2010; Additive composition, Jacob & Willis, 2003; skip counting, Mulligan & Mitchelmore, 1997) |
| | Counts by an equal number, e.g. 4 + 4 + 4 ("doublets" Anghileri, 1989 p. 377) |
| | Uses known additive facts, e.g. 4 + 4 + 4 = 12 (additive calculation, Mulligan & Mitchelmore, 1997; Mulligan & Watson, 1998) |
| Pre-multiplying strategies (Steffe, 1994) | Uses numbers as markers of equal groups, e.g. 1, 2, 3, 4 … |
| | Coordinate a number of equal-sized groups, equal-grouping structure (Cheeseman et al., 2020) |
| | Repeated doubling |
| | Counts by multiples, e.g. 4, 8, 12 |
| Multiplicative strategies | Uses known multiplication facts 2 × 5 = 10 (multiplicative calculation, Mulligan & Watson, 1998) |
| | Uses array structure |
| | Partitions numbers using distributive property — "chunking", e.g. 22 is 12 + 10 then finds multiples of 4 and 2 respectively (wholistic thinking, Downton, 2010) |

## The present study

The children's drawings presented in this paper were taken from existing data (Cartwright, 2019, 2020) collected in primary classrooms in a study regarding mathematical fluency. In this paper, one aspect of that study is considered — children's drawings as a representation of their strategies for solving problems in multiplicative situations. In revisiting this data, $N=72$ kindergarten to Grade 3 (5–8 years old) children's individual and group work samples were selected where children had opportunities to respond to problem-solving tasks using drawings. The drawing artefacts are considered a rich source of data for revealing the ways in which young children make meaning in mathematics (MacDonald, 2013). Within the broader research study, children's drawings were seen as a *support* to numerical, symbolic, written, and oral representations.

The broader study's initial analysis of drawings (pictographic and iconic) was general in nature and identified that 53% of K-6 children (Grades K–3 80%, Grades 4–6 46%) used drawings as part of their working-out processes to solve problems, pictographic representations being more common in Grades K–3. Analysis of representations, particularly young children's drawings, became an important avenue by which to determine a child's mathematical fluency — especially for young children who were generally unable to record lengthy written reasoning. Utilising Way (2018) developmental sequence of mathematical drawings, children's drawings were categorised to indicate a possible progression from emergent to advance stages of representing mathematical situations. The exploration (Cartwright et al., 2021) reported that:

> … drawing ability by itself did not always correspond to a student's mathematical understanding. However, students who made direct links between drawings, numerical, and symbolic representations, showed a higher level of mathematical fluency. Further critical investigation is required to ascertain the benefits of drawings as a data source of mathematical thinking (Cartwright et al., 2021, p. 117).

A gap is identified from the broader study in that the specific multiplicative strategies children displayed in drawings within the problem-solving activities were not analysed, only the efficiency or accuracy of strategies selected. Therefore, the purpose of the present paper is to critically analyse young children's drawings, interpreting them from a mathematical perspective.

## Methodology

### Research design

A qualitative research approach was taken to explore drawing as a representation of children's multiplicative strategies. The study employed deductive analysis (Merriam & Tisdell, 2016) of children's drawings (artefacts), collected as work samples during mathematics problem-solving lessons from the broader study. Analysing children's mathematical drawings happened in two ways: (1) analysis of the drawing types and categories (adapted from Way (2018) drawing framework); (2) analysis of the multiplicative strategy development (using the research-based common strategies for multiplicative situations). Additionally, analysis considered what issues and affordances exist when interpreting children's drawings from a mathematical perspective. This study focused on children's drawings as one mode of external representation (Goldin & Kaput, 1996) where drawings are viewed as "a vehicle for young children to communicate their mathematical thought" (MacDonald & Lowrie, 2011). Cheeseman et al. (2020) used children's drawings as a "research tool to provide insight into their thinking" (p. 3), a strategy mirrored in the present study when examining children's drawings for the two problem-solving tasks.

## Participants

Two schools' data were utilised for analysis from the study. A total of $N=72$ kindergarten to Grade 3 children's (5–8 years old) work samples ($n=39$ individual samples and $n=25$ group samples) from mathematics lessons across 4 classes were analysed. School A is in a medium socio-economic metropolitan area of Sydney, with 32% of children with a non-English speaking background. School B is situated in a low socio-economic urban area of Western Sydney with 93% of children with a non-English speaking background. Drawing work samples from one kindergarten class ($n=16$, individually and in 6 groups) and two Grade 1/2 classes ($n=33$ in 11 groups) from school A, and one Grade 2/3 class ($n=23$ individually and $n=24$ in 8 groups) from school B were analysed.

## Data collection and tools

### The problem-solving lessons and tasks

Two mathematics lessons were taught in each of the four classrooms by the researcher for consistency in lesson delivery. The lessons followed a launch, explore, summarise model (Lampert, 2001). The explore phase of the lesson was when the problem-solving task was introduced, and children could spend time solving the task. Children solved one task individually during one lesson and solved the other in small groups (3–4 children) during a second lesson, see Table 3 for lesson timetable.

Table 4 presents the problem-solving tasks children had to solve (sourced or adapted from nrich.math.org) and the tasks' mathematical focus.

**Table 3** Lesson arrangements and problem-solving tasks

| **Class** | **Lesson 1 individual task and number of children** | **Lesson 2 group task and number of groups** |
|---|---|---|
| Kindergarten (K) | Hens task ($n=16$) | Farmer task ($n=6$) |
| Grade 1/2 M and 1/2S | | Farmer task ($n=11$) |
| Grade 2/3O | Farmer task ($n=23$) | Hens task ($n=8$) |

**Table 4** Problem-solving tasks overview and mathematical focus

| Task | Mathematical focus |
|---|---|
| *Hens task*<br>On a farm there were some hens and sheep. Altogether there were 8 heads and 22 feet. How many hens were there? | Finding the multiplier for two groups<br>Division of whole number (22)<br>Coordinating two composite units |
| *Farmer task*<br>The farmer saw 16 legs in the field. How many animals might he have seen? | Partition division<br>Finding the multiplier and the multiplicand<br>Recognising factors for 16 (2, 4, 8 — not 1 or 16)<br>Partition of numbers using combinations of composite numbers<br>Division of whole number (16) |

Children were encouraged through prompts and questioning to record their thinking and strategies on paper during the lesson. Note that for the Grade 1/2 classes, only group work samples are included here as during their individual problem-solving task a drawn representation was provided. Additionally, it is noted that the kindergarten lessons took place in term 4 at the end of a full year of school learning. Pseudonyms are used for individual children, and groups are referred to using a coding system (grade followed by group number).

Problem-solving tasks employed within the study provided no access to concrete materials and limited visual cues (an image of a sheep and a hen, and an image of a field for the "farmer task"). The purpose of having no concrete materials available was to observe how young children used written and drawn representations during the process of solving the problem as well as when recording their solutions. Kamii et al. (2001) suggest that materials, such as counters, "have properties of their own that interfere with children's representation of their ideas" (p. 34) and that children can think better with the markings they make "by externalizing their own ideas" (p. 34).

## Task settings

The use of individual and group settings provided opportunities for children to tackle a challenging problem at their own pace, to "re-invent mathematical ideas and concepts by themselves" (Takahashi, 2006, p. 3), and to collaboratively construct solutions by sharing their expertise (Goos, 2004) and diverse mathematical strategies (Good et al., 1989). Carruthers and Worthington (2006) reported that "as they make marks on paper, the children's mathematical thinking and understanding supports their meaning. In turn, as their marks and representations are co-constructed and negotiated with others, this extends their ideas" (p. 87).

## Data analysis

To critically analyse how young children's drawings can provide evidence of their multiplicative strategies, several rounds of analysis were employed. The drawings were collected, and the data were coded to the categories of mathematical drawing (Table 1).

Like Way (2018) process, sorting and re-sorting the work samples occurred several times by the researcher as both pictorial and mathematical features of the drawings were noticed and analysed. A second round of deductive analysis then occurred related to multiplicative strategies. Here, the drawings were re-analysed against the common strategies for multiplicative situations framework (Table 2). Analyses of drawings were the focal point of this process; however, numerical representations attached to children's drawings were examined during this process. Goldin and Shteingold (2001) and MacDonald (2013) emphasise the need to "recognise that a mathematical representation cannot be understood in isolation" (MacDonald, 2013, p. 67). Each sample was coded to the most sophisticated strategy visible within the drawings. If individual children or groups displayed a combination of strategies such as repeated addition ($4+4+4+4=16$) and multiplication ($2\times8=16$), in this case, the sample would be aligned with the multiplicative strategies level. The findings obtained through the iterative analysis processes within this study are considered exploratory interpretations as the study and analysis was conducted by one researcher as part of a PhD reviewed by another expert researcher. This limitation is acknowledged later in the paper's conclusion. Having another independent coder work with the data in the future would provide an intercoder reliability measure to the analysis.

## Revising of data analysis tools during the process

Further descriptive details were added to the mathematical drawing framework categories and an additional sub-category was required for Emergent Story to allow for differentiation in how some drawings were sorted (see the Appendix). This subdivision of the Emergent Story category occurred for two reasons; (1) it was noticed by the researcher that differences appeared in the mathematical structural features of children's drawings; (2) the "correctness" of children's drawings, and whether each element in the drawing corresponded to one element of the given problem was important. First, the reclassification of drawings was based on Mulligan (2018) structural levels that are also described as *descriptors of levels of structural development* in Thomas et al. (2002) study, and *broad stages of structural development* from Mulligan et al. (2004, 2005) studies. Second, like Crespo and Kyriakides (2007), it was apparent that supplementary analysis was required for the alignment between the mathematical "correctness" of drawings and answers. These additions provided a more nuanced analysis of the drawings.

## An example of nuanced analysis within the coding

Several Grade 1/2 groups' drawings were initially coded to the same strategy level, pre-multiplying. The drawings included visible grouping by separation, markings indicating *how many in each group* and/or circles indicating the *number of groups*.

**Fig. 1** Partition and Solution — pre-multiplying strategies (G12G4)


Fig. 2 Partition and Solution — multiplicative strategies (G12G3)

On revisiting the supporting representations on the work sample, some drawings were re-aligned to different strategy levels based on additive or multiplicative numerical/symbolic/word representations: G12G4 (Fig. 1) $4+4+4+4=14$; G12G3 (Fig. 2) "counting by eights"; and G12G2 (Fig. 3) $4\times4=16$.

Drawings were compared via their coding to the mathematical drawing categories and cross-referenced with the data on children's multiplicative strategies. The following section presents the findings of analysing the types of mathematical drawings and the multiplicative strategies embedded within children's drawings.

## Results

Of the combined individual and group work samples from kindergarten to Grade 3 children ($n=64$ work samples), 50 included pictographic or iconic drawings. Selected work samples are presented that illustrate specific drawing categories and multiplicative strategies.

### Categorising types of mathematical drawings for multiplicative situations

The determination of children's categories of mathematical drawings was based foremost on their pictographic or iconic representations. Alternate representations such as numerical, symbolic, or written were considered "supporting" representations. The number of children or number of groups mapped to each category is presented in the Appendix. Figure 4 visually represents the drawing categories observed from kindergarten to Grade 2/3.

Fig. 3 Partition and Solution — multiplicative strategies (G12G2)

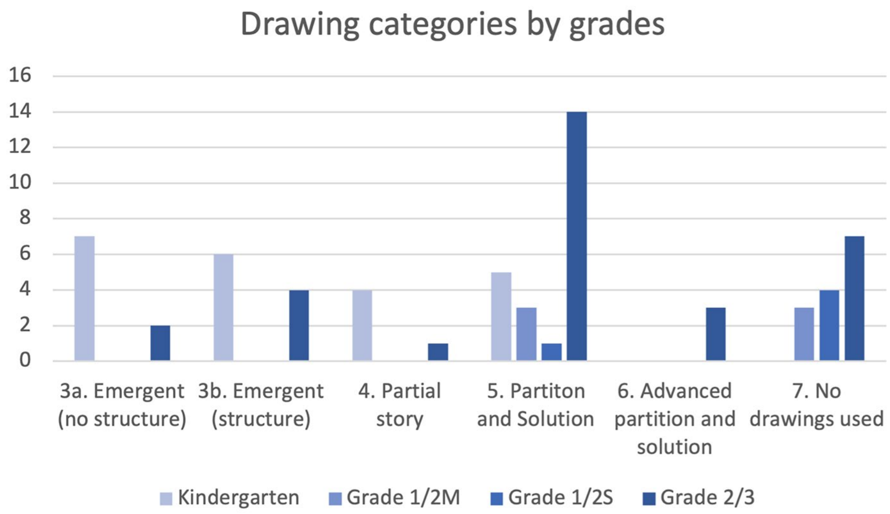



**Fig. 4** Drawing categories by grade

The graph indicates that drawings related to correct mathematical structure were observed mostly in Grade 2/3 children, and that the majority of kindergarten children's drawing types and structures aligned with the Story categories. No children produced scribble or pre-structural pictures to solve either task. All children who drew used pictographic or iconic drawings in some way to resemble parts of the problem.

**Fig. 5** Sean (Grade 2/3)

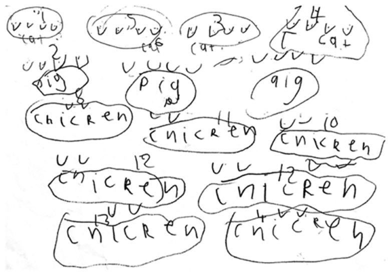


**Fig. 6** Zeke (kindergarten)

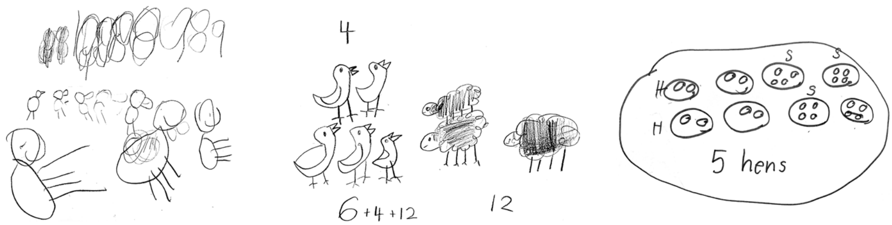
### Emergent Story

For Emergent Story (pre-structural), 7 individual children and 2 groups were unable to represent mathematical structure in their drawings in relation to the problem. Figure 5 depicts an example of Emergent Story (pre-structural). From this drawing, pictographic representations of animals in the problem are visible but no clear link to the mathematical nature of the problem can be observed.

The Emergent Story (structural) category drawings attended to the structural elements of the tasks, particularly the need for grouping or partitioning the animals. This category was evident in kindergarten and Grade 2/3 individual drawings. Ten children represented partitioning/grouping in their drawings; however, the drawings do not show a correct solution. Zeke's drawing in Fig. 6 shows grouping but no solution.

### Partial story, partition and solution, and advanced partition and solution

The remaining categories (Partial Story, Partition and Solution, and Advanced Partition and Solution) all required a correct representation of the problem solution within the drawing itself. Approximately 40% of individual children's drawings and 50% of group drawings showed a correct solution. To differentiate between the three categories however, numerical representations (labels and number sentences) provided the point of difference during analysis in conjunction




a b c

**Fig. 7** **a** Partial Story (Cooper kindergarten). **b** Partition and Solution (Issa kindergarten). **c** Advanced Partition and Solution (G23G1)

**Fig. 8** Partition and Solution (Mary Grade 2/3)

with structural features of the drawings. The continuum of developing mathematical drawings is observable in Fig. 7a–c related to the "hens task". The three drawings have representations of a correct solution of 5 hens and 3 sheep. The structural elements and numerical labelling of both Issa's and the Grade 2/3 group's drawing are more advanced than Cooper's as clear separation of the countable groups can be seen, and the numerical work attached aligns to the task.

## Partition and solution

Two children's drawings from Grade 2/3 that were coded to the Partition and Solution category used the farm context of the story to create an illustration of the narrative. Both had a correct solution visible in their drawing; however, the multiplicative process or action used to find the solution seemed "hidden" (see Fig. 8).

## Advanced partitioning and solution

The Advanced Partitioning and Solution category were only observed in drawing samples from Grade 2/3; one group used iconic representations with advanced structural features (grouping) to solve the "hens task" (see Fig. 7c), and two individual children used array structures when solving the "farmer task" (see example in Fig. 9).

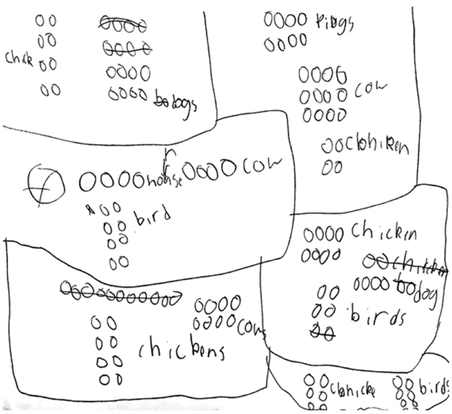



**Fig. 9** Advanced Partitioning and Solution (Rick Grade 2/3)

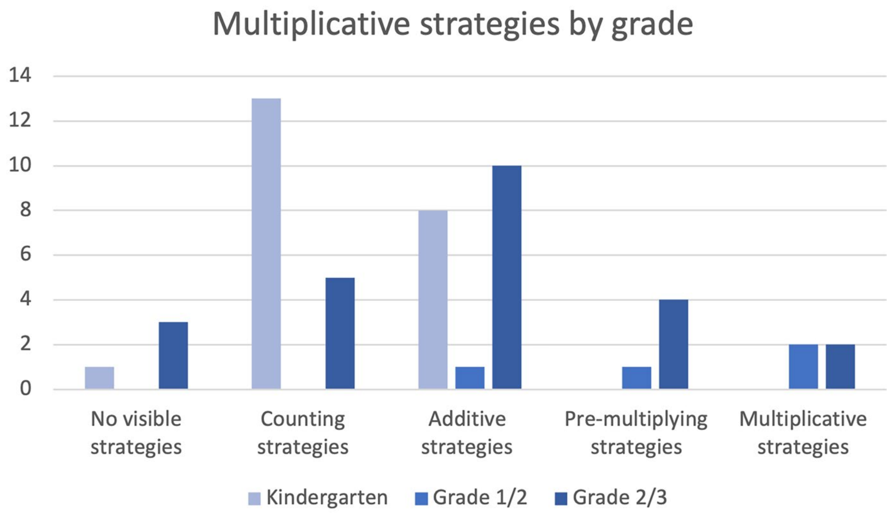



**Fig. 10** Multiplicative strategies by grade ($n = 50$ drawing samples)

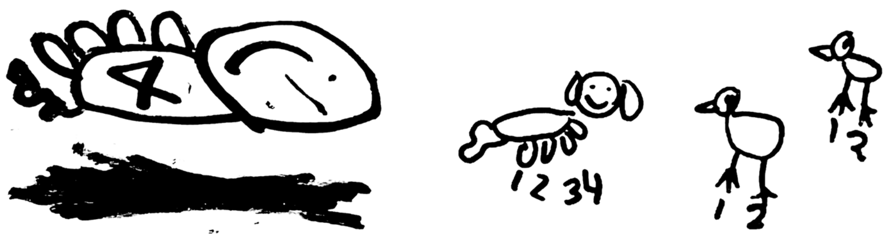
## Categorising types of strategies for multiplicative situations

The four strategies for multiplicative situations were counting, additive, pre-multiplying, and multiplicative strategies. Figure 10 presents strategy-use by children, or groups, visible within their drawings (both group and individual data are presented together as similar patterns appeared in strategy-use).

How children's drawings aligned with the strategies is reported in this section. The strategy results are presented sequentially from least sophisticated to most sophisticated for the multiplicative nature of the tasks.

## Counting strategies

Counting strategies were visible in 18 drawings from the individual and group work samples. Strategies included counting as labels on animals to represent the number of legs (Fig. 11a), counting-by-ones attached to legs, and repeated counting-by-ones for each animal (Fig. 11b).




a b

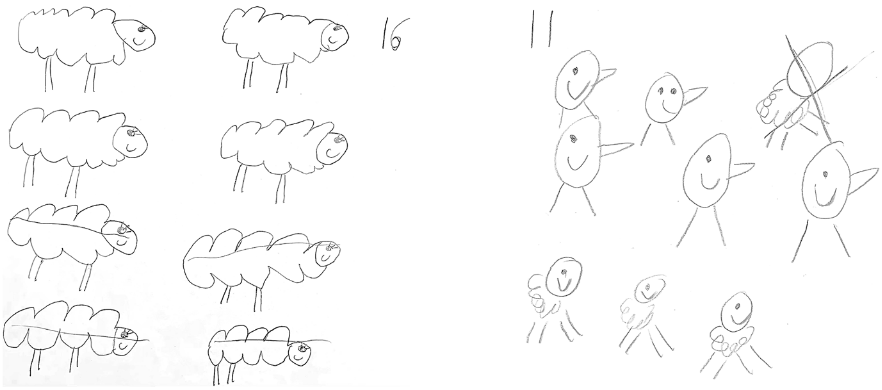
**Fig. 11** **a** Partition and Solution (KG2). **b** Partition and Solution (KG6)


**Fig. 12** Partition and Solution Additive strategies (KG1)

## Additive strategies

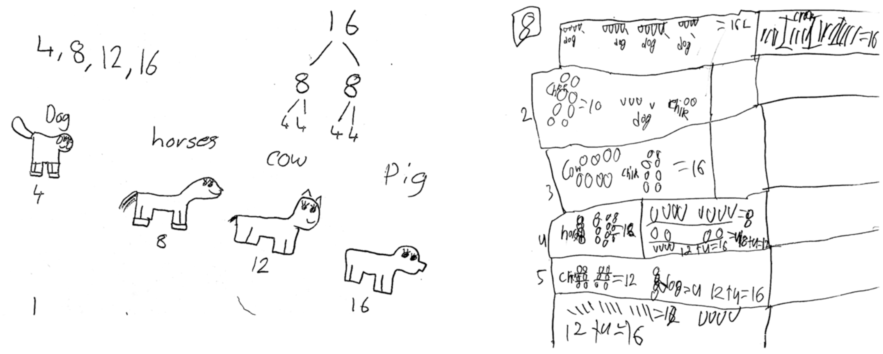
Additive strategies were coded to 19 drawings and were usually visible by the arrangement of legs (in pairs) in pictographic drawings, partitioning of animals into separate groups by space (see Fig. 7b), colour, and enclosures "animal pens" (two for the "hens task" or multiples for the "farmer task"). Partitioning could be interpreted as children's preparation process for a final count. Children's numerical representations that accompanied many drawings may be confirmatory of this analysis (Fig. 12).

Several drawings were analysed where limited or no numerical tagging was present to confirm children's strategies. In these non-numerical cases, assumptions were made about the types of multiplicative strategies these images may illustrate. Two examples are Sammy's response to the "farmer task" (Fig. 13a) and Erin's response to the "hens task" (Fig. 13b). The placement of the legs in pairs suggests a count-by-twos (additive). Both drawings also showed evidence of crossing out which may indicate a self-correction during the counting process.




**a** **b**

**Fig. 13** **a** Partition and Solution (Sammy Grade 2/3). **b** Partial Story (Erin kindergarten)




**a** **b**

**Fig. 14** **a** Partition and Solution pre-multiplying strategies (Ella Grade 2/3). **b** Advanced Partition and Solution multiplicative strategies (Ronnie Grade 2/3)

## Pre-multiplying and multiplicative strategies

A limited number of drawings included pre-multiplying (5 drawings) or multiplicative strategies (4 drawings) and are presented here together. An example of pre-multiplying processes such as counting-by-multiples, doubling and doubling again, and coordinating groups is presented in Ella's drawing in Fig. 14a. Ronnie and Rick's drawings (Fig. 14b and Fig. 9 above) were the only examples of children using an array-structured iconic drawing. Both Ronnie and Rick created multiple solutions.

## Categorising types of mathematical drawings with types of multiplicative strategies

Table 5 presents the strategies for multiplicative situations visible in the work samples cross-referenced with the types of mathematical drawings students' work samples contained. The majority of student work samples aligned with counting or additive strategies and included pictographic or iconic representations aligned with the problem. Children in Emergent or Partial Story drawing categories used counting and additive

**Table 5** Synthesis of mathematical drawing types and multiplicative strategies coding

| | No visible strategies | Counting strategies | Additive strategies | Pre-multiplying strategies | Multiplicative strategies |
|---|---|---|---|---|---|
| 1. Scribble | | | | | |
| 2. Picture | | | | | |
| 3a. Emergent Story (pre-structural) | | 7 | 2 | | |
| 3b. Emergent Story (structural) | 2 | 3 | 5 | | |
| 4. Partial Story (structural) | 2 | 2 | 1 | | |
| 5. Partition and Solution (structural) | 1 | 4 | 13 | 2 | 3 |
| 6. Advanced Partition and Solution (advanced structural) | | | | 1 | 2 |

strategies in most cases whereas children's drawings within the Partition and Solution and Advance Partition and Solution categories appeared to display more sophisticated additive to multiplicative strategies.

## Discussion

The findings illustrate the connection between drawn representation and multiplicative strategies and the importance of question context when posing mathematical questions. Previous research into children's mathematical drawings by MacDonald (2013) reflected that "representations are a powerful tool for accessing the ways children make meaning in mathematics" (p. 72). The focus on multiplicative strategies in drawings within the present study is congruent with Cheeseman et al. (2020) who used open-ended story problems to investigate children's visualisation of equal groups. The purpose of the present study was to critically analyse young children's drawings, interpreting them from a mathematical perspective. The three broad research questions (RQs) are utilised here to organise the discussion of the findings.

### RQ1 What types of drawings do children create to represent multiplicative situations?

#### Story context-focused vs mathematical concept-focused drawings

Subtle differences were observed between how children used the "story" of the problem-solving tasks when drawings were analysed. Issa's (Fig. 7b) drawing included relevant *story contexts from the task* (e.g. different pictographic drawings for the two different animal groups in the "hens task"), and also included partitioning related to the *mathematical concepts* inherent in the task, while some children like Mary (Fig. 8) used the problem's story more as a theme to create illustrations *of the story context itself*. For Mary, it is proposed that she is yet to realise the story presented in the problem need not be *re*presented in her mathematical solutions. Her drawings need to be simple, "more like diagrams than lifelike depictions of the objects and information provided in word problems" (Crespo & Kyriakides, 2007, p. 121). Carraher et al. (2006) suggest that as children become increasingly fluent in mathematics "they will be able to rely relatively less on the semantics of the problem situation to solve problems" (p. 110).

Observations also revealed some children's pictographic drawings were almost indecipherable in relation to the problem's mathematical *concept* (Sean, Fig. 5) whereas Ella and Ronnie (Fig. 14a, b) used drawings to model the multiplicative *concepts* of the task. Crespo and Kyriakides (2007) state that educators "seem to take for granted that students will know how to draw for the purposes of solving a mathematics problem … we cannot assume that students will know what to do when asked to draw a picture" (p. 124). Author et al. likewise noted that for some children the "difficulty actually arose from the lack of experience in drawing *for the purpose of mathematical representation* [emphasis added]. The children were daunted by the change in the function of drawing from a personal expression to a communication of precise meanings (Machón, 2013)" (p 91). From the present study's analysis, it is suggested that guidance is needed for young children in how to use their drawings to represent their mathematical ideas, processes, and strategies in either pictographic or iconic forms.

### Pictographic vs iconic drawings

There was no evidence to suggest that iconic drawings—although in some instances more closely aligned with a multiplicative structure or array model—were more advanced than children's pictographic representations in communicating their strategies. Not all iconic drawings represented multiplicative structure well. Alternately, several children used pictographic drawings well to represent multiplicative strategies. These findings are consistent with Dahl (2019) results where children's iconic drawings were no more sophisticated in showing multiplicative strategies than children who drew more pictographic representations. Like the current study, Dahl categorised children's drawings into pictographic and iconic groupings. However, in Dahl's findings, she reflects that "classifying drawings from pictographic to iconic on a continuum, is more fruitful than seeing it as a dichotomy" (p. 6).

### RQ2 What strategies for multiplicative situations can be observed in children's drawings when problem solving?

The range of multiplicative strategies used by children is discussed (organised by grade), then two sections follow reflecting on the structural nature of drawings and multiplicative strategies, and then on the impact of the problem-solving questions posed.

### Kindergarten's multiplicative strategies

Kindergarten children were able to represent multiplicative situations on paper, without the use of concrete materials, using counting and additive strategies (such as counting by ones, or skip counting by twos) for solving multiplicative problems. It is therefore argued that for these kindergarten children, number relationships such as counting, part-whole, and coordinating a count of composite units had already been constructed. This finding is confirmatory of previous research by Cheeseman et al. (2020) where children's drawings represented abstract mathematical thinking and called into question "the accepted view of the way early multiplication typically develops" (p. 14). Like Cheeseman et al., this current research suggests that drawings, as an alternate approach to using manipulatives, are useful in noticing children's complex early multiplicative strategies.

### Grade 1/2's multiplicative strategies

Only four of the 11 Grade 1/2 groups used drawings in representing their mathematical actions. Space does not permit for a lengthy discussion on why children in some groups did not use drawings. One hypothesised reason is the opportunity for oral interactions (Thompson & Rubenstein, 2000) and alternate representations such as gesture that group collaborations bring may have influenced their use, or lack of use, of drawings. Group work affords children with time to discuss, conjecture, challenge, and confer with peers prior to putting pen to paper.

Iconic drawings created by Grade 1/2 students reflected additive and multiplicative strategies of equal grouping, repeating units (tallies), and an array structure within the equal grouping.

### Grade 2/3's multiplicative strategies

The majority of Grade 2/3 drawings aligned with pre-multiplying or multiplicative strategies. Evidence of more advanced structural iconic drawings was observed in the Grade 2/3 individual drawing samples of Rick (Fig. 9) and Ronnie (Fig. 14b). Interestingly, Rick's work sample only contained iconic drawings, and no numerical representations accompanied the images. His strategies were observable as he drew different array structures for his solutions.

### The structural nature of iconic drawings and observing multiplicative strategies

It is proposed that multiplicative strategies were more noticeable in iconic drawings where the structure of grouping, tallying, arrays, and partitioning is inherently numerical. The structural nature of the drawings can be interpreted as strong evidence towards their development of multiplicative strategies. Although not generalisable from the current study's small sample, these findings add to and illustrate previous research by Mulligan and Mitchelmore (2009) who suggested that "that students who recognise the structure of mathematical processes and representations acquire deep conceptual understanding" (p. 33).

The synthesis of all work samples with drawings revealed a potential connection between children's representation of structural features (partitioning, grouping, or arrays) in their drawings and what multiplicative strategies can be observed (see Table 5 in the "Results" section). Whether the drawings were pictographic or iconic in nature did not affect the sophistication of multiplicative strategies observed—only the structural elements. It is implied that to observe advanced multiplicative strategies (from drawings alone), the array structure needs to be present. Within these work samples, all arrays were iconic representations. The intended purpose of the analysis process was not to make broad statements for immediate generalisation related to drawing and strategy-use. However, this finding on array structure and iconic representations in analysing the data requires further interrogation.

In drawings that were more narrative, representing the story, mathematical *solutions* were visible (by correct number of animals); however, mathematical strategies were not observed. In these cases, analysis of other representations was necessary. It is proposed that drawings alone can be used to interpret children's multiplicative strategies when mathematical structures are visible. Conversely, where numerical labels or notations accompany iconic or pictographic drawings, alignment is needed. These findings confirm Bakar et al. (2016) statement that "a correct drawing does not necessarily result in a correct solution" (p. 86).

### The impact the wording has on strategy choice in problem-solving situations

The problem-solving tasks employed in this study provided the numerical "total" within the stated question and therefore may be interpreted as requiring division or partitioning of 22 or 16 rather than a building up (Downton, 2010) of equal groups. The researcher acknowledges that the language of "altogether" presented in the wording of the "hens task" may have influenced children's choice of counting and additive strategies (actions) in solving the task. The Grade 2/3 groups were the only children who recognised the problem as requiring division — in one case they saw it as repeated subtraction using only numerical representations (number sentences). The wording of "how many" in the "farmer task" may have prevented children from representing the act of "sharing" or "dividing" through drawing representations. Research conducted by Way (2018) for subtraction utilised the words "flew away" in the problem. In Way's study, children drew arrows or lines to indicate movement related to subtraction. This brings to the fore division as an area for future investigation using problems worded specifically for division contexts. This suggestion reiterates Cheeseman et al. (2020) reflection that further studies are needed that "reveal how young children imagine equal groups to solve multiplication *or division problems* [emphasis added]" (p. 3).

### RQ3 What issues and affordances are present when interpreting children's drawings from a mathematical perspective?

Evidence was found that suggested numerous benefits to using drawings to observe children's multiplicative strategies. Likewise, several issues were also recorded. Here, issues could be rephrased as "considerations" when utilising drawings as a data source. "Issues" suggests that there may be a time when drawings are *not* useful. It is proposed that children's drawings (as representations) are *always* useful as a vehicle for young children to communicating their mathematical ideas—particularly when other communication forms (such as oral or numerical) are still developing. Drawing is an opportunity for young children to visualise and communicate mathematical concepts (Cheeseman et al., 2020) and make connections among the problems they are trying to solve (Crespo & Kyriakides, 2007).

Affordances when interpreting drawings from a mathematical perspective:

- Indicate "the presence or absence of important developing structural features such as equal grouping, partitioning, [and] array structure" (Mulligan, 2018, p. 109). Presence of array structural quality of iconic drawings is visible in Figs. 9 and 14b and early grouping structures were visible in pictographic drawings (Figs. 11a and 14a). Whereas in Figs. 5 and 7a, structural features were absent
- Beneficial for observing early multiplicative strategies such as skip counting by ones, twos, or fours when mathematical aspects of the problem are present such as the grouping of legs into pairs (see Figs. 7c, 12, and 13a)
- Valuable when numerical, written, and symbolic representations may not be as developed (incorrect or not present). Figure 9 uses no numerical representation to indicate multiplication, yet the array structure of their iconic drawings provided
- Useful to look for alignment between external representations. Figure 14b shows alignment between Ronnie's iconic drawings (arrays) and his numerical solution. These representations provide artefacts for future discourse with the child in relation to how he reached a solution of 16

Considerations when interpreting drawings from a mathematical perspective:

- Worded problems may result in drawings illustrating story more so than the problem's mathematical processes (Fig. 8). These children may not yet realise how their drawings in mathematical contexts need to represent mathematical ideas in a meaningful way
- Children may draw for different purposes, (1) to show their processes of solving the problem *as they are thinking*, (2) as the end product or artefact *after solving* the problem mentally (Saundry & Nicol, 2006). Talking with children at various times during lessons about their drawings may assist in connecting children's "image-making and mathematical understanding" (Saundry & Nicol, 2006, p. 61)

### Interpreting children's drawings from a mathematical perspective

Deguara and Nutbrown (2018) highlight that the content of children's drawings is influenced by their experiences, interests, and internal schemas. Children's drawings, as noted by MacDonald and Murphy (2018), are open to external influences such as their ability to draw and their interpretation of the task itself. The data analysis in the present study shows a potential alignment between the development of mathematical structures in drawings (both pictographic and iconic) and the ability of the observer to notice more sophisticated strategies for multiplicative situations. This analysis is not suggesting children's multiplicative strategies increase as drawing development increases, simply that as children's drawings become more structural, schematic in nature, it may be easier for children to *show* their understanding of the structural elements of multiplicative relationships. Research by Carraher et al. (2006) and Brizuela et al. (2000) encouraged children to integrate schematic diagrams and notations into their representations in problems requiring algebraic reasoning. Over the duration of their study, Brizuela et al. noticed how children's representations "became more and more context independent" (p. 3). They reflected that although children's early representations of "story" elements of the problems served a purpose, they would not be as useful when representing problems with underlying arithmetic structure. Although children are encouraged to move towards using iconic (schematic) notations as an abstraction of mathematical concepts (Brizuela et al., 2000), not all iconic drawings within the present study indicated advanced multiplicative strategies. The current study's data both adds to, and brings new questions to, Brizuela et al.'s findings. Evidence to support children being taught how to use more schematic representations for representing multiplicative structure is visible. Yet children's "readiness" to use iconic representations may still be open to question, creating a gap between what children are capable of drawing and how children communicate multiplicative strategies. There may be validity in supporting children to develop their own "invented" pictographic representations to communicate mathematical meaning more clearly as a bridge between pictures and iconic representations. Children may also be able to (verbally) reason about mathematical concepts but not yet represent them; therefore, verbal explanations need to be triangulated with other representational modes. The current analysis was limited to observing multiplicative strategies solely from drawings. The wider study analysed oral responses alongside drawings. The inclusion of analysing children's accompanying verbal explanations would potentially generate more robust findings when considering children's multiplicative strategies.

## Limitations of the study

Limitations exist for the current study. Comparable to Thomas et al. (2002) study, the analysis in the current paper is only a snapshot in time, where "methods of inferring aspects of children's internal representations from their externally produced representations are still exploratory, and not yet subject to tests of validity or inter-researcher reliability" (p. 130). Inter-researcher reliability was not possible as there was only one researcher conducting the study. A wider group of researchers is required to test the validity of the coding process. The analysis is only based on a small cohort of children; therefore, these findings are exploratory regarding the use of children's drawings to notice multiplicative strategies. Another limitation is the wording of the questions used in the problem-solving tasks. The wider study (Cartwright, 2019) which the drawing data had been selected from was not specifically exploring multiplicative strategies. A similar data analysis could be replicated using problems that feature words such as "times as many", "shared", "for each", or "divided". Mulligan and Mitchelmore (1997) list of "multiplication and division word problems" (p. 314) would be appropriate for further investigations.

## Implications for teaching

The findings from the present study have implications for future teaching regarding young children's mathematical drawings. Questions related to the affordances of using drawings emerged: How do we use story context in problems to allow children to understand the purpose and real-world application of mathematics (Calabrese et al., 2020)? How can we develop their ability to abstract their drawings to more "useful" mathematical models? How do we draw their attention to the mathematics of the problem? Further questions were raised about interpreting children's drawings when supporting numerical representations were incorrect (Crespo & Kyriakides, 2007).

### Implications for future research

Future research could explore if or when do numerical representations take precedence over drawings, investigating when do representational connections (translations) become a necessity. Do drawings require confirmation through numerical representations? Way (2018) posed a related question worth answering, "how much attention do educators give to actually teaching children to make explicit connections between mathematics concepts and drawn representations?" (p. 747). Implications for research on drawing development in mathematics in relation to pictographic and iconic representations also emerged from the results. Developmentally, children progress from using context-dependent "pictures" in mathematics towards more iconic, schematic diagrams. The question remains, when does this shift need to occur? Many children in the current study were able to produce correct solutions and strategies using pictographic representations. Dahl (2019) concluded about children's strategies and use of drawings that:

> ... even though the pupils are capable of using more advanced strategies, they prefer drawing … they consider drawings more suitable for this purpose than number symbols … [they] regard drawing as a legitimate way to reason … they need to model the situations in order to fully grasp the meaning of the numbers and the relations between them (p. 7)

## Conclusion

Analysis revealed important findings related to young children's drawings and multiplicative strategies: (1) some children are not aware that in mathematics, the purpose of their drawings need to shift to represent mathematics in a meaningful way, beyond "drawing as a personal expression" (Way, 2018, p. 98); (2) both children's pictographic and iconic drawings are useful for noticing multiplicative strategies *when* mathematical elements of the problem are visible; (3) it was easier to notice advanced multiplicative strategies in drawings that included the multiplicative structure of arrays. The present study reiterates the importance of meaning-making from children's representations. The analysis of children's drawings in this paper may assist teachers in understanding that much of mathematics learning, "is learning about representations" (Diezmann & McCosker, 2011, p. 168). Consequently, through noticing children's drawings from a mathematical perspective teachers can improve their ability in "'reading' and responding to their student-created representations" (Diezmann & McCosker, 2011, p. 169).

This paper adds to the existing mathematical drawing research base (Bakar et al., 2016; Brizuela et al., 2000; Cheeseman et al., 2020; Crespo & Kyriakides, 2007; Dahl, 2019; MacDonald, 2013; Mulligan, 2018; Way, 2018) by providing pictographic and iconic drawings that depict the range of strategies children use in multiplicative situations. The results add to Cheeseman et al. (2020) and Calabrese et al. (2020) request for research illustrating children's multiplicative thinking and representation of problems requiring the operation of multiplication. The addition of mathematical structural features to the drawing development framework brings a new perspective to observing mathematical drawing that requires structure to be attended to. In conclusion, from the example drawings analysed in the current paper, I wonder if a child may be able to understand and draw a representation of the mathematics, but not be aware of the need to convey structural *as well as* numerical features.

## Appendix. Drawings mapped to elaborated categories (Cartwright et al., 2021; Way, 2018)

| Category | Category description | K ($n=16$) | G 2/3 ($n=8$) | K ($n=6$) | G 1/2 ($n=11$) | G 2/3 ($n=23$) |
|---|---|---|---|---|---|---|
| | | Hens task | | Farmer task | | |
| 1. Scribble | Incoherent, no representation of the mathematical story | 0 | 0 | 0 | 0 | 0 |
| 2. Picture (pre-structural) | Shows pictures from the story problem | 0 | 0 | 0 | 0 | 0 |
| | Drawing does not portray a correct solution | | | | | |
| | No numerical or symbolic representations | | | | | |
| 3a. Emergent Story (pre-structural) | Shows pictures or iconic representations of the story problem | 6 | 1 | 1 | 0 | 1 |
| | Drawings do not resemble numerical structure needed for the problem (i.e. grouping) and do not portray a correct solution | | | | | |
| | Numerical process and/or solution visible but incorrect in general | | | | | |

| Category | Category description | K ($n=16$) | G 2/3 ($n=8$) | K ($n=6$) | G 1/2 ($n=11$) | G 2/3 ($n=23$) |
|---|---|---|---|---|---|---|
| | | Hens task | | Farmer task | | |
| 3b. Emergent Story (structural)<br>Some features relevant to the task | Shows pictures or iconic representations of the story problem<br>Drawing/s structurally align to problem but do not portray correct solution<br>Numerical process and/or solution visible but incorrect | 6 | 0 | 0 | 0 | 4 |
| 4. Partial Story (structural)<br>Most features relevant to the task | Uses pictures or iconic representations to show process of solving the problem<br>Drawing representation is accurately aligned with correct solution<br>Numerical labels, processes or solutions present, show correct process but incorrect/incomplete solution or correct solution with incomplete/incorrect process, but not both | 3 | 1 | 1 | 0 | 0 |
| 5. Partition and Solution (structural) | Uses pictures or iconic representations to show process of solving the problem<br>Drawing representation is accurately aligned with correct solution<br>Numerical values show a correct process and solution<br>May include multiple solutions | 1 | 2 | 4 | 4 | 12 |

| Category | Category description | K (*n* = 16) | G 2/3 (*n* = 8) | K (*n* = 6) | G 1/2 (*n* = 11) | G 2/3 (*n* = 23) |
|---|---|---|---|---|---|---|
| | | **Hens task** | | **Farmer task** | | |
| 6. Advanced Partition and Solution (advanced structural) | Uses iconic representations that represent the mathematical structure of the problem, e.g. arrays<br>Numerical values to show correct process and solution and align to drawings<br>May include multiple solutions or patterns to find solutions | 0 | 1 | 0 | 0 | 2 |
| No drawings | | 0 | 3 | 0 | 7 | 4 |
| Individual (I) or group task (G) | | I | G | G | G | I |

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## Declarations

**Ethical approval** Ethics approval was provided by University of Sydney Humans Research Ethics Committee (Project No. 2018/810) and from the NSW Department of Education (SERAP2018806).

**Informed consent** Informed consent was provided by parents of students through Participant Consent Forms collected under the ethic approval number listed above.

**Conflict of interest** The author declares no competing interests.



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